# EXAM <br> MVE166/MVE165/MMG631 <br> Linear and integer optimization with applications 

- Date: 2023-06-01
- Hours: 08:30-12:30
- Aids: Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- Number of questions: 5
- questions are not ordered by difficulty


## - Requirements

- To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
- To pass a question requires at least two (2) points out of three (3) points
- For higher grades (i.e., 4,5 , or VG) at most two (2) bonus points can be counted towards the grade
- Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
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- Phone: 0705-273645


## General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.


## Question 1

## [3p]

## Modelling of paper production

Paper can be produced from either pulp, recycled high quality paper, or recycled newsprint paper. Pulp costs 700 SEK/tonne, recycled high quality paper costs $450 \mathrm{SEK} /$ tonne, and recycled newsprint paper costs 150 SEK/tonne. Five different processes can be used for the production of paper. Producing one tonne of paper requires different quantities of the raw materials, depending on which process is used, according to the following table (numbers in tonnes):

| Process | Pulp | Recycled high quality | Recycled newsprint |
| :---: | :---: | :---: | :---: |
| 1 | 3 |  |  |
| 2 | 1 | 4 |  |
| 3 | 1 |  | 12 |
| 4 |  | 8 |  |
| 5 | 2 | 2 | 5 |

The supply of pulp is limited to 75 tonnes and at least $50 \%$ of the raw material used must come from recycled paper. The goal is to produce 100 tonnes of paper at minimum cost. Formulate this problem as a linear program.

## Question 2

## Simplex algorithm

Consider the following linear optimization problem:

$$
\begin{array}{lrl}
\operatorname{maximize} & z=3 x_{1}+9 x_{2}, & \\
\text { subject to } & x_{1}+4 x_{2} \leq 8, \\
& x_{1}+2 x_{2} \leq 4, \\
& x_{1}, & x_{2} \leq 0 .
\end{array}
$$

(a) $[1.5 p]$

Solve the problem using the simplex method. State all optimal solutions to the problem, report your calculations, and explain your reasoning.
(b) $[1.5 \mathrm{p}]$

Formulate a linear optimization dual of this problem and derive all optimal dual solutions. Verify the optimality of the dual solution(s) identified using relevant theory.

## Question 3

## [3p]

## Convexity of the feasible set of a linear optimization problem

Consider a linear optimization problem, stated as

$$
\begin{array}{ll}
\operatorname{minimize} & z=\sum_{j=1}^{n} c_{j} x_{j}, \\
\text { subject to } & \sum_{j=1}^{n} a_{i j} x_{j} \leq b_{i}, \\
& i=1, \ldots, m, \\
& x_{j} \geq 0, \\
& j=1, \ldots, n .
\end{array}
$$

Theorem 4.1 in the course book expresses that the feasible set of this problem is a convex set. State and prove this theorem.

## Question 4

LP SENSITIVITY AND INTEGRALITY
Consider the linear optimization problem to

$$
\begin{array}{rr}
\operatorname{maximize} \quad z=6 x_{1}+14 x_{2}+13 x_{3}, \\
\text { subject to } & x_{1}+4 x_{2}+2 x_{3} \leq 12, \\
& x_{1}+2 x_{2}+4 x_{3} \leq 15, \\
& x_{1},
\end{array} x_{2}, \quad x_{3} \geq 0, ~ l
$$

with optimal solution $x_{1}^{*}=9, x_{2}^{*}=0, x_{3}^{*}=1.5$, and $z^{*}=73.5$.
The subquestions (a)-(c) should be solved independent of each other.
(a) $[\mathbf{1 p}]$

How does the optimal value change when the right hand side coefficient of the first inequality constraint is increased from 12 by a small amount $\delta>0$ ?
(b) $[1 \mathrm{p}]$

Assume that the problem is extended by a new variable, $x_{\text {new }} \geq 0$, with constraint vector $A_{\text {new }}=\left[\begin{array}{l}3 \\ 5\end{array}\right]$. For what values of the objective coefficient $c_{\text {new }}$ will the variable $x_{\text {new }}$ enter the basis?
(c) $[\mathbf{1 p}]$

Assume that the variables should take integer values. Utilize the optimal solution to find a linear constraint that cuts off the fractional point, but no integer points.

