EXAM
MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2022-06-02
  - **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3) points
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg
  - **Phone:** 0705-273645

Due to a misprint in the exam, the proposed solution to Question 2 was erroneous. It has been corrected below

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

MODELLING

[3p]

A transport company needs to ship n items in containers. Item j weighs a_j tonnes, $j = 1, \dots, n$. There are m containers available, and container i can carry b_i tonnes, $i = 1, \dots, m$. The items are very heavy in comparison to their sizes, and therefore it is only the weight capacity of each container that is limiting what items can be packed in which container. The cost of using container i is c_i SEK, regardless of its weight, $i = 1, \dots, m$. The transport company wants to choose containers such that the total cost is minimized.

Help the transport company to formulate a *binary (integer) linear optimization model* of the problem described. Declare and describe your variables and constraints carefully.

[*Hint*: Use one set of variables for choosing containers and another set of variables to describe what item is packed in which container.]

Question 2

THE SIMPLEX ALGORITHM AND SENSITIVITY ANALYSIS

(a) **[2p]**

Solve the following linear optimization problem using the simplex method.

$$\begin{array}{llcrcrcrcrr} \text{maximize} & z & = & 6x_1 & + & 14x_2 & + & 13x_3 & & & \quad (1a) \\ \text{subject to} & & & x_1 & + & 4x_2 & + & 2x_3 & \leq & 48 & \quad (1b) \\ & & & x_1 & + & 2x_2 & - & 4x_3 & \leq & 60 & \quad (1c) \\ & & & x_1 & , & x_2 & , & x_3 & \geq & 0 & \quad (1d) \end{array}$$

[*Hint*: In the first iteration, let x_1 be the entering variable.]

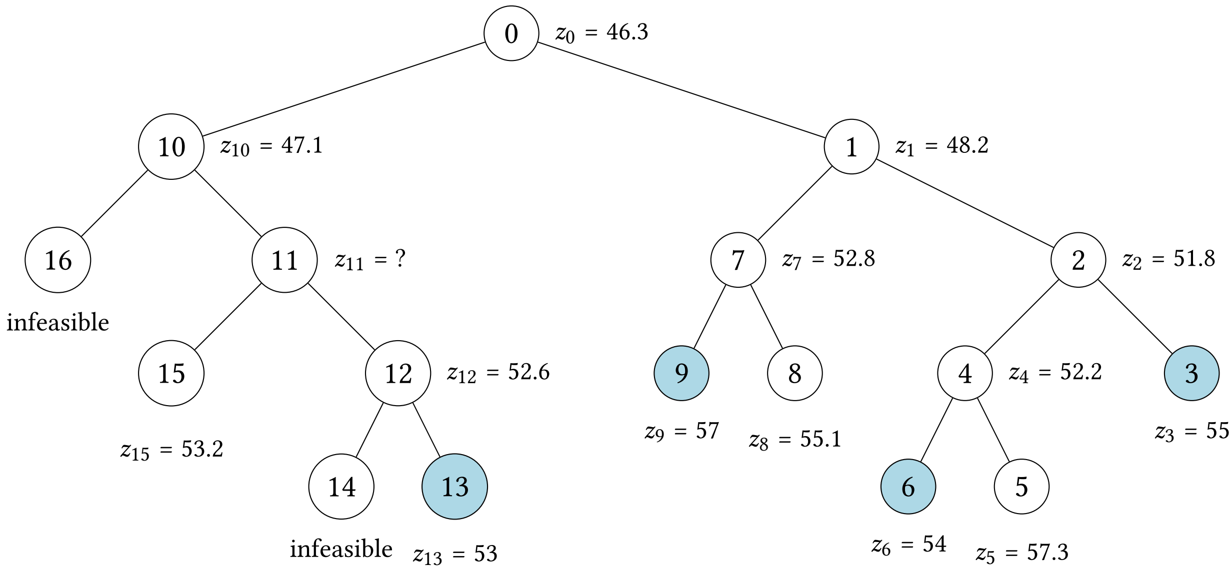
(b) **[1p]**

Assume that the problem is extended with a new variable, x_{new}, with constraint coefficients $\mathbf{A}_{\text{new}} = (3, 5)^\top$. For what values of its objective function coefficient, c_{new}, will the optimal objective function value calculated in (a) change?

