EXAM
MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2022-08-25
  - **Hours:** 14:00–18:00
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg (available only via the mobile number below)
  - **Phone:** 0705-273645

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

[3p]

A carpentry manufactures three types of mats that are cut from boards of plywood. The boards come in two different sizes: 2 m × 2 m and 1.5 m × 1.5 m, which cost SEK 700 and SEK 590, respectively. The mats to be cut are of three different profiles: a square, a rectangle, and a circle. Each board can be cut in a number of patterns (indexed A–G) and each pattern yields a certain number of each of the mat profiles, according to the table below. At least 500 squares, 700 rectangles, and 280 circles must be manufactured.

Formulate an *integer linear optimization problem* that determines how many of each size of the boards the carpentry should purchase in order to minimize its expenses?

The problem should not be solved.

Board size	pattern	# squares	# rectangles	# circles
2 × 2	A	13	1	7
2 × 2	B	0	20	3
2 × 2	C	2	2	25
2 × 2	D	20	5	5
1.5 × 1.5	E	7	2	12
1.5 × 1.5	F	0	18	1
1.5 × 1.5	G	16	1	0

Question 2

Consider the following linear optimization problem

$$
\begin{array}{llrrrcll}
\max & z = & x_1 & +2x_2 & +x_3, & & & \quad (1a)\\
\text{s.t.} & & 2x_1 & +x_2 & -x_3 & \leq & 2, & \quad (1b)\\
 & & 2x_1 & -x_2 & +5x_3 & \leq & 6, & \quad (1c)\\
 & & 4x_1 & +x_2 & +x_3 & \leq & 6, & \quad (1d)\\
 & & x_1, & x_2, & x_3 & \geq & 0. & \quad (1e)
\end{array}
$$

(a) **[2p]**

Formulate and solve the above problem using the simplex algorithm. At termination, verify using linear optimization theory that the solution found is optimal.

(b) **[1p]**

Now, consider varying the right–hand–side of the constraint (1c) with $\delta \in \mathbb{R}$, i.e., replacing the constraint (1c) by the inequality $2x_1 - x_2 + 5x_3 \leq 6 + \delta$.

Express the optimal value of the problem (1) as a function of δ.

For what values of δ is the optimal basis in (a) feasible in (1)?

Question 3

Consider the following constraints of a linear optimization problem:

$$
\begin{array}{rrrrcl}
-x_1 & +x_2 & +x_3 & & = & 1 \\
 & x_2 & & +x_4 & = & 3 \\
x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
\begin{array}{r}
(2a) \\ (2b) \\ (2c)
\end{array}
$$

(a) **[1p]**

For each of the points $\mathbf{x}^1 = (2,3,0,0)$, $\mathbf{x}^2 = (0,3,-2,0)$, $\mathbf{x}^3 = (2,1,2,2)$, and $\mathbf{x}^4 = (3,3,1,0)$, determine whether a suitable non-constant linear objective function could make the point optimal or uniquely optimal, or neither of these.

For each conclusion, explain your reasoning.

[Hint: *you may sketch the feasible set defined by these constraints in a suitable 2-dimensional plot.*]

(b) **[1p]**

For each of the following sets of variables: $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_2, x_4\}$, $\{x_4\}$, and $\{x_1, x_2, x_4\}$, determine whether the corresponding columns of the equality constraints (2a)–(2b) form a basis.

(c) **[1p]**

For each of the sets in (b) that does form a basis, determine the corresponding basic solution and classify it as feasible or infeasible.

Question 4

[3p]

Consider the following integer linear optimization problem (a so-called binary knapsack problem):

$$
\begin{array}{rrrrrcl}
z^* := \text{maximum} & 5x_1 & +7x_2 & +3x_3 & +9x_4, & & \\
\text{subject to} & 2x_1 & +4x_2 & +2x_3 & +3x_4 & \leq & 8, \\
 & x_1, & x_2, & x_3, & x_4 & \in & \{0,1\}.
\end{array}
\begin{array}{r}
(3a) \\ (3b) \\ (3c)
\end{array}
$$

State and solve a linear optimization relaxation of the problem (3).

