EXAM
MVE166/MVE165/MMG631
Linear and integer optimization with applications

2024-01-03

- **Date:** 2024-01-03
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg (available only via the mobile number below)
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

[3p]

MODELING: CHOOSE STORAGE LOCATIONS THAT TOGETHER COVER ALL FACTORIES

A large manufacturing company decides to construct a number of storage facilities, in order to supply its factories.

After a careful investigation, n different geographical locations are selected as candidates for the storage facilities. Denote the factories by $F_1, F_2, \dots, F_m$, where m is the number of factories. Due to physical and geographical limitations, each storage location can supply only a specific subset of factories. Denote by $S_k \subseteq \{F_1, F_2, \dots, F_m\}$ the subset of factories that storage location k may cover. Also, denote by $c_k > 0$ the cost of constructing a storage facility at the kth location. It has to be ensured that each factory can be supplied by at least one storage facility.

The problem is to select a number of candidate locations, such that their total construction cost is minimized. Formulate an integer (or binary) linear optimization model that solves the problem described above.

[Hint: Assign a binary variable $x_k \in \{0, 1\}$ to each candidate location $k \in \{1, \dots, n\}$ and define parameters needed to formulate the model.]

Question 2

LINEAR OPTIMIZATION DUALITY AND OPTIMALITY CONDITIONS

Consider the following linear optimization problem

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^3}{\text{maximize}} \quad & -5x_1 + 8x_2 + 4x_3, && \text{(1a)}\\
\text{subject to} \quad & x_1 + x_2 = 2, && \text{(1b)}\\
& x_2 - x_3 \leq 3, && \text{(1c)}\\
& 2x_1 - x_3 \geq -1, && \text{(1d)}\\
& x_1 \geq 0, && \text{(1e)}\\
& x_2 \in \mathbb{R}, && \text{(1f)}\\
& x_3 \leq 0. && \text{(1g)}
\end{aligned}
$$

(a) **[1.5p]**

Formulate the linear optimization dual of the problem (1).

(b) **[1.5p]**

Utilize linear optimization complementarity (Theorem 6.5) to conclude whether or not the point $\bar{\mathbf{x}}$, defined by $\bar{x}_1 = 0$, $\bar{x}_2 = 2$, and $\bar{x}_3 = -1$, is optimal in (1).

Question 3

THE SIMPLEX ALGORITHM

(a) **[2p]**

Solve the linear optimization problem to

$$
\begin{array}{llrrrcr}
\text{maximize} & z = & 3x_1 & & +x_3, & & \\
\text{subject to} & & x_1 & +2x_2 & & \leq & 4, \\
& & 2x_1 & +x_2 & +x_3 & \leq & 10, \\
& & x_1, & x_2, & x_3 & \geq & 0,
\end{array}
$$

using the simplex algorithm. Explain all the steps in the pre-processing as well as in the algorithm.

(b) **[1p]**

Express all optimal solutions to the problem and explain the properties leading to your expression(s).

Question 4

INTEGER LINEAR OPTIMIZATION MODELLING

(a) **[1p]**

An optimization problem comprises the variables x_1, x_2, x_3, and y, which are all restricted to the values 0 or 1. The relations between these three variables should be the following:

$$
y = \begin{cases} 1 \text{ if } x_1 = x_2 = x_3 = 0, \\ 0 \text{ otherwise.} \end{cases}
$$

Model these relations using *linear* constraints.

(b) **[1p]**

In a linear optimization problem with non-negative variables, it is known that neither of the variables x_1 and x_2 can take a value larger than $M \gg 1$. It is also required that

either the constraint $3x_1 + x_2 \leq 4$ *or* the constraint $x_1 + 2x_2 \geq 10$

is fulfilled, but not both. Model this requirement using additional binary variables and linear constraints.

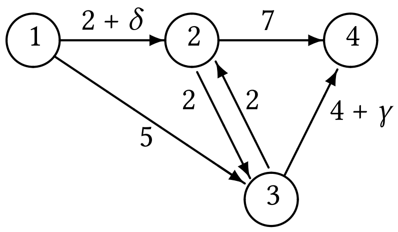
(c) **[1p]**

An optimization problem contains three binary variables, $x_1, x_2, x_3 \in \{0, 1\}$. Construct *linear constraints* that make the solutions $(x_1, x_2, x_3) = (0, 1, 0)$ and $(x_1, x_2, x_3) = (1, 0, 1)$ infeasible, but such that *no other* binary points are cut off.

Question 5

Basic feasible solutions and sensitivity analysis for a structured LP

Consider the graph below, consisting of a set of four nodes, $\mathcal{N} = \{1, 2, 3, 4\}$ and a set of six directed arcs $\mathcal{A} = \{(1,2), (1,3), (2,3), (2,4), (3,2), (3,4)\}$, with a corresponding vector of arc lengths $\mathbf{c} = (2 + \delta, 5, 2, 7, 2, 4 + \gamma)^\top$, where $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ are parameters.