Question 3

The Branch & Bound algorithm

The complete search tree below was produced using the branch & bound algorithm to solve an integer linear minimization problem. The nodes are numbered according to the search order (depth-first strategy over the ≥-branch). For each node $k = 0, \ldots, 12, 13, 15$, z_k denotes the optimal objective value of the corresponding LP-relaxed problem. The coloured nodes indicate feasible integer solutions. Assume that all objective function coefficients are integers.



(a) **[1p]**

Why have the nodes 5, 8, and 16 been cut? What is the optimum objective function value?

(b) **[1p]**

What is the interval for z_{11} (node 11)?

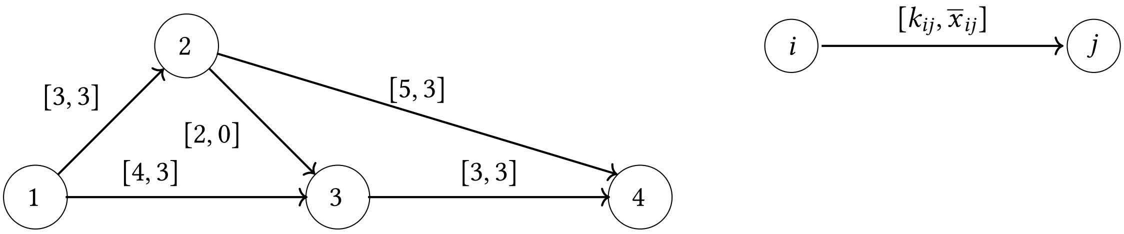
(c) **[1p]**

Assume that the tree search had been stopped directly after node 9 was evaluated. What would be the interval for the optimal objective value?

Question 4

Maximum flow problem and the complementary theorem

Consider a *maximum flow problem* instance on a graph $G = (\mathcal{N}, \mathcal{A}, \mathbf{k})$, given by



- with the set of nodes $\mathcal{N} := \{1, 2, 3, 4\}$,
- the set of directed arcs/links $\mathcal{A} := \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$, and
- the link capacities $\mathbf{k} = (k_{ij})_{(i,j)\in\mathcal{A}} := (3, 4, 2, 5, 3)$.

A feasible flow in the graph G is given by $\overline{\mathbf{x}} = (\overline{x}_{ij})_{(i,j)\in\mathcal{A}} := (3, 3, 0, 3, 3)$.

The problem of maximizing the flow from node 1 to node 4 is modelled as the following linear optimization problem:

$$
\begin{array}{lrrrrrrl}
\text{maximize}_{\mathbf{x},v} & v, & & & & & & \\
\text{subject to} & -x_{12} & -x_{13} & & & & +v = & 0, \\
 & +x_{12} & & -x_{23} & -x_{24} & & = & 0, \\
 & & +x_{13} & +x_{23} & & -x_{34} & = & 0, \\
 & & & & +x_{24} & +x_{34} & -v = & 0, \\
 & & & & 0 \leq & x_{ij} & \leq & k_{ij}, \quad (i,j) \in \mathcal{A}.
\end{array}
\tag{2}
$$

Its linear optimization dual, the so-called *minimum cut problem*, is defined as follows:

$$
\begin{array}{lrrrrrl}
\text{minimize}_{\boldsymbol{\pi},\boldsymbol{\gamma}} & 3\gamma_{12} & +4\gamma_{13} & +2\gamma_{23} & +5\gamma_{24} & +3\gamma_{34}, & \\
\text{subject to} & -\pi_1 & +\pi_2 & & & +\gamma_{12} \geq & 0, \\
 & -\pi_1 & & +\pi_3 & & +\gamma_{13} \geq & 0, \\
 & & -\pi_2 & +\pi_3 & & +\gamma_{23} \geq & 0, \\
 & & -\pi_2 & & +\pi_4 & +\gamma_{24} \geq & 0, \\
 & & & -\pi_3 & +\pi_4 & +\gamma_{34} \geq & 0, \\
 & +\pi_1 & & & -\pi_4 & = & 1, \\
 & & & & & \gamma_{ij} \geq & 0, \quad (i,j) \in \mathcal{A}.
\end{array}
\tag{3}
$$

(a) **[2p]**

State the *complementary theorem of primal and dual optimal solutions*, and the resulting, explicit formulas when it is applied to the primal–dual pair of models (2)–(3).

