Chalmers/GU
Mathematical Sciences

EXAM

MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2022-01-03
- **Hours:** 08:30–12:30
- **Aids:** Text memory-less calculator; English-Swedish dictionary; pens; paper; ruler
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3)
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade
  - Bonus points (from assignments) are valid for the three first exam occasions, counted from the course round when they were gained (i.e., the ordinary exam and the two following re-exam occasions)
- **Examiner:** Ann-Brith Strömberg (0705-273645)

General instructions for the exam

When answering the questions

- use generally valid theory and methodology
- state your methodology carefully
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged
- do not use a red pen
- do not answer more than one question per sheet

Question 1

[3p]

A company produces two types of candy, *toffee* and *sweety*, both of which contains *sugar*, *nuts*, and *chocolate*.

The company has in stock 100 kg of sugar, 20 kg of nuts, and 30 kg of chocolate. The mixture for *toffee* must contain at least 20 % nuts. The mixture for *sweety* must contain at least 10 % nuts and at least 10 % chocolate.

Each hg of *toffee* can be sold for SEK 25 and each hg of *sweety* can be sold for SEK 20. It is assumed that the demand for candy is at least as large as the quantity that can be produced in the factory.

Formulate a linear optimization model to find the recipes for the mixtures of *toffee* and *sweety* such that only the stock content is used for the production and such that the company's revenue from candy sales is maximized.

Declare and describe your variables and constraints carefully.

Question 2

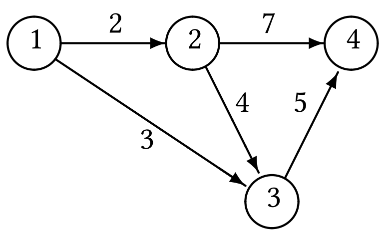
Consider a graph consisting of a set $\mathcal{N}$ of nodes, a set $\mathcal{A}$ of directed arcs, and a vector $\mathbf{c}$ of positive lengths corresponding to the arcs in $\mathcal{A}$. Hence, for each arc $(i, j) \in \mathcal{A}$, $c_{ij} > 0$ denotes its length.

(a) **[1p]**

For a *general graph* as defined above, give a linear optimization formulation of the problem to find the shortest path from node $s \in \mathcal{N}$ to node $t \in \mathcal{N}$.

(b) **[1p]**

The *specific graph* defined by $\mathcal{N} := \{1, 2, 3, 4\}$, $\mathcal{A} := \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$, and $\mathbf{c} := (2, 3, 4, 7, 5)$ is illustrated below. Let $s = 1$ and $t = 4$ and formulate the linear optimization dual (LP dual) of your model in (a) for this specific graph.



(c) **[1p]**

Utilize linear optimization strong duality to verify that $1 - 3 - 4$ is a shortest path from node 1 to node 4 in the graph above.

State carefully the properties and assumptions referred to.

Question 3

Consider the linear optimization problem to

$$
\begin{array}{lcccccccccl}
\text{maximize} & z & = & 5x_1 & + & 4x_2 & + & x_3 & & & \quad (1a)\\
\text{s.t.} & & & 2x_1 & - & x_2 & & & \leq & 1 & \quad (1b)\\
& & & x_1 & + & x_2 & - & x_3 & \leq & 3 & \quad (1c)\\
& & & x_1 & , & x_2 & , & x_3 & \geq & 0 & \quad (1d)
\end{array}
$$

(a) **[1p]**

Formulate the problem (1) on the standard form.

(b) **[2p]**

Solve the problem (1) using the simplex method. At termination, what can be concluded about the properties of an optimal solution to the problem (1)?

Question 4

Consider the linear optimization problem (LP)

$$
\begin{array}{lccccccccl}
z^*_{\text{LP}} & := & \max\limits_{x_1,x_2} & 2x_1 & + & c_2x_2, & & & & \quad (2a)\\
& & \text{s.t.} & x_1 & + & x_2 & \leq & 5, & & \quad (2b)\\
& & & 4x_1 & + & 2x_2 & \leq & 17, & & \quad (2c)\\
& & & 2x_1 & + & 5x_2 & \leq & b_3, & & \quad (2d)\\
& & & x_1 & , & x_2 & \geq & 0, & & \quad (2e)
\end{array}
$$

and let x_3, x_4, and x_5 denote the (non-negative) slack variables of the constraints (2b), (2c), and (2d), respectively.