Defining the variables $x_{ij} \geq 0$, for all $(i, j) \in \mathcal{A}$, the problem of finding a shortest path from node 1 to node 4 is formulated as to

$$\begin{array}{llllllllll}
\underset{\mathbf{x}\in\mathbb{R}^6}{\text{minimize}} & (2+\delta)x_{12} & +5x_{13} & +2x_{23} & +7x_{24} & +2x_{32} & +(4+\gamma)x_{34}, & & & \text{(2a)} \\
\text{subject to} & +x_{12} & & -x_{23} & -x_{24} & +x_{32} & & = & 0, & \text{(2b)} \\
 & & +x_{13} & +x_{23} & & -x_{32} & -x_{34} & = & 0, & \text{(2c)} \\
 & & & & +x_{24} & & +x_{34} & = & 1, & \text{(2d)} \\
 & x_{12}, & x_{13}, & x_{23}, & x_{24}, & x_{32}, & x_{34} & \geq & 0. & \text{(2e)}
\end{array}$$

Note that the flow balance constraint of node 1, i.e., $-x_{12} - x_{13} = -1$ is omitted, since it equals a linear combination of the balance constraints of nodes 2–4 (i.e., of the constraints (2b)–(2d)).

The model (2) is a special case of a linear optimization problem $\min_{\mathbf{x}\geq\mathbf{0}}\{\mathbf{c}^\top\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$, where (2b)–(2d) defines the equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, (2e) represents $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{b} = (0, 0, 1)^\top$. The matrix $\mathbf{A}$ can be partitioned into $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, where $\mathbf{B}$ is a basis matrix.

(a) **[1p]**

For parameter values $\delta = 2$ and $\gamma = 3$, an optimal basis in (2) is composed by the variables x_{12}, x_{13}, and x_{24}, thus corresponding to $\mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

Which is the corresponding shortest path, and what is its length (optimal value)? Show that the shortest path corresponds to a *basic feasible solution*. Motivate your answer theoretically.

(b) **[2p]**

For what values of the parameters $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ is the path $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$ optimal? What is then the length of the optimal path, as a function of δ and γ?

Motivate your answer by careful derivations.

You may utilize the following: The path $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$ is characterized by the values of the corresponding basic variables being $x_{12} = x_{23} = x_{34} = 1$, while $x_{13} = x_{24} = x_{32} = 0$ (the non-basic variables). This corresponds to $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution proposals to EXAM 2024-01-03

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These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

Solutions to Question 1

Define for $i \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$ parameters a_{ik}, such that $a_{ik} = 1$ if $F_i \in S_k$ and $a_{ik} = 0$ if $F_i \notin S_k$. The problem is modeled as

$$\begin{aligned} \min \quad & \sum_{k=1}^{n} c_k x_k, \\ \text{s.t.} \quad & \sum_{k=1}^{n} a_{ik} x_k \geq 1, \qquad i = 1, \dots, m, \\ & x_k \in \{0, 1\}, \quad k = 1, \dots, n. \end{aligned}$$

Solutions to Question 2

(a) The LP dual is stated as

$$\begin{aligned} \min_{\mathbf{y} \in \mathbb{R}^3} \quad & 2y_1 + 3y_2 - y_3, && && \text{(3a)} \\ \text{s.t.} \quad & y_1 + 2y_3 && \geq -5, && \text{(3b)} \\ & y_1 + y_2 && = 8, && \text{(3c)} \\ & -y_2 - y_3 && \leq 4, && \text{(3d)} \\ & y_1 && \in \mathbb{R}, && \text{(3e)} \\ & y_2 && \geq 0, && \text{(3f)} \\ & y_3 && \leq 0. && \text{(3g)} \end{aligned}$$

(b) The complementarity conditions applied to a pair $(\mathbf{x}, \mathbf{y})$, such that $\mathbf{x}$ is feasible in (1) and $\mathbf{y}$ is feasible in (3), are stated as: $(x_1 + x_2 - 2)y_1 = 0$; $(x_2 - x_3 - 3)y_2 = 0$; $(2x_1 - x_3 + 1)y_3 = 0$; $(y_1 + 2y_3 + 5)x_1 = 0$; $(y_1 + y_2 - 8)x_2 = 0$; $(-y_2 - y_3 - 4)x_3 = 0$.

