RE-EXAM
MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2021-08-26
  - **Hours:** 14:00–18:00
- **Examiner:** Ann-Brith Strömberg
- **Aids:** All aids are allowed, but cooperation is not allowed
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least seven (7) out of fifteen (15) points (not including bonus points) and at least two (2) passed questions
  - To pass a question requires at least two (2) points out of three (3)
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

An exam is composed of three exercises that should be scored. The total number of points should amount to 100. Since the three exercises are differently extensive, exercises 1 and 2 should amount to at least 20 points each, while exercise 3 should amount to at least 30 points. Exercise 1 contains 25 % of questions that require more than basic knowledge. The corresponding share for exercise 2 (3) is 50 % (50 %). At least 40 points out of the 100 should require more than basic knowledge. It is expected that the examinees will receive on average 50 % of the total number of points for exercise 1, 40 % of the total number of points for exercise 2, and 30 % of the total number of points for exercise 3. The totally 100 points should be distributed such that the expected average number of points received by the examinees is as high as possible.

(a) **[2p]**

Formulate the problem described above as a linear optimization problem.

(b) **[1p]**

It can be shown that the optimal scoring of exercises 1, 2, and 3 is 40, 30, and 30 points, respectively. How will the expected average number of points received by the examinees change if the number of points that require more that basic knowledge is reduced from 40 to 35?

Question 2

Consider the following system of constraints:

$$
\begin{aligned}
4x_1 - x_2 - 2x_3 - 5x_4 + 7x_5 &= 8 && (2.1a)\\
-2x_1 + 2x_2 + x_3 + 4x_4 - 5x_5 &= -4 && (2.1b)\\
x_1,\ x_2,\ x_3,\ x_4,\ x_5 &\geq 0 && (2.1c)
\end{aligned}
$$

For each of the questions below, motivate your answer using theory from the course.

(a) **[1p]**

Is the point $\mathbf{x} = (1, 1, 1, 0, 1)^\top$ an extreme point to the set in $\mathbb{R}^5$ defined by the system (2.1)? Why/why not?

(b) **[1p]**

Does it hold that $x_1 = x_3 = x_5 = 0$ in any extreme point to the set in $\mathbb{R}^5$ defined by the system (2.1)? Why/why not?

(c) **[1p]**

Is the point $\mathbf{x} = (3, 0, 2, 0, 0)^\top$ an extreme point to the set in $\mathbb{R}^5$ defined by the system (2.1)? Why/why not?

Question 3

(a) **[1p]**

An optimization problem comprises the variables x_1, x_2, and y, which are all restricted to the values 0 or 1. The relations between these three variables should be the following:

$$y = \begin{cases} 1 \text{ if } x_1 = x_2 = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Model these relations using *linear* constraints.

(b) **[1p]**

In an optimization problem there is a requirement that *either* the constraints

$$\begin{cases} 3x_1 + x_2 \leq 5 \\ x_1 + 2x_2 \leq 6 \end{cases}$$

should be fulfilled, *or* the constraints

$$\begin{cases} 2x_1 + x_2 \geq 18 \\ x_1 + 2x_2 \geq 10 \end{cases}$$

should be fulfilled. It is known that both x_1 and x_2 can attain values in the interval $[0, M]$, where $M \gg 1$, but no values outside of this interval. (Obviously then, the two sets of constraints cannot be fulfilled simultaneously.) Model the requirement stated above using additional binary variables and linear constraints.

(c) **[1p]**

An optimization problem contains three binary variables, $x_1, x_2, x_3 \in \{0, 1\}$. Construct a *linear constraint* that makes the solution $(x_1, x_2, x_3) = (0, 1, 1)$ infeasible, but such that *no other* binary point is cut off.

Question 4

Consider the following linear optimization problem

$$
\begin{array}{llcrcll}
z^\star := \max & c_1x_1 & + & c_2x_2, & & & (4.1a)\\
\text{s.t.} & 3x_1 & + & x_2 & \le & 4, & (4.1b)\\
 & x_1 & + & 2x_2 & \le & 4, & (4.1c)\\
 & x_1, & & x_2 & \ge & 0, & (4.1d)
\end{array}
$$

where the parameters c_1 and c_2 are assumed to be non-negative.

For each of the questions below, you should motivate your answer by theory and calculations.

(a) **[1p]**

For what values of the parameters c_1 and c_2 is the point $x_1 = \frac{4}{5}$, $x_2 = \frac{8}{5}$ optimal?

(b) **[1p]**

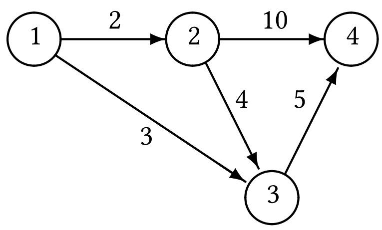
Determine values of the parameters c_1 and c_2 such that the point $x_1 = \frac{4}{5}$, $x_2 = \frac{8}{5}$ is the only optimal point.

(c) **[1p]**

Assume that the value of the right-hand-side of the constraint (4.1c) is changed from 4 to $4 + \delta$. For what values of δ is the optimal basis in (a) infeasible?

Question 5

Consider the following network, with the link capacities k_{ij} denoted on the directed links.