(a) **[1p]**

For which values of the coefficient b_3 is the point $(x_1, x_2) = (2, 3)$ feasible in (2)?

(b) **[1p]**

For which values of the coefficient b_3 is $\mathbf{x}_B = (x_1, x_2, x_4)$ a feasible basis for the linear optimization problem (2) (formulated on the standard form)?

(c) **[1p]**

For which values of the coefficients b_3 and c_2 is $\mathbf{x}_B = (x_1, x_2, x_4)$ an optimal basis for the linear optimization problem (2) (formulated on the standard form)?

Question 5

Consider the integer linear optimization problem (ILP)

$$
\begin{aligned}
z^* \quad := \quad \max_{x_1,x_2} \quad & 2x_1 + 3x_2, && \text{(3a)}\\
\text{s.t.} \quad & x_1 + 2x_2 \leq 8, && \text{(3b)}\\
& 2x_1 + 2x_2 \leq 11, && \text{(3c)}\\
& x_1 \;,\; x_2 \geq 0 \text{ and integer.} && \text{(3d)}
\end{aligned}
$$

(a) **[2p]**

Use the Branch-and-bound algorithm to solve the problem (3).

Use depth-first search and search the $\leq$-branch first. In case more than one variable has a fractional value in the solution to a node subproblem, branch over the variable having the lowest index. The node subproblems may be solved graphically.

State the optimal solution obtained.

(b) **[1p]**

Consider now a constant parameter $M \gg 1$ and the ILP

$$
\begin{aligned}
z_M^* \quad := \quad \max_{x_1,x_2,y} \quad & 2x_1 + 3x_2, && \text{(4a)}\\
\text{s.t.} \quad & x_1 + 2x_2 - My \leq 8, && \text{(4b)}\\
& 2x_1 + 2x_2 + My \leq 11 + M, && \text{(4c)}\\
& x_1 \;,\; x_2 \geq 0 \text{ and integer,} && \text{(4d)}\\
& y \in \{0, 1\}. && \text{(4e)}
\end{aligned}
$$

Do not solve the problem (4)*!*

Give the best possible (upper and/or lower) bounds on z_M^* based on your solution to the problem (3) and/or the value z^*.

Motivate your conclusion(s) carefully.

Solution proposals

Note that some of these solutions are quite brief and that more explanations may be needed to pass some of the (part) questions.

Solution to Question 1

Let x_{ij} be the amount (in kg) of ingredient i used for candy j.

$i = 1$: sugar, $i = 2$: nuts, $i = 3$: chocolate $j = 1$: toffee, $j = 2$: sweety

$$\begin{array}{llrcll}
\text{maximize} & 25(x_{11}+x_{21}+x_{31}) + 20(x_{12}+x_{22}+x_{32}) & & & & \\
\text{subject to} & & x_{11}+x_{12} & \leq & 100 & \\
 & & x_{21}+x_{22} & \leq & 20 & \\
 & & x_{31}+x_{32} & \leq & 30 & \\
 & & 0.2(x_{11}+x_{21}+x_{31}) & \leq & x_{21} & \iff \quad 0.2x_{11} - 0.8x_{21} + 0.2x_{31} \leq 0 \\
 & & 0.1(x_{12}+x_{22}+x_{32}) & \leq & x_{22} & \iff \quad 0.1x_{12} - 0.9x_{22} + 0.1x_{32} \leq 0 \\
 & & 0.1(x_{12}+x_{22}+x_{32}) & \leq & x_{32} & \iff \quad 0.1x_{12} + 0.1x_{22} - 0.9x_{32} \leq 0 \\
 & & x_{ij} & \geq & 0 \quad \forall i,j &
\end{array}$$

Solution to Question 2

(a)

$$\begin{array}{rrcl}
\min & \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}, & & \\
\text{s.t.} & \sum_{i\in\mathcal{N}:(i,k)\in\mathcal{A}} x_{ik} - \sum_{j\in\mathcal{N}:(k,j)\in\mathcal{A}} x_{kj} & = & \begin{cases} -1, & k = s, \\ 1, & k = t, \\ 0, & k \in \mathcal{N}\setminus\{s,t\}, \end{cases} \\
 & x_{ij} & \geq & 0, \quad (i,j)\in\mathcal{A}
\end{array}$$

