EXAM
MVE166/MVE165/MMG631
Linear and integer optimization with applications

2024-05-30

- **Date:** 2024-05-30
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

[3p]

MODELLING

Suppose we wish to perform four projects, each of which will run for three consecutive years. Each project generates an expected return (measured in MSEK) by the end of the third year, while it requires a certain amount of capital (measured in MSEK) per year during its three years. There is further a maximum available capital to invest in the projects during each of those three years. The numbers are given in the table below.

		Capital requirements (MSEK)			
Project #	Expected return (MSEK)	Year	#1	#2	#3
1	4.2		0.5	0.3	0.2
2	3.3		1.0	0.8	0.2
3	4.5		1.5	1.5	0.3
4	1.1		0.1	0.4	0.1
		Available capital (MSEK)	3.1	2.5	0.4

Assuming that we can only start a project during the current year (#1), which of the projects should be chosen in order to maximise the total expected return from the projects?

Formulate this problem as an integer linear optimization problem.

Do *not* solve the problem.

Question 2

SIMPLEX ALGORITHM

Consider the linear optimization problem to

$$\begin{array}{lrcrcrcr} \text{maximize} & z & = & 6x_1 & + & 5x_2 & & \\ \text{s.t.} & & & x_1 & - & 2x_2 & \leq & 6 \\ & & & x_1 & & & \leq & 10 \\ & & & & & x_2 & \geq & 1 \\ & & & x_1 & , & x_2 & \geq & 0 \end{array}$$

(a) **[1p]**

Reformulate this optimization problem on the standard form, such that it can be solved using the simplex method.

(b) **[2p]**

Solve the problem using the simplex method. At termination, what can be concluded about the properties of an optimal solution to this problem?

Question 3

Optimality and sensitivity analysis

Consider the linear optimization problem given by

$$\begin{array}{llrcrcrcrcrcrcl}
\text{maximize} & z = & 2x_1 & + & x_2 & + & 5x_3 & + & 6x_4, & & & & & & & \text{(1a)}\\
\text{s.t.} & & 2x_1 & & & + & x_3 & + & x_4 & + & x_5 & & & = 8, & & \text{(1b)}\\
& & 2x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & & & + & x_6 & = 12, & & \text{(1c)}\\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq 0, & & \text{(1d)}
\end{array}$$

where the variables x_5 and x_6 are slack variables.

(a) **[1p]**

Show that the variables (x_3, x_4) define an optimal basis for this problem. Explain your reasoning.

(b) **[1p]**

Consider a new variable, $x_7 \geq 0$, entering the problem, with objective coefficient c and the constraint vector $\mathbf{a}_7 = (d, e)^\top$, resulting in the following extended problem.

$$\begin{array}{llrcrcrcrcrcrcrl}
\text{maximize} & z^{\text{ext}} = & 2x_1 & + & x_2 & + & 5x_3 & + & 6x_4 & & & & & + & c \cdot x_7, & \\
\text{s.t.} & & 2x_1 & & & + & x_3 & + & x_4 & + & x_5 & & & + & d \cdot x_7 & = 8,\\
& & 2x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & & & + & x_6 & + & e \cdot x_7 & = 12,\\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & \geq 0.
\end{array}$$

What relations between the coefficients c, d, and e must hold for the basis defined by the variables (x_3, x_4) to be an optimal basis in the extended problem?

(c) **[1p]**

This subquestion should be solved independently of 3(b).

Assume that the right–hand–sides in the problem (1) are altered from $\mathbf{b} = (8, 12)^\top$ to $\mathbf{b} = (8 + \delta, 12)^\top$, where $\delta \in \mathbb{R}$. For what values of δ will the basis defined by the variables (x_3, x_4) become infeasible in (1)?

Question 4

Valid Inequalities

Define the set $X_{\text{IP}} := \left\{ \mathbf{x} \in \mathbb{Z}_+^2 \,\middle|\, x_1 - x_2 \leq 1, x_1 + 2x_2 \leq 6 \right\}$.

(a) **[0.5p]**

Draw an illustration of the set X_{IP} and its *convex hull*, conv (X_{IP}), in $\mathbb{R}^2$.

(b) **[1.5p]**

Show that the inequality $x_1 + x_2 \leq 4$ is a *valid inequality* for the set X_{IP}.

Hint: combine the inequality constraints in the definition of the set X_{IP} to derive a useful inequality.

(c) **[1p]**

Does the inequality $x_1 + x_2 \leq 4$ define a *face* of the set conv (X_{IP})?