(b) **[1p]**

Then, utilize the complementary theorem to show that the flow defined by the solution $\overline{\mathbf{x}}$ is optimal in the model (2).

[*Hint*: You may utilize a cut that partitions the nodes of the graph into the two sets $\{1, 3\}$ (comprising the origin node 1) and $\{2, 4\}$ (comprising the destination node 4).

You may also utilize the fact that there always exists a solution to (3) for which $\pi_4 = 0$.]

Question 5

LP DUALITY

To determine whether there exists a feasible solution to the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 2x_4 &= 3 && \text{(4a)} \\ x_1 + x_2 - 2x_3 + 2x_4 &= 5 && \text{(4b)} \\ x_1, x_2, x_3, x_4 &\geq 0 && \text{(4c)} \end{aligned}$$

one can introduce the artificial variables a_1 and a_2, and solve the phase-I problem to

$$\begin{aligned} \text{minimize} \quad v = & \quad a_1 + a_2 && \text{(5a)} \\ \text{subject to} \quad & 2x_1 + 3x_2 - x_3 + 2x_4 + a_1 = 3 && \text{(5b)} \\ & x_1 + x_2 - 2x_3 + 2x_4 + a_2 = 5 && \text{(5c)} \\ & x_1, x_2, x_3, x_4, a_1, a_2 \geq 0 && \text{(5d)} \end{aligned}$$

(a) **[1p]**

State the linear optimization dual to the problem (5).

(b) **[1p]**

Solve the dual problem graphically.

(c) **[1p]**

Use the optimal solution to the dual problem (as computed in (b)) to determine whether or not there exists a feasible solution to the system (4).

Refer to the theoretical properties/results that are utilized for your conclusion(s).

Solution proposals

Note that some of these solutions are quite brief and that more explanations may be needed to pass some of the (part) questions.

Solutions to Question 1

Variables:

$$y_i = \begin{cases} 1, & \text{if container } i,\, i = 1, \dots, m, \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1, & \text{if item } j, j = 1, \dots, n, \text{ is packed in container } i,\, i = 1, \dots, m \\ 0, & \text{otherwise} \end{cases}$$

Model:

$$\begin{array}{llcll} \text{minimize } z = & \sum_{i=1}^{m} c_i y_i & & & \\ \text{subject to} & \sum_{j=1}^{n} a_j x_{ij} & \leq & b_i y_i, & i = 1, \dots, m \\ & \sum_{i=1}^{m} x_{ij} & = & 1, & j = 1, \dots, n \\ & x_{ij} & \in & \{0, 1\} & \forall\, i, j \\ & y_i & \in & \{0, 1\} & \forall\, i \end{array}$$

The first constraint implies that no container is packed with too much weight, and no item is packed in a container which is not used.

The second constraint implies that every item is packed in exactly one container.

Solutions to Question 2 — Corrected 220602, evening

(a) Introduce the slack variables $x_4, x_5 \geq 0$, corrseponding to the first and second constraint, respectively. Using x_1 as entering variable, x_4 will leave. In the next iteration, x_3 will enter and x_1 leaves, which yields the optimal solution $x_1^* = 0, x_2^* = 0, x_3^* = 24 \implies z^* = 312$

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$
z	1	**-6**	-14	-13	0	0	0
x_4	0	**1**	4	2	1	0	48
x_5	0	1	2	-4	0	1	60
z	1	0	10	**-1**	6	0	288
x_1	0	1	4	**2**	1	0	48
x_5	0	0	-2	-6	-1	1	12
z	1	1/2	12	0	13/2	0	312
x_3	0	1/2	2	1	1/2	0	24
x_5	0	3	10	0	2	1	156

(b) Optimal dual solution: $(\mathbf{y}^*)^\top = \mathbf{c}_B^\top \mathbf{B}^{-1} = (13, 0)\begin{pmatrix} 1/2 & 0 \\ 2 & 1 \end{pmatrix} = (13/2, 0)$.

Reduced costs for x_{new}: $\bar{c}_{\text{new}} = c_{\text{new}} - (\mathbf{y}^*)^\top \mathbf{A}_{\text{new}} = c_{\text{new}} - (13/2, 0)\begin{pmatrix} 3 \\ 5 \end{pmatrix} = c_{\text{new}} - \frac{3 \cdot 13}{2} > 0$.