## Question 5

## [3p]

A Branch-And-Bound algorithm for the TSP
Consider an undirected graph with node set $\mathcal{N}=\{1, \ldots, N\}$, edge set $\mathcal{E}=\{1, \ldots, K\}$, and edge costs $c_{k}>0, k \in \mathcal{E}$. Let $\mathcal{E}_{j}$ denote the set of edges that connect to node $j \in \mathcal{N}$, and let $\mathcal{E}(S)$ denote the set of all edges that connect any two nodes in the subset $S \subset \mathcal{N}$.
A traveling salesperson problem (TSP) is defined on this graph, such that each node should be visited exactly once and that the tour should form a cycle. Defining variables as

$$
x_{k}=\left\{\begin{array}{ll}
1, & \text { if edge } k \text { is in the TSP solution, } \\
0, & \text { otherwise }
\end{array} \quad k \in \mathcal{E},\right.
$$

this problem can be formulated as an integer linear optimization problem as

$$
\begin{array}{lll}
\operatorname{minimize} & z= & \sum_{k \in \mathcal{E}} c_{k} x_{k}, \\
\text { subject to } & \sum_{k \in \mathcal{E}_{j}} x_{k}=2, & j \in \mathcal{N}, \\
& \sum_{k \in \mathcal{E}(S)} x_{k} \leq|S|-1, & S \subset \mathcal{N}: 1<|S|<N, \\
& x_{k} \in\{0,1\}, & k \in \mathcal{E} . \tag{1d}
\end{array}
$$

The constraints 1b) make sure that exactly two edges are connected to each node. The subtour constraints (1c) imply that the edges of a solution form a connected graph, that is, a spanning tree plus one extra edge due to the constraints (1b).

Consider the following tailored Branch-and-bound ( BnB ) procedure for an undirected TSP.

1. Relax the constraints (1b). The subproblem solved in each node of the BnB tree is then to find a minimum spanning tree (MST), and then add one edge (the edge $\ell \notin$ MST with the lowest $\operatorname{cost} c_{\ell}$ ). Denote this solution by a binary vector $\mathbf{x}^{\text {MST }+1}$ such that $x_{k}^{\text {MST }+1}:=1$ if edge $k \in \mathrm{MST} \cup\{\ell\}$ and $x_{k}^{\mathrm{MST}+1}:=0$ otherwise, $k \in \mathcal{E}$.
2. The branching in the BnB tree is then determined by a node $j \in \mathcal{N}$ such that the constraint (1b) is not fulfilled, i.e., a $j$ such that $\sum_{k \in \mathcal{E}_{j}} x_{k}^{\text {MST }+1}>2$. In each branch, one variable $x_{k}$ such that $x_{k}^{\mathrm{MST}+1}=1$ is then fixed to the value $x_{k}=0$. Note that the number of branches equals the number of edges that connect to the BnB-node $j$ chosen.

Apply this procedure and solve the undirected TSP given by the following graph:


Perform at most three branchings, and cut branches due to the following criteria: a) the BnBnode problem is infeasible; $b$ ) the solution to the BnB-node problem is a tour; $c$ ) the lower bound in the BnB-node is higher than the value of a known feasible solution.

## Solution proposals to EXAM 2023-06-01

## MVE166/MVE165/MMG631 Linear and integer optimization with applications

These solutions may be brief in relation to the requirements on your answers, in particular regarding motivations.

## Solutions to Question 1

Let $x_{j}$ be the amount (in tonnes) of paper produced using process $j \in\{1,2,3,4,5\}$.

$$
\text { minimize } 700\left(3 x_{1}+x_{2}+x_{3}+2 x_{5}\right)+450\left(4 x_{2}+8 x_{4}+2 x_{5}\right)+150\left(12 x_{3}+5 x_{5}\right)
$$

subject to

$$
\begin{aligned}
x_{1}+x_{2}+x_{3}+x_{4}+x_{5} & =100 \quad \text { (prod.req.) } \\
3 x_{1}+x_{2}+x_{3}+2 x_{5} & \leq 75 \quad(\text { pulp }) \\
\left(4 x_{2}+8 x_{4}+2 x_{5}\right)+\left(12 x_{3}+5 x_{5}\right)-\left(3 x_{1}+x_{2}+x_{3}+2 x_{5}\right) & \geq 0 \quad(\geq 50 \% \text { from recycl.) } \\
x_{j} & \geq 0 \quad \forall j
\end{aligned}
$$