Motivate your answer.

Question 5

[3p]

Lagrangean weak duality

Consider an integer linear optimization problem stated as

$$\begin{array}{rcllcl} z_{\text{IP}}^* & = & \min & \mathbf{c}^\top \mathbf{x}, & & \\ & & \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b}, \\ & & & \mathbf{x} & \in & X, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $X = \left\{ \mathbf{x} \in \mathbb{Z}_+^n \,\middle|\, D\mathbf{x} \leq \mathbf{d} \right\}$.

A Lagrangean relaxation of the constraints $A\mathbf{x} \leq \mathbf{b}$ defines the Lagrangean dual function $h : \mathbb{R}^m \mapsto \mathbb{R}$, given by

$$h(\mathbf{u}) = \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^\top \mathbf{x} + \mathbf{u}^\top (A\mathbf{x} - \mathbf{b}) \right\}.$$

Prove Theorem 17.1, that weak duality holds, i.e., show that $h(\mathbf{u}) \leq z_{\text{IP}}^*$ for all $\mathbf{u} \geq \mathbf{0}^m$.

Solution proposals to EXAM 2024-05-30

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These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

Solutions to Question 1

$x_j = 1$ if we decide to do project j, $j = 1, \dots, 4$.
$x_j = 0$ otherwise, i.e., not do project j, $j = 1, \dots, 4$.

$$\begin{array}{llllllll}
\max & 4.2x_1 & +3.3x_2 & +4.5x_3 & +1.1x_4 & & & \\
 & 0.5x_1 & +1.0x_2 & +1.5x_3 & +0.1x_4 & \le & 3.1 & \text{(year \#1)} \\
 & 0.3x_1 & +0.8x_2 & +1.5x_3 & +0.4x_4 & \le & 2.5 & \text{(year \#2)} \\
 & 0.2x_1 & +0.2x_2 & +0.3x_3 & +0.1x_4 & \le & 0.4 & \text{(year \#3)} \\
 & & & & x_j & \in & \{0,1\}, \quad j = 1, \dots, 4. &
\end{array}$$

Solutions to Question 2

(a) Standard form: include slack variables x_3, x_4 and x_5 to get equality constraints:

$$\begin{array}{lllllllllllll}
\text{maximize} & z & = & 6x_1 & + & 5x_2 & & & & & & & \\
\text{s.t.} & & & x_1 & - & 2x_2 & + & x_3 & & & & = & 6 \\
 & & & x_1 & & & & & + & x_4 & & = & 10 \\
 & & & & & x_2 & & & & & - x_5 & = & 1 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & , x_5 & \ge & 0
\end{array}$$

Add an artificial variable, x_6, to find a feasible solution by minimizing the sum of the artificial variables (phase I), which equals $x_6 = 1 - x_2 + x_5$.

$$\begin{array}{llllllllllllll}
\text{minimize} & w & = & & - & x_2 & & & & & + x_5 & & + & 1 \\
\text{s.t.} & & & x_1 & - & 2x_2 & + & x_3 & & & & & = & 6 \\
 & & & x_1 & & & & & + & x_4 & & & = & 10 \\
 & & & & & x_2 & & & & & - x_5 & + x_6 & = & 1 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & , x_6 & , x_6 & \ge & 0
\end{array}$$

(b) Simplex solve:

$\mathbf{x}_B$	w	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{B}^{-1}\mathbf{b}$
w	1	0	**1**	0	0	-1	0	1
x_3	0	1	-2	1	0	0	0	6
x_4	0	1	0	0	1	0	0	10
x_6	0	0	**1**	0	0	-1	1	1
w	1	0	0	0	0	0	-1	0
x_3	0	1	0	1	0	-2	2	8
x_4	0	1	0	0	1	0	0	10
x_2	0	0	1	0	0	-1	1	1

entering variable: x_2
leaving variable: x_6

optimum phase I: $w = x_6 = 0$

Reinsert the original objective function $z = 6x_1 + 5x_2 = 6x_1 + 5(1 + x_5 - x_6)$ and remove x_6 $\Longrightarrow$ phase II