(b) For the graph defined by $\mathcal{N} = \{1,2,3,4\}$, $\mathcal{A} = \{(1,2),(1,3),(2,3),(2,4),(3,4)\}$, and $\mathbf{c} = (2,3,4,7,5)$ the LP dual is formulated as

$$\begin{array}{rrcl}
\max & y_4 - y_1, & & \\
\text{s.t.} & y_2 - y_1 & \leq & 2, \\
 & y_3 - y_1 & \leq & 3, \\
 & y_3 - y_2 & \leq & 4, \\
 & y_4 - y_2 & \leq & 7, \\
 & y_4 - y_3 & \leq & 5
\end{array}$$

(c) The path $1 - 3 - 4$ corresponds to the primal solution $x_{12} = 0$, $x_{13} = 1$, $x_{23} = 0$, $x_{24} = 0$, $x_{34} = 1$, which is feasible in the model in (a) for the specific graph in (b).

The complementarity conditions are given by $x_{ij} \cdot (y_j - y_i - c_{ij}) = 0$ for all $(i,j) \in \mathcal{A}$. Since $x_{13} = x_{34} = 1 > 0$, it thus must hold that $y_3 - y_1 = c_{13}$ and $y_4 - y_3 = c_{34}$, i.e., $y_3 - y_1 = 3$ and $y_4 - y_3 = 5$. Without loss of generality, we may assume that $y_1 = 0$, which yields that $y_3 = 3$ and $y_4 = 8$. The optimal dual solution must be feasible, which means that the inequalities $y_2 - y_1 \leq 2$, $y_3 - y_2 \leq 4$, and $y_4 - y_2 \leq 7$ must hold, i.e., $y_2 - 0 \leq 2 \iff y_2 \leq 2$, $3 - y_2 \leq 4 \iff y_2 \geq -1$, and $8 - y_2 \leq 7 \iff y_2 \geq 1$. Hence, an optimal dual solution is given by $(y_1, y_2, y_3, y_4) = (0, 1, 3, 8)$. The optimal value is 8.

Solution to Question 3

(a) Introduce the slack variables x_4 and x_5:

$$
\begin{array}{lccccccccccccc}
\text{maximize} & z & = & 5x_1 & + & 4x_2 & + & x_3 & & & & & & \\
\text{s.t.} & & & 2x_1 & - & x_2 & & & + & x_4 & & & = & 1 \\
 & & & x_1 & + & x_2 & - & x_3 & & & + & x_5 & = & 3 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

(b) Simplex iterations are performed according to the following:

$\mathbf{x}_B$	z	x_1	x_2	x_3	x_4	x_5	$\mathbf{B}^{-1}\mathbf{b}$
z	1	**-5**	-4	-1	0	0	0
x_4	0	**2**	-1	0	1	0	1
x_5	0	1	1	-1	0	1	3
z	1	0	**-13/2**	-1	5/2	0	5/2
x_1	0	1	-1/2	0	1/2	0	1/2
x_5	0	0	**3/2**	-1	-1/2	1	5/2
z	1	0	0	**-16/3**	1/3	13/3	40/3
x_1	0	1	0	-1/3	1/3	1/3	4/3
x_2	0	0	1	-2/3	-1/3	2/3	5/3

The variable x_3 should enter the basis, since its reduced cost is positive ($-\bar{c} = -16/3$). None of the basic variables can be detected to leave the basis, since all coefficients of the step direction corresponding to the basic variables are negative, i.e., $(d_1, d_2) = (-1/3, -2/3)$. It follows that the problem has an unbounded solution and that the objective value can grow infinitely large.

Solution to Question 4

(a) The constraint (2d) is fulfilled by $(x_1, x_2) = (2, 3)$ whenever $b_3 \geq 19$. The constraints (2b), (2c), and (2e) are fulfilled by $(x_1, x_2) = (2, 3)$.

(b) The basic matrix corresponding to the basis $\mathbf{x}_B = (x_1, x_2, x_4)$ is given by $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ 2 & 5 & 0 \end{pmatrix}$ and $\mathbf{b} = (5, 17, b_3)$. The basis inverse is computed as $\mathbf{B}^{-1} = \frac{1}{3}\begin{pmatrix} 5 & 0 & -1 \\ -2 & 0 & 1 \\ -16 & 3 & 2 \end{pmatrix}$.

