EXAM
MVE165/MMG631
Linear and integer optimization with applications
7.5 hp

- **Date:** 2023-01-03
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg (available only via the mobile number below)
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

Optimization modelling

A grocery store has estimated the total number of working hours needed over the next five months. The current, trained staff consists of twelve persons, and each employee can work up to 160 hours per month. In the beginning of each month there is a possibility to hire new staff. A newly hired person is not considered to perform any work during his/her first month, but requires instead 50 hours of supervision from a colleague who is already trained. After one month, the new employee is considered to work (up to) full time. The salary cost during the first month of a newly hired person is 75% of the salary cost for a trained staff. Any hired person is employed during an integer number of months, but in the end of each month it is assumed that 15% of the trained staff terminate their employment—for various reasons.

month	number of working hours
February	600
March	750
April	850
May	900
June	800

(a) **[2.5p]**

Formulate a linear optimization model for minimizing the total salary cost during the five month period, such that all working hours are staffed.

You should not solve the model.

(b) **[0.5p]**

Adjust your model for the case that all hired staff has the right to work full time (i.e., 160 hours per month).

Question 2

Optimal basic solutions and integrality of a network flow problem

For a general primal–dual pair, (P) and (D), of linear optimization problems

$$\text{(P):}\quad z^* = \max\{\, \mathbf{c}^\top \mathbf{x} \,:\, \mathbf{A}\mathbf{x} \le \mathbf{b};\, \mathbf{x} \ge \mathbf{0} \,\} \qquad \text{and} \qquad \text{(D):}\quad w^* = \min\{\, \mathbf{b}^\top \mathbf{y} \,:\, \mathbf{A}^\top \mathbf{y} \ge \mathbf{c};\, \mathbf{y} \ge \mathbf{0} \,\},$$

Theorem 6.4 states that if $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ is an optimal basic solution to (P) then $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1}$ is optimal in (D) and $z^* = w^*$.

Now, consider the linear optimization problem to

$$\begin{array}{llllllcrr}
\text{minimize} & x_1 & +3x_2 & +6x_3 & +2x_4 & +2x_5, & & & \text{(1a)} \\
\text{subject to} & -x_1 & -x_2 & -x_3 & & & = & -1, & \text{(1b)} \\
 & x_1 & & & -x_4 & & = & 0, & \text{(1c)} \\
 & & x_2 & & +x_4 & -x_5 & = & 0, & \text{(1d)} \\
 & & & x_3 & & +x_5 & = & 1, & \text{(1e)} \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \ge & 0, & \text{(1f)}
\end{array}$$

where the constraint (1e) is redundant and can be removed, since it equals a linear combination of the constraints (1b)–(1d). The feasible set to the problem (1) can thus be expressed as

$$\left\{ \mathbf{x} \in \mathbb{R}^5_+ \;\middle|\; \text{the constraints (1b)–(1d) are fulfilled} \right\}.$$

(a) **[2p]**

Consider the three bases $\mathbf{x}_B^1 = (x_1, x_4, x_5)$, $\mathbf{x}_B^2 = (x_2, x_4, x_5)$, and $\mathbf{x}_B^3 = (x_1, x_2, x_3)$.

Utilize Theorem 6.4 to determine which of the three bases that are optimal to the problem of minimizing the objective (1a) subject to the constraints (1b)–(1d) and (1f).

(b) **[1p]**

Is the point $\bar{\mathbf{x}} = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 1)$ an optimal extreme point to the problem (1)? Motivate your answer.

Question 3

THE SIMPLEX ALGORITHM

(a) **[2p]**

Solve the linear optimization problem to

$$\begin{array}{llrrrcr} \text{maximize} & z = & 2x_1 & & +x_3, & & \\ \text{subject to} & & x_1 & +2x_2 & & \leq & 4, \\ & & 2x_1 & +x_2 & +x_3 & \leq & 10, \\ & & x_1, & x_2, & x_3 & \geq & 0, \end{array}$$

using the simplex algorithm. Explain all the steps in the pre-processing as well as in the algorithm.

(b) **[1p]**

Express all optimal solutions to the problem and explain the properties leading to your expression(s).