Alternate model: Define also $y_{i}=$ amount of raw material $i$ used (these variables are not necessary)

$$
\begin{aligned}
\operatorname{minimize} 700 y_{1}+450 y_{2}+150 y_{3} & \\
3 x_{1}+x_{2}+x_{3}+2 x_{5} & =y_{1} \\
4 x_{2}+8 x_{4}+2 x_{5} & =y_{2} \\
12 x_{3}+5 x_{5} & =y_{3} \\
x_{1}+x_{2}+x_{3}+x_{4}+x_{5} & =100 \\
y_{1} & \leq 75 \\
y_{2}+y_{3}-y_{1} & \geq 0 \\
x_{j} & \geq 0 \quad \forall j
\end{aligned}
$$

## Solutions to Question 2

(a) Introduce two slack variables, $x_{3}$ and $x_{4}$. Solution course by the simplex method.:

|  | $-z$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $\bar{b}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $-z$ | 1 | 3 | 9 | 0 | 0 | 0 |
| $x_{3}$ | 0 | 1 | 4 | 1 | 0 | 8 |
| $x_{4}$ | 0 | 1 | 2 | 0 | 1 | 4 |
| $-z$ | 1 | $3 / 4$ | 0 | $-9 / 4$ | 0 | -18 |
| $x_{2}$ | 0 | $1 / 4$ | 1 | $1 / 4$ | 0 | 2 |
| $x_{4}$ | 0 | $1 / 2$ | 0 | $-1 / 2$ | 1 | 0 |
| $-z$ | 1 | 0 | 0 | $-3 / 2$ | $-3 / 2$ | -18 |
| $x_{2}$ | 0 | 0 | 1 | $1 / 2$ | $-1 / 2$ | 2 |
| $x_{1}$ | 0 | 1 | 0 | -1 | 2 | 0 |

The optimal solution is $\mathbf{x}^{*}=(0,2,0,0)^{\top}$ and there are two optimal bases: $\left(x_{2}, x_{4}\right)=(2,0)$ and $\left(x_{2}, x_{1}\right)=(2,0)$. Optimal objective value $z^{*}=18$. The optimal extreme point is degenerate.
(b) The LP dual is given by

$$
\begin{array}{rrl}
\operatorname{minimize} \quad w=8 y_{1}+4 y_{2}, & \\
\text { subject to } & y_{1}+y_{2} & \geq 3, \\
4 y_{1}+2 y_{2} & \geq 9, \\
& y_{1}, & y_{2}
\end{array} \geq 0 .
$$

Optimal dual solutions are found using complementary and dual feasibility conditions:

$$
\begin{aligned}
& 0 \cdot\left(y_{1}+y_{2}-3\right)=0 \\
& 2 \cdot\left(4 y_{1}+2 y_{2}-9\right)=0 \text { complementary } \\
& y_{1}+y_{2} \geq 3 \\
& y_{1}, y_{2} \geq 0 \\
& \text { dual feasibility } \\
& \text { dual feasibility }
\end{aligned}
$$

$\Longrightarrow 4 y_{1}+2 y_{2}=9, y_{1}+y_{2} \geq 3, y_{1} \geq 0, y_{2} \geq 0 \Longleftrightarrow y_{2}=\frac{9}{2}-2 y_{1}, 0 \leq y_{1} \leq \frac{3}{2}, w=8 y_{1}+4 y_{2}=18$
$\Longrightarrow$ The set of optimal solutions equals the convex hull of the dual (extreme) points $\left(y_{1}, y_{2}\right)=\left(0, \frac{9}{2}\right)$ and $\left(y_{1}, y_{2}\right)=\left(\frac{3}{2}, \frac{3}{2}\right)$, i.e., the set
$\left\{\left(y_{1}, y_{2}\right) \in \mathbb{R}^{2} \left\lvert\, y_{1}=\alpha \cdot 0+(1-\alpha) \cdot \frac{3}{2}\right., y_{2}=\alpha \cdot \frac{9}{2}+(1-\alpha) \cdot \frac{3}{2} ; 0 \leq \alpha \leq 1\right\}$
$=\left\{\left.\left(\frac{3}{2}(1-\alpha), \frac{3}{2}(1+2 \alpha)\right) \in \mathbb{R}^{2} \right\rvert\, 0 \leq \alpha \leq 1\right\}$. Hence, there are multiple optimal solutions to the dual.