If $\bar{x}_1 = 0$, $\bar{x}_2 = 2$, and $\bar{x}_3 = -1$ corresponds to an optimal point in (1) the complementarity conditions must hold for $\mathbf{x} = \bar{\mathbf{x}}$ and $\mathbf{y} = \bar{\mathbf{y}}$, which leads to: $(0+2-2)\bar{y}_1 = 0 \cdot \bar{y}_1 = 0$; $(2-(-1)-3)\bar{y}_2 = 0 \cdot \bar{y}_2 = 0$; $(2 \cdot 0 - (-1) + 1)\bar{y}_3 = 2\bar{y}_3 = 0 \Longrightarrow \boxed{\bar{y}_3 = 0}$; $(\bar{y}_1 + 2\bar{y}_3 + 5) \cdot 0 = 0$; $(\bar{y}_1 + \bar{y}_2 - 8) \cdot 2 = 0 \Longrightarrow \boxed{\bar{y}_1 + \bar{y}_2 = 8}$; $(-\bar{y}_2 - \bar{y}_3 - 4) \cdot (-1) = 0 \Longrightarrow \boxed{\bar{y}_2 + \bar{y}_3 = -4}$. The framed equalities $\Longrightarrow \bar{y}_1 = 12$ ((3e) holds); $\bar{y}_2 = -4$ ((3f) does not hold); $\bar{y}_3 = 0$ ((3g) holds). Since the point $\bar{\mathbf{y}}$ is not feasible in (3), complementarity does not hold for the pair $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

We conclude that the point $\bar{\mathbf{x}}$ is *not* optimal in (1).

Solutions to Question 3

(a) Include slack variables x_4 and x_5 in the inequality constraints $\Longrightarrow$ the following LP:

$$\begin{aligned} \max z = \; & 3x_1 && + x_3, \\ \text{s.t.} \; & x_1 + 2x_2 && + x_4 = 4, \\ & 2x_1 + x_2 + x_3 && + x_5 = 10, \\ & x_1, \; x_2, \; x_3, \; x_4, \; x_5 \geq 0. \end{aligned}$$

Simplex iterations:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$
z	1	**-3**	0	-1	0	0	0
x_4	0	**1**	2	0	1	0	4
x_5	0	2	1	1	0	1	10
z	1	0	6	**-1**	3	0	12
x_1	0	1	2	0	1	0	4
x_5	0	0	-3	**1**	-2	1	2
z	1	0	3	0	1	1	14
x_1	0	1	2	0	1	0	4
x_3	0	0	-3	1	-2	1	2

The last tableau is optimal since all reduced costs are ≤ 0. An optimal solution is given by $\mathbf{x} = \begin{pmatrix} 4 & 0 & 2 & 0 & 0 \end{pmatrix}^\top$ with optimal value $z^* = 14$.

(b) Since the reduced cost of all non-basic variables (x_2, x_4, and x_5) are strictly positive, the extreme point $\mathbf{x} = \begin{pmatrix} 4 & 0 & 2 & 0 & 0 \end{pmatrix}^\top$ is the unique optimal solution.

Solution to Question 4

(a)

$$\begin{aligned} y + x_1 + x_2 + x_3 &\geq 1, \\ y + x_1 &\leq 1, \\ y + x_2 &\leq 1, \\ y + x_3 &\leq 1, \\ (\&\ x_1, x_2, x_3, y &\in \{0,1\}) \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 + x_2 &\leq 4 + 4My \\ x_1 + 2x_2 &\geq 10y \\ 0 \leq x_1, x_2 &\leq M \\ y &\in \{0,1\} \end{aligned}$$

(c)

$$\begin{aligned} x_1 + (1 - x_2) + x_3 &\geq 1 \\ \iff x_1 - x_2 + x_3 &\geq 0 \\ (1 - x_1) + x_2 + (1 - x_3) &\geq 1 \\ \iff x_1 - x_2 + x_3 &\leq 1 \\ x_1, x_2, x_3 &\in \{0,1\} \end{aligned}$$

Solutions to Question 5

(a) The inverse of the basis matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The corresponding basic solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Hence, $x_{12} = x_{24} = 1$ and $x_{13} = 0$ which corresponds to the path $1 \longrightarrow 2 \longrightarrow 4$, which, by construction, corresponds to a BFS.

(b) The inverse of the basis matrix $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

The reduced costs of the non-basic variables are then given by
$\bar{\mathbf{c}}_N^\top = \mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N}$
$= (5,7,2) - (2+\delta, 2, 4+\gamma) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
$= \ldots = (5,7,2) - (4+\delta, 6+\gamma, -2)$
$= (1-\delta, 1-\gamma, 4)$.
The path $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$ is optimal when the reduced costs are nonnegative, i.e., when $\delta \leq 1$ and $\gamma \leq 1$.

The length of the optimal path is $2 + \delta + 2 + 4 + \gamma = 8 + \delta + \gamma$. It holds that this path is a shortest path when $\delta \leq 1$ and $\gamma \leq 1$ hold.