Question 4

BINARY KNAPSACK PROBLEM

Consider the following binary knapsack problem:

$$\begin{array}{rcrrrrcl} z^* := & \max & 10x_1 & +20x_2 & +15x_3 & +20x_4, & & \\ & \text{s.t.} & x_1 & +3x_2 & +2x_3 & +4x_4 & \leq & 5, \\ & & x_1, & x_2, & x_3, & x_4 & \in & \{0,1\}. \end{array}$$

(a) **[1p]**

Verify that the constraint $x_1 + x_2 + x_3 + x_4 \leq 2$ is a *valid inequality* (VI) of the convex hull of the feasible set to the problem stated above. State carefully all properties utilized in your derivations.

(b) **[2p]** [SUB-QUESTION (b) CAN BE SOLVED INDEPENDENTLY OF (a)]

Solve the problem using the branch–and–bound algorithm.

Use a depth-first strategy and always search the 1-branch first. You may utilize the valid inequality $x_1 + x_2 + x_3 + x_4 \leq 2$ when pruning branches in the search tree.

Question 5

LP DUALITY END SENSITIVITY ANALYSIS

Consider the linear optimization problem to

$$
\begin{array}{llrrrcl}
\text{maximize} & z = & 4x_1 & +2x_2 & +x_3, & & \\
\text{subject to} & & x_1 & +2x_2 & +x_3 & \leq & 4, \\
& & 2x_1 & +x_2 & -x_3 & \leq & 2, \\
& & x_1, & x_2, & x_3 & \geq & 0.
\end{array}
$$

(a) **[1p]**

Formulate the linear optimization dual (LP dual) problem and solve it graphically. State the optimal dual solution and the optimal value.

(b) **[1p]**

Utilize complementarity and a graphic examination of the LP dual to answer the following question.

Suppose that the right–hand–sides of the primal constraints are changed, in such a way that the primal feasible set stays non-empty. Which primal constraint will always be fulfilled with equality in an optimal solution?

(c) **[1p]**

Utilize complementarity and a graphic examination of the LP dual to answer the following question.

Suppose that the objective coefficient $c_1 = 4$ of the variable x_1 is changed. For which values of c_1 is x_1 *not* part of an optimal basis?

Solutions

Solutions to Question 1

(a) Variable definition:

$$
\begin{aligned}
x_j &= \text{number of newly hired persons in month } j = 1, \dots, 5,\\
y_j &= \text{number of trained staff available during month } j = 1, \dots, 5,\\
w_j &= \text{number of hours worked by newly hired persons during month } j = 1, \dots, 5,\\
z_j &= \text{number of hours worked by trained staff during month } j = 1, \dots, 5.
\end{aligned}
$$

Parameter definition: $(d_1, d_2, d_3, d_4, d_5) = (600, 750, 850, 900, 800)$.

$$
\begin{array}{llll}
\min & \sum_{j=1}^{5} c \cdot \left(0.75x_j + y_j\right), & & \\
\text{s.t.} & z_j - 50x_j & \geq d_j, & j = 1, \dots, 5, \\
& x_j + 0.85y_j & \geq y_{j+1}, & j = 1, \dots, 4, \\
& z_j & \leq 160y_j, & j = 1, \dots, 5, \\
& 50x_j \leq w_j & \leq 160x_j, & j = 1, \dots, 5, \quad \text{(remove)} \\
& y_1 & = 12, & \\
& x_j, y_j & \geq 0 \text{ and integer}, & j = 1, \dots, 5, \\
& w_j, z_j & \geq 0, & j = 1, \dots, 5.
\end{array}
$$

If it is assumed that the salary cost for the staff is proportional to the number of working hours, then the objective is altered to:

$$
\min \sum_{j=1}^{5} c \cdot \left(0.75 \cdot \frac{w_j}{160} + \frac{z_j}{160}\right)
$$

(b) Replace the constraints $z_j \leq 160y_j$ by $z_j = 160y_j$. Replace the constraints $50x_j \leq w_j \leq 160x_j$ by $w_j = 160x_j$.

Solutions to Question 2

(a) Theorem 6.4 states that if $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ is an optimal basic solution to the primal linear optimization problem, then the point $\mathbf{y}^\top = \mathbf{c}_B^\top\mathbf{B}^{-1}$ is optimal in the dual problem, and the two optimal solutions have the same objective value. The linear optimization dual of the model (1a)–(1d), (1f) is given by

$$\begin{array}{lrrrcl}
\max & -y_1, & & & & \\
\text{s.t.} & -y_1 & +y_2 & & \leq & 1, \\
& -y_1 & & +y_3 & \leq & 3, \\
& -y_1 & & & \leq & 6, \\
& & -y_2 & +y_3 & \leq & 2, \\
& & & -y_3 & \leq & 2.
\end{array}$$

The basis $\mathbf{x}_B^1 = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix}$.