The dual optimal value equals $w^{*}=\frac{3}{2}(8(1-\alpha)+4(1+2 \alpha))=18$ for all $\alpha \in[0,1]$.

## Solutions to Question 3

See the course book, Theorem 4.1 and its proof.

## Solutions to Question 4

(a) The optimal basis is $\left(x_{1}, x_{3}\right)$. Hence, the basic matrix is $\mathbf{B}=\left(\begin{array}{ll}1 & 2 \\ 1 & 4\end{array}\right)$ with inverse $\mathbf{B}^{-1}=$ $\left(\begin{array}{cc}2 & -1 \\ -\frac{1}{2} & \frac{1}{2}\end{array}\right)$, and the vector $\mathbf{c}_{B}=\binom{6}{13}$. The optimal value as a function of $\delta \in \mathbb{R}$ is given by $z^{*}(\delta)=\mathbf{c}_{B}^{\top} \mathbf{B}^{-1}\left(\mathbf{b}+\binom{\delta}{0}\right)=73.5+\left(\begin{array}{ll}6 & 13\end{array}\right)\left(\begin{array}{cc}2 & -1 \\ -\frac{1}{2} & \frac{1}{2}\end{array}\right)\binom{\delta}{0}=73.5+5.5 \delta$.
Conclusion: a small positive increase of the RHS of the first constraint by $\delta$ yields an increase of the optimal value by $5.5 \delta$. (This holds for any value of $\delta$ such that $\left.\mathbf{B}^{-1}\left(\mathbf{b}+\binom{\delta}{0}\right) \geq\binom{ 0}{0} \Longleftrightarrow-\frac{9}{2} \leq \delta \leq 3.\right)$
(b) The new variable will enter the basis if its reduced cost $\bar{c}_{\text {new }} \geq 0 . \bar{c}_{\text {new }}=c_{\text {new }}-\mathbf{c}_{B}^{\top} \mathbf{B}^{-1} A_{\text {new }}=$ $c_{\text {new }}-\left(\begin{array}{ll}6 & 13\end{array}\right)\left(\begin{array}{cc}2 & -1 \\ -\frac{1}{2} & \frac{1}{2}\end{array}\right)\binom{3}{5}=c_{\text {new }}-19 . \Longrightarrow$ The new variable will enter the basis if $c_{\text {new }} \geq 19$.
(c) Any basic solution can be expressed as $\mathbf{x}_{B}+\mathbf{B}^{-1} \mathbf{N x}_{N}=\mathbf{B}^{-1} \mathbf{b}$. Since the basic variable $x_{3}$ has a fractional value, we apply Gomory's method to the corresponding constraint (the second constraint, corresponding to the second variable in the optimal basis): $x_{i}+$ $\sum_{j \in N}\left(\mathbf{B}^{-1} \mathbf{N}\right)_{i j} x_{j}=\left(\mathbf{B}^{-1} \mathbf{b}\right)_{i} \Longleftrightarrow x_{3}+\left(-x_{2}-\frac{1}{2} x_{4}+\frac{1}{2} x_{5}\right)=\frac{3}{2}$, where the variables $x_{4}$ and $x_{5}$ are slack variables corresponding to the first and second constraint, respectively.