Then, make a Lagrangean relaxation of the constraint (3b) with multiplier $u \geq 0$. Evaluate the Lagrangean dual function $h(u)$ for the values $u = 1$, $u = 2$, and $u = 3$. (Note that each function evaluation involves the solution of a subproblem in the x-variables.)

Utilizing the information from the totally four solutions to relaxations of the problem (3), what upper and/or lower bounds on the optimal value z^* can be stated?

Question 5

[3p]

Consider the following linear optimization problem:

$$
\begin{aligned}
z^* := \max_{\mathbf{x}\in\mathbb{R}^n} \quad & \mathbf{c}^\top \mathbf{x}, && (4a)\\
\text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b}, && (4b)\\
& \mathbf{x} \geq \mathbf{0}, && (4c)
\end{aligned}
$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. Suppose that the feasible set $\{\, \mathbf{x} \geq \mathbf{0} \mid A\mathbf{x} \leq \mathbf{b} \,\}$ is nonempty and that also the feasible set of the linear optimization dual of the problem (4) is nonempty.

Formulate the corresponding linear optimization dual problem and prove that *weak duality* holds between the two problems.

Solution proposals

Note that some of these solutions are quite brief and that more explanations may be needed to pass some of the (part) questions.

Solutions to Question 1

Let x_j denote the number of boards that are cut according to pattern $j \in \{A, B, C, D, E, F, G\}$.

The problem can then be modelled as follows

$$\begin{array}{lrcl}
\text{minimize} & 700(x_A + x_B + x_C + x_D) + 590(x_E + x_F + x_G) & & \\
\text{subject to} & 13x_A + 0x_B + 2x_C + 20x_D + 7x_E + 0x_F + 16x_G & \geq & 500 \\
& 1x_A + 20x_B + 2x_C + 5x_D + 2x_E + 18x_F + 1x_G & \geq & 700 \\
& 7x_A + 3x_B + 25x_C + 5x_D + 12x_E + 1x_F + 0x_G & \geq & 280 \\
& x_j & \geq & 0 \quad \text{and integer, } \forall j
\end{array}$$

The number of boards to buy of size 2 m × 2 m is then given by $x_A + x_B + x_C + x_D$ and the number of boards to buy of size 1.5 m × 1.5 m is given by $x_E + x_F + x_G$, where the variable values come from an optimal solution to the ILP.

Solutions to Question 2

(a) Introduce slack variables, x_4, x_5, and x_6:

$$\begin{array}{llrrrrrrcl}
\max & z = & x_1 & +2x_2 & +x_3, & & & & & \\
\text{s.t.} & & 2x_1 & +x_2 & -x_3 & +x_4 & & & = & 2, \\
& & 2x_1 & -x_2 & +5x_3 & & +x_5 & & = & 6, \\
& & 4x_1 & +x_2 & +x_3 & & & +x_6 & = & 6, \\
& & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq & 0.
\end{array}$$

Simplex iterations:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{B}^{-1}\mathbf{b}$
z	1	-1	**-2**	-1	0	0	0	0
x_4	0	2	**1**	-1	1	0	0	2
x_5	0	2	-1	5	0	1	0	6
x_6	0	4	1	1	0	0	1	6
z	1	3	0	**-3**	2	0	0	4
x_2	0	2	1	-1	1	0	0	2
x_5	0	4	0	4	1	1	0	8
x_6	0	2	0	**2**	-1	0	1	4
z	1	6	0	0	1/2	0	3/2	10
x_2	0	3	1	0	1/2	0	1/2	4
x_5	0	0	0	0	3	1	-2	0
x_3	0	1	0	1	-1/2	0	1/2	2

The optimal solution is $\mathbf{x} = (0, 4, 2, 0, 0, 0)$.

Optimal basic solution: $\mathbf{x}_B = (x_2, x_5, x_3) = (4, 0, 2) \geq \mathbf{0}$; $\mathbf{x}_N = (x_1, x_4, x_6) = \mathbf{0}$.