Complementary dual solution: $(\mathbf{y}^1)^\top = \mathbf{c}_B^\top\mathbf{B}^{-1} = (1 \quad 2 \quad 2) \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} = (-5 \quad -4 \quad -2)$,

which is feasible in the dual. Primal and dual objective value: $\mathbf{c}_B^\top\mathbf{B}^{-1}\mathbf{b} = (1 \quad 2 \quad 2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5$.

The basis $\mathbf{x}_B^2 = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix}$.

Complementary dual solution: $(\mathbf{y}^2)^\top = \mathbf{c}_B^\top\mathbf{B}^{-1} = (3 \quad 2 \quad 2) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} = (-5 \quad -4 \quad -2)$,

which is feasible in the dual. Primal and dual objective value: $\mathbf{c}_B^\top\mathbf{B}^{-1}\mathbf{b} = (3 \quad 2 \quad 2) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 5$.

The basis $\mathbf{x}_B^3 = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Complementary dual solution: $(\mathbf{y}^3)^\top = \mathbf{c}_B^\top\mathbf{B}^{-1} = (1 \quad 3 \quad 6) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} = (-6 \quad -5 \quad -3)$,

which is not feasible in the dual. Primal/dual objective value: $\mathbf{c}_B^\top\mathbf{B}^{-1}\mathbf{b} = (1 \quad 3 \quad 6) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 6$.

It follows that the bases $\mathbf{x}_B^1$ and $\mathbf{x}_B^2$ are optimal with corresponding optimal solutions $\mathbf{x}^1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{x}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \end{pmatrix}$.

(b) Theorem 8.1 states that a minimum cost network flow problem with integer constants in all RHS:s has integer extreme points. Since (1) models a shortest path problem, being a network flow problem, all its extreme points have integer values. Hence, the point $\bar{\mathbf{x}}$ is *not* an extreme point; since it is feasible in the primal with objective value 5, it is an optimal point, being a convex combination of the solutions $\mathbf{x}^1$ and $\mathbf{x}^2$.

Solutions to Question 3

(a) Include slack variables x_4 and x_5 in the inequality constraints $\Longrightarrow$ the following LP:

$$\begin{array}{llrrrrrrl} \text{maximize} & z = & 2x_1 & & +x_3, & & & & \\ \text{subject to} & & x_1 & +2x_2 & & +x_4 & & = & 4, \\ & & 2x_1 & +x_2 & +x_3 & & +x_5 & = & 10, \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0, \end{array}$$

Simplex iterations:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$	
z	1	**-2**	0	-1	0	0	0	
x_4	0	**1**	2	0	1	0	4	
x_5	0	2	1	1	0	1	10	
z	1	0	4	**-1**	2	0	8	
x_1	0	1	2	0	1	0	4	
x_5	0	0	-3	**1**	-2	1	2	
z	1	0	1	0	**0**	1	10	optimal since all reduced costs are ≤ 0
x_1	0	1	2	0	**1**	0	4	
x_3	0	0	-3	1	-2	1	2	

An optimal solution is given by $\mathbf{x} = \begin{pmatrix} 4 & 0 & 2 & 0 & 0 \end{pmatrix}^\top$ with optimal value $z^* = 10$.

(b) Since the reduced cost $\bar{c}_4 = 0$, there is at least one other optimal solution. Let the variable x_4 enter the basis, such that the variable x_1 has to leave the basis:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$	
z	1	0	1	0	**0**	1	10	optimal since all reduced costs are ≤ 0
x_1	0	1	2	0	**1**	0	4	
x_3	0	0	-3	1	-2	1	2	
z	1	**0**	1	0	0	1	10	optimal since all reduced costs are ≤ 0
x_4	0	**1**	2	0	1	0	4	
x_3	0	2	1	1	0	1	10	