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$
z	1	**-6**	0	0	0	-5	5
x_3	0	**1**	0	1	0	-2	8
x_4	0	1	0	0	1	0	10
x_2	0	0	1	0	0	-1	1
z	1	0	0	6	0	**-17**	53
x_1	0	1	0	1	0	-2	8
x_4	0	0	0	-1	1	**2**	2
x_2	0	0	1	0	0	-1	1
z	1	0	0	**-5/2**	17/2	0	70
x_1	0	1	0	0	1	0	10
x_5	0	0	0	-1/2	1/2	1	1
x_2	0	0	1	-1/2	1/2	0	2

entering variable: x_1
leaving variable: x_3

entering variable: x_5
leaving variable: x_4

entering variable: x_3
leaving variable: none, unbounded solution

Solutions to Question 3

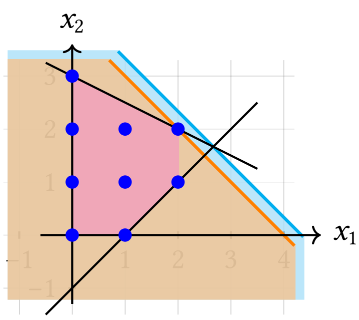
(a) Basis $(x_3, x_4) \Longrightarrow \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Longrightarrow \mathbf{B}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \Longrightarrow \mathbf{x}_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\mathbf{x}_N^\top = (x_1, x_2, x_5, x_6) = (0, 0, 0, 0)$. Hence the basis is feasible.

Reduced costs: $\bar{\mathbf{c}}_N^\top = \mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = (2, 1, 0, 0) - (5, 6)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} = (-8, -1, -4, -1) \leq \mathbf{0}$. Hence, the basis is optimal.

(b) If the reduced cost of the variable x_7 is non-positive, the current basis (x_3, x_4) will stay optimal: $\bar{c}_7 = c - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{a}_7 = c - (5, 6)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} d \\ e \end{pmatrix} = c - (4, 1)\begin{pmatrix} d \\ e \end{pmatrix} = c - 4d - e \leq 0$. The basis is optimal if it holds that $c \leq 4d + e$.

(c) The basis is infeasible if $\mathbf{x}_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 8+\delta \\ 12 \end{pmatrix} = \begin{pmatrix} 4+2\delta \\ 4-\delta \end{pmatrix} \ngeq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\Longrightarrow$ The basis is infeasible if any of the inequalities $\delta < -2$ or $\delta > 4$ hold, i.e., whenever $\delta \notin [-2, 4]$.

Solutions to Question 4



(a) X_{IP}: blue filled circles. $\text{conv}(X_{\text{IP}})$: pink polyhedron.

(b) According to Def. 14.3, a valid inequality is a linear inequality that is satisfied for all $\mathbf{x} \in X_{\text{IP}}$.

Any convex combination of the four inequality constraints $x_1 - x_2 - 1 \leq 0$, $x_1 + 2x_2 - 6 \leq 0$, $-x_1 \leq 0$, and $-x_2 \leq 0$ defines a valid inequality for the set X_{IP}.

For $\alpha_i \geq 0$, $i = 1, \dots, 4$, $\sum_{i=1}^{4} \alpha_i = 1$:

$$\alpha_1(x_1 - x_2 - 1) + \alpha_2(x_1 + 2x_2 - 6) + \alpha_3(-x_1) + \alpha_4(-x_2) \leq 0$$
$$\iff$$
$$(\alpha_1 + \alpha_2 - \alpha_3)x_1 + (-\alpha_1 + 2\alpha_2 - \alpha_4)x_2 \leq \alpha_1 + 6\alpha_2$$

Choosing $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$, and $\alpha_3 = \alpha_4 = 0$ then yields the inequality $x_1 + x_2 \leq \frac{13}{3}$ (illustrated in the figure by the turquoise halfspace).

Since x_1 and x_2 are both integers in any point in the set X_{IP}, the right–hand–side $\frac{13}{3}$ can be rounded down to 4 while still being valid for the set X_{IP}. This yields the valid inequality $x_1 + x_2 \leq 4$ (illustrated in the figure by the orange halfspace).

(c) The intersection of the set $\mathrm{conv}(X_{\mathrm{IP}})$ and $\left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, x_1 + x_2 = 4 \right\}$ is a face of $\mathrm{conv}(X_{\mathrm{IP}})$. The point $\bar{\mathbf{x}} = \binom{2}{2} \in X_{\mathrm{IP}} \subset \mathrm{conv}(X_{\mathrm{IP}})$ and it holds that $\bar{x}_1 + \bar{x}_2 = 4$. Hence, the inequality $x_1 + x_2 \leq 4$ defines a face of $\mathrm{conv}(X_{\mathrm{IP}})$.

Solutions to Question 5

See the course book.