The basis is feasible whenever $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0} \iff b_3 \leq 25$, $b_3 \geq 10$, and $b_3 \geq \frac{29}{2}$. Hence, the basis is feasible for $\frac{29}{2} \leq b_3 \leq 25$.

(c) Any optimal basis must also be feasible; hence the constraints on b_3 should be as in (b), i.e., $\frac{29}{2} \leq b_3 \leq 25$.

For the basis to be optimal, the reduced costs should be non-positive (maximization problem), i.e., $\mathbf{c}_N^\top - \mathbf{c}_B^\top \mathbf{B}^{-1}\mathbf{N} \leq \mathbf{0}^\top$.

Since $\mathbf{N} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{c}_B = \begin{pmatrix} 2 \\ c_2 \\ 0 \end{pmatrix}$, and $\mathbf{c}_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, it thus holds that

$$\begin{pmatrix}0 & 0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2 & c_2 & 0\end{pmatrix}\begin{pmatrix}5 & 0 & -1\\ -2 & 0 & 1\\ -16 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}2 & c_2 & 0\end{pmatrix}\begin{pmatrix}5 & -1\\ -2 & 1\\ -16 & 2\end{pmatrix}$$
$$= -\frac{1}{3}\begin{pmatrix}10 - 2c_2 & -2 + c_2\end{pmatrix} \leq \begin{pmatrix}0 & 0\end{pmatrix}.$$

It follows that the basis $\mathbf{x}_B = (x_1, x_2, x_4)$ is optimal when $\frac{29}{2} \leq b_3 \leq 25$ and $2 \leq c_2 \leq 5$.

Solution to Question 5

(a) The first node subproblem (P0) is the LP relaxation of (3):

$$\begin{array}{rrrrrrr} z^*_{\text{LP}} & := & \max\limits_{x_1,x_2} & 2x_1 & + & 3x_2, & \\ & & \text{s.t.} & x_1 & + & 2x_2 & \leq & 8, \\ & & & 2x_1 & + & 2x_2 & \leq & 11, \\ & & & x_1 & , & x_2 & \geq & 0. \end{array}$$

P0 = LP relaxation: node solution: $\mathbf{x}^*_{\text{P0}} = \begin{pmatrix}3\\ \frac{5}{2}\end{pmatrix}$, $z^*_{\text{P0}} = \frac{27}{2} \Longrightarrow \bar{z}_{\text{P0}} = 13$

P1 = P0 & "$x_2 \leq 2$": node solution: $\mathbf{x}^*_{\text{P1}} = \begin{pmatrix}\frac{7}{2}\\ 2\end{pmatrix}$, $z^*_{\text{P1}} = 13 \Longrightarrow \bar{z}_{\text{P1}} = 13$

P2 = P1 & "$x_1 \leq 3$": node solution: $\mathbf{x}^*_{\text{P2}} = \begin{pmatrix}3\\ 2\end{pmatrix}$, $z^*_{\text{P2}} = 12 \Longrightarrow \underline{z}_{\text{P2}} = 12$
Integer solution $\Longrightarrow$ cut the branch

P3 = P1 & "$x_1 \geq 4$": node solution: $\mathbf{x}^*_{\text{P3}} = \begin{pmatrix}4\\ \frac{3}{2}\end{pmatrix}$, $z^*_{\text{P3}} = \frac{25}{2} \Longrightarrow \bar{z}_{\text{P3}} = 12$
Cannot contain any solution better than $\mathbf{x}^*_{\text{P2}} \Longrightarrow$ cut the branch

P4 = P0 & "$x_2 \geq 3$": node solution: $\mathbf{x}^*_{\text{P4}} = \begin{pmatrix}2\\ 3\end{pmatrix}$, $z^*_{\text{P4}} = 13 \Longrightarrow \underline{z}_{\text{P4}} = 13$
Integer solution $\Longrightarrow$ cut the branch

The optimal solution to (3) is $\mathbf{x}^* = \begin{pmatrix}2\\ 3\end{pmatrix}$ with optimal value $z^* = 13$.

(b) For a large enough value of $M \gg 1$, the problem (4) is a relaxation of (3). Hence, it holds that $z^*_M \geq z^* = 13$.