If $\bar{c}_{\text{new}} > 0 \iff c_{\text{new}} > 19.5$, then x_{new} will enter and z^* increases. (Note that the optimum is non-degenerate.)

$\therefore \quad c_{\text{new}} > 19.5 \implies z^*$ changes.

Solutions to Question 3

(a) Node 5: $z_5 = 57.3 > \bar{z} = z_3 = 55$

Node 8: $z_8 = 55.1 > \bar{z} = z_6 = 54$

Node 16: the LP-relaxed problem is infeasible, and thus also the original integer problem

The optimum objective function value is $z^* = z_{13} = 53$.

(b) $47.1 \leq z_{11} \leq 52.6$

(c) $\lceil 46.3 \rceil = 47 \leq z^* \leq 54$

Solutions to Question 4

The complementarity theorem applied to the given maximum flow problem instance states the following: Assume that $\overline{\mathbf{x}}$ and (π, γ) are feasible points in the primal and dual problems, respectively. Then, they are also optimal in their respective problems if and only if the following constraints hold:

$$
\begin{array}{rcl}
(-\pi_1 + \pi_2 + \gamma_{12}) \cdot \overline{x}_{12} = 0 & \& & \gamma_{12} \cdot (3 - \overline{x}_{12}) = 0 \\
(-\pi_1 + \pi_3 + \gamma_{13}) \cdot \overline{x}_{13} = 0 & \& & \gamma_{13} \cdot (4 - \overline{x}_{13}) = 0 \\
(-\pi_2 + \pi_3 + \gamma_{23}) \cdot \overline{x}_{23} = 0 & \& & \gamma_{23} \cdot (2 - \overline{x}_{23}) = 0 \\
(-\pi_2 + \pi_4 + \gamma_{24}) \cdot \overline{x}_{24} = 0 & \& & \gamma_{24} \cdot (5 - \overline{x}_{24}) = 0 \\
(-\pi_3 + \pi_4 + \gamma_{34}) \cdot \overline{x}_{34} = 0 & \& & \gamma_{34} \cdot (3 - \overline{x}_{34}) = 0 \\
(\pi_1 - \pi_4 - 1) \cdot \overline{v} = 0 & \& & \overline{v} = \overline{x}_{12} + \overline{x}_{13}
\end{array}
$$

where . Inserting $\overline{\mathbf{x}}$ and $\overline{v}$ in the above yields

$$
\begin{array}{rcrcrcl}
(-\pi_1 + \pi_2 + \gamma_{12}) \cdot 3 = 0 & \Rightarrow & -\pi_1 + \pi_2 + \gamma_{12} = 0 & \& & \gamma_{12} \cdot (3 - 3) = 0 & \Rightarrow & \gamma_{12} \geq 0 \\
(-\pi_1 + \pi_3 + \gamma_{13}) \cdot 3 = 0 & \Rightarrow & -\pi_1 + \pi_3 + \gamma_{13} = 0 & \& & \gamma_{13} \cdot (4 - 3) = 0 & \Rightarrow & \gamma_{13} = 0 \\
(-\pi_2 + \pi_3 + \gamma_{23}) \cdot 0 = 0 & \Rightarrow & -\pi_2 + \pi_3 + \gamma_{23} \geq 0 & \& & \gamma_{23} \cdot (2 - 0) = 0 & \Rightarrow & \gamma_{23} = 0 \\
(-\pi_2 + \pi_4 + \gamma_{24}) \cdot 3 = 0 & \Rightarrow & -\pi_2 + \pi_4 + \gamma_{24} = 0 & \& & \gamma_{24} \cdot (5 - 3) = 0 & \Rightarrow & \gamma_{24} = 0 \\
(-\pi_3 + \pi_4 + \gamma_{34}) \cdot 3 = 0 & \Rightarrow & -\pi_3 + \pi_4 + \gamma_{34} = 0 & \& & \gamma_{34} \cdot (3 - 3) = 0 & \Rightarrow & \gamma_{34} \geq 0 \\
(\pi_1 - \pi_4 - 1) \cdot 6 = 0 & \Rightarrow & \pi_1 - \pi_4 = 1 & \& & \overline{v} = 3 + 3 & \Rightarrow & \overline{v} = 6
\end{array}
$$