In the next iteration, the variable x_1 will enter the basis, and the variable x_4 will leave the basis, which leads back to a basis that has already been explored. The set of optimal solutions to the LP equals the convex hull of the two optimal solutions $\mathbf{x}^1 = \begin{pmatrix} 4 & 0 & 2 & 0 & 0 \end{pmatrix}^\top$ and $\mathbf{x}^2 = \begin{pmatrix} 0 & 0 & 10 & 4 & 0 \end{pmatrix}^\top$. It follows that the optimal set is given by $X^* = \left\{ \alpha \begin{pmatrix} 4 & 0 & 2 & 0 & 0 \end{pmatrix}^\top + (1-\alpha) \begin{pmatrix} 0 & 0 & 10 & 4 & 0 \end{pmatrix}^\top \,\middle|\, 0 \leq \alpha \leq 1 \right\} = \left\{ \begin{pmatrix} 4\alpha & 0 & 10-8\alpha & 4-4\alpha & 0 \end{pmatrix}^\top \,\middle|\, 0 \leq \alpha \leq 1 \right\}$

Solutions to Question 4

(a) The binary knapsack problem can be equivalently stated as

$$
\begin{array}{rlccccclr}
z^* = & \max & 10x_1 & +20x_2 & +15x_3 & +20x_4, & & & \text{(2a)}\\
& \text{s.t.} & x_1 & +3x_2 & +2x_3 & +4x_4 & \le & 5, & \text{(2b)}\\
& & x_1 & & & & \le & 1, & \text{(2c)}\\
& & & x_2 & & & \le & 1, & \text{(2d)}\\
& & & & x_3 & & \le & 1, & \text{(2e)}\\
& & & & & x_4 & \le & 1, & \text{(2f)}\\
& & x_1, & x_2, & x_3, & x_4 & \in & \mathbb{Z}_+. & \text{(2g)}
\end{array}
$$

Aggregating non-negative multiples the constraints (2b)–(2f)—with coefficients 1, 3, 1, 2, and 0, respectively—yields the inequality

$$1 \cdot (x_1 + 3x_2 + 2x_3 + 4x_4) + 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 + 0 \cdot x_4 \le 1 \cdot 5 + 3 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 + 0 \cdot 1 = 11$$

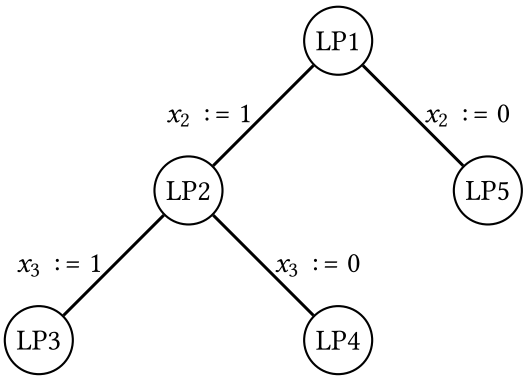
$$\iff \quad 4 \cdot (x_1 + x_2 + x_3 + x_4) \le 11 \quad \iff \quad x_1 + x_2 + x_3 + x_4 \le \frac{11}{4} = 2 + \frac{3}{4} < 3$$

By the constraint (2g), in every feasible point all variables possess integer values. It follows that the inequality $x_1 + x_2 + x_3 + x_4 \le 2$ holds for all feasible points, i.e., it is a VI for the convex hull of all feasible points.

(b) In the search tree, the root node is denoted LP1; corresponding relaxed problem (LP1):

$$
\begin{array}{rlccccccl}
z^* \le z^{\text{LP1}} = & \max & 10x_1 & +20x_2 & +15x_3 & +20x_4, & & \\
& \text{s.t.} & x_1 & +3x_2 & +2x_3 & +4x_4 & \le & 5, \\
& & x_1, & x_2, & x_3, & x_4 & \in & [0,1].
\end{array}
$$

Relaxing the integrality requirements, and branching on fractional variable values, yields the following BnB-tree:



Solutions to the node problems and details:

LP1: $\mathbf{x}^{\text{LP1}} = \begin{pmatrix} 1 & \frac{2}{3} & 1 & 0 \end{pmatrix}^\top$ with objective value $z^{\text{LP1}} = 10 + \frac{40}{3} + 15 = 38 + \frac{1}{3} \ge z^* \Longrightarrow z^* \le 38$

LP2: $\mathbf{x}^{\text{LP2}} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & 0 \end{pmatrix}^\top$ with objective value $z^{\text{LP2}} = 10 + 20 + \frac{15}{2} = 37 + \frac{1}{2}$

LP3: $\mathbf{x}^{\text{LP3}} = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^\top$ with objective value $z^{\text{LP3}} = 20 + 15 = 35$. Since $\mathbf{x}^{\text{LP3}}$ is feasible in the original problem, it is a candidate for optimum and yields a lower bound:

$z^* \geq 35$. The branch is cut.