Rearrange this equation as $x_{3}-x_{2}-x_{4}+0 \cdot x_{5}-1=-\frac{1}{2} x_{4}-\frac{1}{2} x_{5}+\frac{1}{2}$, where the RHS is $\leq \frac{1}{2}$ since both slack variables should be non-negative. Since the LHS must take an integer value for any feasible point, the same must hold for the RHS, which then implies that $-\frac{1}{2} x_{4}-\frac{1}{2} x_{5}+\frac{1}{2} \leq 0 \Longleftrightarrow x_{4}+x_{5} \geq 1$.
Hence, it holds that $x_{4}+x_{5}=\left(12-x_{1}-4 x_{2}-2 x_{3}\right)+\left(15-x_{1}-2 x_{2}-4 x_{3}\right)=27-2 x_{1}-6 x_{2}-6 x_{3} \geq 1$ $\Longleftrightarrow x_{1}+3 x_{2}+3 x_{3} \leq 13$, which is the linear constraint sought.
Check: The optimal LP solution $\mathbf{x}^{*}$ does not fulfill this constraint and will be cut off, since $x_{1}^{*}+3 x_{2}^{*}+3 x_{3}^{*}=9+0+4.5=13.5>13$

## Solutions to Question 5

Denote the optimal value by $z^{*}$. Illustration of the BnB-tree:


The branching in BnB-node P0 is based on the constraint in (1b) corresponding to the TSP-node $\mathrm{C}: x_{\mathrm{AC}}+x_{\mathrm{BC}}+x_{\mathrm{CD}}=2$, which is not fulfilled by the subproblem solution. This constraint is implied by fixing each of the variables $x_{\mathrm{AC}}=0, x_{\mathrm{BC}}=0$, and $x_{\mathrm{CD}}=0$ in the branches, P1, P2, and P3, respectively.

Solutions to the BnB-node subproblems (minimize (1a) subject to (1c) and (1d)):
P0: The solution does not fulfill $(1 \mathrm{~b}) \Longrightarrow \underline{z}_{0}=2+3+4+5=14 \leq z^{*}$.
Branching based on the constraint $x_{\mathrm{AC}}+x_{\mathrm{BC}}+x_{\mathrm{CD}}=2$
P1: $\left(x_{\mathrm{AC}}=0\right)$ Solution is feasible in $(1) \Rightarrow \bar{z}=3+4+5+6=18 \geq z^{*}$. Cut the branch
P2: $\left(x_{\mathrm{BC}}=0\right)$ The solution does not fulfill 1 B$) \Longrightarrow \underline{z}_{2}=2+4+6+5=17$.
Branching based on the constraint $x_{\mathrm{AB}}+x_{\mathrm{AC}}+x_{\mathrm{AD}}=2$
P3: $\left(x_{\mathrm{CD}}=0\right)$ The solution does not fulfill $(1 \mathrm{~b}) \Longrightarrow \underline{z}_{3}=2+3+5+6=16$.
Branching based on the constraint $x_{\mathrm{AC}}+x_{\mathrm{AD}}+x_{\mathrm{AB}}=2$
P4: $\left(x_{\mathrm{BC}}=x_{\mathrm{AB}}=0\right)$ The feasible set is empty $\Longrightarrow$ cut the branch
P5: $\left(x_{\mathrm{BC}}=x_{\mathrm{AC}}=0\right)$ The feasible set is empty $\Longrightarrow$ cut the branch
P6: $\left(x_{\mathrm{BC}}=x_{\mathrm{AD}}=0\right)$ Solution is feasible in (1) $\Rightarrow \bar{z}=2+4+6+7=19 \geq z^{*}$. Cut the branch
P7: $\left(x_{\mathrm{CD}}=x_{\mathrm{AC}}=0\right)$ The feasible set is empty $\Longrightarrow$ cut the branch
P8: $\left(x_{\mathrm{CD}}=x_{\mathrm{AD}}=0\right)$ The feasible set is empty $\Longrightarrow$ cut the branch
P9: $\left(x_{\mathrm{CD}}=x_{\mathrm{AB}}=0\right)$ Solution is feasible in $(1) \Rightarrow \bar{z}_{5}=2+3+5+7=17 \geq z^{*}$. Cut the branch

Since all branches in the BnB-tree are cut, the optimal solution is the feasible solution with the lowest cost, which is found in P9.

The optimal TSP tour is given by the sequence A-C-B-D-A, with total cost $2+3+7+5=17=z^{*}$.
The solutions to the subproblems are illustrated below: The thin solid lines represent the MST while the thick line is the added edge. The dotted lines represent variables $x_{i j}$ fixed to 0 in the respective subproblem.

P5

P1

P2

PT

P3

P8

P4
 Pg