Optimality holds, since all reduced costs are negative:

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = (1, 0, 0) - (2, 0, 1)\begin{pmatrix} 1/2 & 0 & 1/2 \\ 3 & 1 & -2 \\ -1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix} = (-6, -1/2, -3/2)$$

Alternative solution course (leading to an alternative optimal basis):

Simplex iterations:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{B}^{-1}\mathbf{b}$
z	*1*	*-1*	**-2**	*-1*	*0*	*0*	*0*	*0*
x_4	*0*	*2*	**1**	*-1*	*1*	*0*	*0*	*2*
x_5	*0*	*2*	*-1*	*5*	*0*	*1*	*0*	*6*
x_6	*0*	*4*	*1*	*1*	*0*	*0*	*1*	*6*
z	*1*	*3*	*0*	**-3**	*2*	*0*	*0*	*4*
x_2	*0*	*2*	*1*	*-1*	*1*	*0*	*0*	*2*
x_5	*0*	*4*	*0*	*4*	*1*	*1*	*0*	*8*
x_6	*0*	*2*	*0*	**2**	*-1*	*0*	*1*	*4*
z	*1*	*6*	*0*	*0*	*11/4*	*3/4*	*0*	*10*
x_2	*0*	*3*	*1*	*0*	*5/4*	*1/4*	*0*	*4*
x_3	*0*	*1*	*0*	*1*	*1/4*	*1/4*	*0*	*2*
x_6	*0*	*0*	*0*	*0*	*-3/2*	*-1/2*	*1*	*0*

The optimal solution is $\mathbf{x} = (0, 4, 2, 0, 0, 0)$.

Optimal basic solution: $\mathbf{x}_B = (x_2, x_3, x_6) = (4, 2, 0) \geq \mathbf{0}$*;* $\mathbf{x}_N = (x_1, x_4, x_5) = \mathbf{0}$.

Optimality holds, since all reduced costs are negative:

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} = (1, 0, 0) - (2, 1, 0)\begin{pmatrix} 5/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix} = (-6, -11/4, -3/4)$$

(b) The optimal value is expressed as $z(\delta) = z^* + \mathbf{c}_B^\top \mathbf{B}^{-1}\Delta$, where $\Delta = (0, \delta, 0)^\top$.

Hence, $z(\delta) = 10 + (2, 0, 1)\begin{pmatrix} 1/2 & 0 & 1/2 \\ 3 & 1 & -2 \\ -1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = 10 + (1/2, 0, 3/2)\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = 10.$

The optimal basis is feasible when $\mathbf{B}^{-1}(\mathbf{b} + \Delta) \geq \mathbf{0}$, i.e., when

$$\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & 1/2 \\ 3 & 1 & -2 \\ -1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ \delta \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the optimal basis $\mathbf{x}_B = (x_2, x_5, x_3)$ is feasible for all $\delta \geq 0$.

Analysis corresponding to the alternative optimal basis in a):

The optimal value is expressed as $z(\delta) = z^* + \mathbf{c}_B^\top \mathbf{B}^{-1}\Delta$*, where* $\Delta = (0, \delta, 0)^\top$.

Hence, $z(\delta) = 10 + (2,1,0)\begin{pmatrix} 5/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = 10 + (11/4, 3/4, 0)\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = 10 + \frac{3}{4}\delta.$

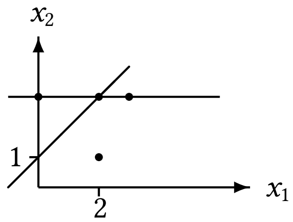
The optimal basis is feasible when $\mathbf{B}^{-1}(\mathbf{b} + \Delta) \geq \mathbf{0}$*, i.e., when*

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 5/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} 4 + \delta/4 \\ 2 + \delta/4 \\ 0 - \delta/2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the optimal basis $\mathbf{x}_B = (x_2, x_3, x_6)$ *is feasible for all* δ *that fulfills* $\delta \geq -16$, $\delta \geq -8$, *and* $\delta \leq 0$*, i.e., whenever the inequalities* $-8 \leq \delta \leq 0$ *hold.*

Solutions to Question 3

(a) Illustration of the feasible set in the (x_1, x_2)-space:



- $\mathbf{x}^1 = (2,3,0,0)$ is an extreme point. E.g. the objective $\max\{-x_1 + 2x_2\} \Longrightarrow$ unique optimum at $\mathbf{x}^1$.
- $\mathbf{x}^2 = (0,3,-2,0)$ is infeasible. Hence it can neither be optimal.
- $\mathbf{x}^3 = (2,1,2,2)$ is an interior point. Hence it cannot be optimal in an LP with a non-constant objective.
- $\mathbf{x}^4 = (3,3,1,0)$ is a non-extreme boundary point. E.g., the objective $\max\{x_2\} \Longrightarrow$ non-unique optimum at $\mathbf{x}^4$