This reduces to the following equations and inequalities:

$$
\begin{gathered}
-\pi_1 + \pi_3 = 0; \quad -\pi_2 + \pi_4 = 0; \quad +\pi_1 - \pi_4 = 1; \\
\gamma_{13} = \gamma_{23} = \gamma_{24} = 0; \\
\gamma_{12} = \pi_1 - \pi_2 \geq 0; \quad -\pi_2 + \pi_3 \geq 0; \quad \gamma_{34} = \pi_3 - \pi_4 \geq 0.
\end{gathered}
$$

This yields the relations $\pi_3 = \pi_1$ and $\pi_2 = \pi_4 = \pi_1 - 1$.

Setting $\pi_4 = 0$ leads to $\pi_1 = \pi_3 = 1$ and $\pi_2 = 0$, and further that $\gamma_{12} = 1$, $\gamma_{34} = 1$.

By, instead, utilizing the cut, one can compute the dual variable values according to the following.

For nodes $i \in \{1, 3\}$, $\pi_i = 1$, while for nodes $i \in \{2, 4\}$, $\pi_i = 0$.

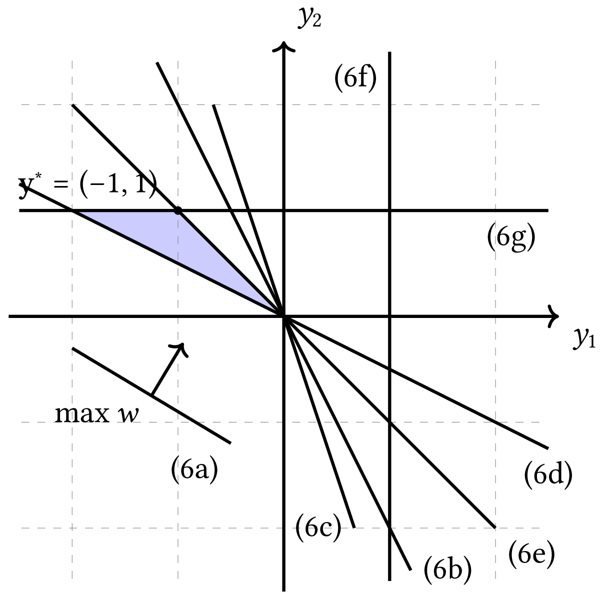
For links (i, j) passing over the cut, i.e., such that $i \in \{1, 3\}$ and $j \in \{2, 4\}$, $\gamma_{ij} = 1$. For all other links, $\gamma_{ij} = 0$. (Then, the dual objective value equals the sum of capacities of the links passing over the cut.) It follows that $\gamma_{13} = \gamma_{23} = \gamma_{24} = 0$ and $\gamma_{12} = \gamma_{34} = 1$.

Solutions to Question 5

(a)

$$
\begin{array}{llcrcrcr}
\text{maximize} & w & = & 3y_1 & + & 5y_2 & & \\
\text{subject to} & & & 2y_1 & + & y_2 & \leq & 0 \\
 & & & 3y_1 & + & y_2 & \leq & 0 \\
 & & - & y_1 & - & 2y_2 & \leq & 0 \\
 & & & 2y_1 & + & 2y_2 & \leq & 0 \\
 & & & y_1 & & & \leq & 1 \\
 & & & & & y_2 & \leq & 1
\end{array}
\begin{array}{r}
(6a)\\(6b)\\(6c)\\(6d)\\(6e)\\(6f)\\(6g)
\end{array}
$$

(b) Optimal solution to the dual problem: $y_1^* = -1$, $y_2^* = 1$, $w^* = 2 > 0$. Equality holds for the constraints (6e) and (6g).



(c) Strong duality yields that $w^* = v^* = 2 > 0$, which means that in the problem (5) at least one of the artificial variables has a non-zero optimal value. Hence, the system (4) has no feasible solution.

[Using the complementarity condition yields the following optimal solution to the phase-I problem: $x_1^* = x_2^* = x_3^* = 0$, $x_4^* = \frac{3}{2}$, $a_1 = 0$, $a_2 = 2$, and $v^* = 2$.]