Note: since in this branch $x_2 = x_3 = 1$, the VI $x_1 + x_2 + x_3 + x_4 \leq 2$ yields that $x_1 = x_4 = 0$, which also means that the branch can be cut.

LP4: $\mathbf{x}^{\mathrm{LP4}} = \begin{pmatrix}1 & 1 & 0 & \frac{1}{4}\end{pmatrix}^\top$ with objective value $z^{\mathrm{LP4}} = 10 + 20 + \frac{20}{4} = 35$. We know that there is a feasible solution, $\mathbf{x}^{\mathrm{LP3}}$, with objective value 35. The current branch cannot contain a feasible solution with a value < 35; hence, the branch is cut

LP5: $\mathbf{x}^{\mathrm{LP5}} = \begin{pmatrix}1 & 0 & 1 & \frac{1}{2}\end{pmatrix}^\top$ with objective value $z^{\mathrm{LP4}} = 10 + 15 + \frac{20}{2} = 35$. We know that there is a feasible solution, $\mathbf{x}^{\mathrm{LP3}}$, with objective value 35. The current branch cannot contain a feasible solution with a value < 35; hence, the branch is cut

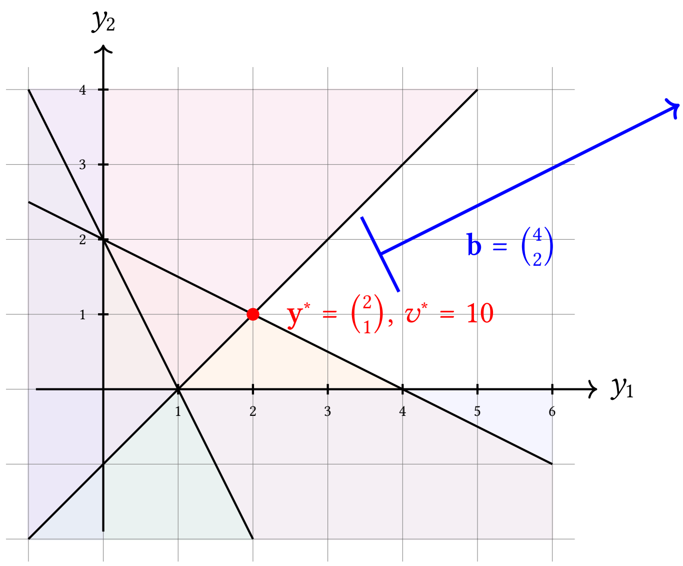
The optimal solution is $\mathbf{x}^* = \begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}^\top$ and the optimal objective value is $z^* = 20 + 15 = 35$

Solutions to Question 5

(a) LP dual:

$$
\begin{array}{lrrcl}
v^* = \min & 4y_1 & +2y_2, & & \\
\text{s.t.} & y_1 & +2y_2 & \geq & 4, \\
 & 2y_1 & +y_2 & \geq & 2, \\
 & y_1 & -y_2 & \geq & 1, \\
 & y_1, & y_2 & \geq & 0.
\end{array}
$$

Graphic solution:



(b) Altering the RHS:s of the primal constraints means that the coefficients of the dual objective changes.

In every dual feasible solution, it holds that $y_1 > 0$. Due to complementarity of an optimal solution, $y_1(x_1 + 2x_2 + x_3 - 4) = 0$. Therefore, for any optimal primal solution, it must hold that $x_1 + 2x_2 + x_3 = 4$, i.e., the first primal constraint will be fulfilled with equality in any optimal solution.

(c) For $c_1 < 1$ the first dual constraint $y_1 + 2y_2 \geq c_1$ is not active, i.e., for any dual feasible solution $y_1 + 2y_2 > c_1$ will hold. Due to complementarity of an optimal solution, $x_1(y_1 + 2y_2 - c_1) = 0$, which means that $x_1 = 0$ will hold in any optimal solution. Hence, x_1 cannot be part of an optimal basis when $c_1 < 1$.