(b) The following sets form bases:

- $\{x_2, x_3\} \Longleftrightarrow \mathbf{x} = (0,3,-2,0)$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
- $\{x_3, x_4\} \Longleftrightarrow \mathbf{x} = (0,0,1,3)$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\{x_2, x_4\} \Longleftrightarrow \mathbf{x} = (0,1,0,2)$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

The following sets do not form bases:

- $\{x_1, x_3\}$ is not a basis, since the corresponding matrix $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ is singular
- $\{x_4\}$ is not a basis, since it contains only one variable Bases have $m = 2$ variables, where m is the number of equality constraints
- $\{x_1, x_2, x_4\}$ is not a basis, since it contains three variables

(c) Feasible $\Longleftrightarrow$ all values ≥ 0

- $\{x_2, x_3\} \Longleftrightarrow \mathbf{x} = (0,3,-2,0)$: an infeasible basis

- $\{x_3, x_4\} \iff \mathbf{x} = (0, 0, 1, 3)$: a feasible basis
- $\{x_2, x_4\} \iff \mathbf{x} = (0, 1, 0, 2)$: a feasible basis

Solutions to Question 4

$$\begin{array}{rlllllll} z^*_{\text{LP}} := \text{maximum} & 5x_1 & +7x_2 & +3x_3 & +9x_4, & & \\ \text{subject to} & 2x_1 & +4x_2 & +2x_3 & +3x_4 & \leq & 8, \\ & x_1, & x_2, & x_3, & x_4 & \in & [0,1]. \end{array}$$

This continuous knapsack problem can be solved to optimality by sorting the ratios:
$\frac{c_j}{a_j} \in \left\{\frac{5}{2}, \frac{7}{4}, \frac{3}{2}, \frac{9}{3}\right\} = \{2.5, 1.75, 1.5, 3\}$.
Optimal LP-solution: $\mathbf{x}^*_{LP} = (1, \frac{3}{4}, 0, 1)$. $z^*_{\text{LP}} = 5 + 7 \cdot \frac{3}{4} + 0 + 9 = 19.25 \geq z^*$.

Lagrangean dual function:

$h(u) = 8u + \max\limits_{x_j \in \{0,1\}, j=1,\dots,4} \big((5-2u)x_1 + (7-4u)x_2 + (3-2u)x_3 + (9-3u)x_4\big)$

$h(1) = 8 + \max\limits_{x_j \in \{0,1\}, j=1,\dots,4} \big(3x_1 + 3x_2 + 1x_3 + 6x_4\big) = 8 + 3 + 3 + 1 + 6 = 21 \geq z^*$.
Subproblem solution $\mathbf{x}(u) = \mathbf{x}(1) = (1, 1, 1, 1)$ is infeasible in (3).

$h(2) = 16 + \max\limits_{x_j \in \{0,1\}, j=1,\dots,4} \big(1x_1 - 1x_2 - 1x_3 + 3x_4\big) = 16 + 1 + 0 + 0 + 3 = 20 \geq z^*$.
Subproblem solution $\mathbf{x}(u) = \mathbf{x}(2) = (1, 0, 0, 1)$ is feasible in (3) $\iff z^* \geq 5 + 0 + 0 + 9 = 14$.

$h(3) = 24 + \max\limits_{x_j \in \{0,1\}, j=1,\dots,4} \big(-1x_1 - 5x_2 - 3x_3 + 0x_4\big) = 24 + 0 + 0 + 0 + 0 = 24 \geq z^*$.
Subproblem solution $\mathbf{x}(u) = \mathbf{x}(3) = (0, 0, 0, 0)$ is feasible in (3) $\iff z^* \geq 0 + 0 + 0 + 0 = 0$.

We conclude that $\max\{14, 0\} \leq z^* \leq \min\{19.25, 21, 20, 24\} \iff 14 \leq z^* \leq 19$ (since z^* must be an integer)

Solutions to Question 5

See the course book, Theorem 6.1 and its proof.