(a) **[1.5p]**

Give a linear optimization formulation of the problem to find the maximum total flow from node 1 to node 4, specifically for this network instance.

Note that you should *not present a general* maximum flow model.

(b) **[1.5p]**

Formulate the linear optimization dual problem of your model in (a). What does the dual problem optimize? Give an interpretation.

Solution proposals

Note that some of these solutions are quite brief and that more explanations may be needed to pass some of the (part) questions.

Solution to Question 1

(a) Let the variable x_j represent the number of points assigned to exercise $j = 1, 2, 3$. The model is then given by:

$$
\begin{aligned}
z^\star := \max \quad & 0.5x_1 + 0.4x_2 + 0.3x_3 && (5.1a)\\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 100 && (5.1b)\\
& 0.25x_1 + 0.5x_2 + 0.5x_3 \geq 40 && (5.1c)\\
& x_1 \geq 20 && (5.1d)\\
& x_2 \geq 20 && (5.1e)\\
& x_3 \geq 30 && (5.1f)\\
& (x_1, \quad x_2, \quad x_3 \geq 0) && (5.1g)
\end{aligned}
$$

(b) Transform the variables according to $v_1 = x_1 - 20$, $v_2 = x_2 - 20$, and $v_3 = x_3 - 30$. The model in (a) is then expressed as

$$
\begin{aligned}
z^\star := \max \quad & 0.5v_1 + 0.4v_2 + 0.3v_3 + 27 && (5.2a)\\
\text{s.t.} \quad & v_1 + v_2 + v_3 = 30 && (5.2b)\\
& 0.25v_1 + 0.5v_2 + 0.5v_3 \geq 10 && (5.2c)\\
& v_1, \quad v_2, \quad v_3 \geq 0 && (5.2d)
\end{aligned}
$$

The given optimal solution $(x_1^\star, x_2^\star, x_3^\star) = (40, 30, 30)$ to the model (5.1) corresponds to the optimal solution $(v_1^\star, v_2^\star, v_3^\star) = (20, 10, 0)$ to the model (5.2).

Let the LP dual variables y_1 and y_2 correspond to the constraints (5.2b) and (5.2c), respectively. The optimal basis for the model (5.2) is given by $\mathbf{v}_B = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. The corresponding optimal dual solution is given by $\mathbf{y}^\top = \mathbf{c}_B^\top B^{-1} = \begin{pmatrix} 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0.25 & 0.5 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.4 \end{pmatrix}$.

The current basis is feasible if
$B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq \mathbf{0} \iff \begin{pmatrix} 2 & -4 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 30 \\ 10 - \delta \end{pmatrix} = \begin{pmatrix} 60 - 40 + 4\delta \\ -30 + 40 - 4\delta \end{pmatrix} = \begin{pmatrix} 20 + 4\delta \\ 10 - 4\delta \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -5 \leq \delta \leq \frac{5}{2}$.

A reduction of the requirement of 40 points to 35 points corresponds to $\delta = 5$, which means that the current basis will not stay feasible. For $\delta = \frac{5}{2}$ (the largest value of δ for which the current basis is feasible) the expected average number of points will increase by $\mathbf{c}_B^\top B^{-1}\Delta\mathbf{b} = \begin{pmatrix} 0.6 & -0.4 \end{pmatrix} \begin{pmatrix} 0 \\ -\delta \end{pmatrix} = 0.4 \cdot \delta = 0.4 \cdot \frac{5}{2} = 1$ points. For $\delta = 5 > \frac{5}{2}$ the expected average number of points will then increase by at least 1 point (from the original optimal value), since the reduction from 37.5 to 35 will further enlarge the feasible set.

Solution to Question 2

(a) No, since any extreme point corresponds to (at least) one basic solution. For this system (with two equality constraints and non-negative variables), any basic solution can have at most two non-zero variable values.

(b) No, since for $x_1 = x_3 = x_5 = 0$ it must hold that $x_2 = 2$ and $x_4 = -2 \not\geq 0$, which is not feasible in (2.1c).

(c) No, since the columns $a_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $a_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are linearly dependent.

Solution to Question 3

(a)

$$\begin{aligned} y &\geq x_1 + x_2 - 1 \\ y &\leq x_1 \\ y &\leq x_2 \\ x_1, x_2, y &\in \{0, 1\} \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 + x_2 &\leq 5 + 4My \\ x_1 + 2x_2 &\leq 6 + 3My \\ 2x_1 + x_2 &\geq 18y \\ x_1 + 2x_2 &\geq 10y \\ 0 \leq x_1, x_2 &\leq M \\ y &\in \{0, 1\} \end{aligned}$$

(c)

$$\begin{aligned} x_1 + (1 - x_2) + (1 - x_3) &\geq 1 \\ x_1, x_2, x_3 &\in \{0, 1\} \end{aligned}$$

Solution to Question 4

(a) The variables x_1 and x_2 are basic variables in the optimal point, while the slack variables for the respective constraints (4.1b) and (4.1c), are non-basic. The reduced costs for the non-basic variables are given by $\bar{\mathbf{c}}_N^\top = \mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} N = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2c_1 + c_2 & c_1 - 3c_2 \end{pmatrix}$. The point $x_1 = \frac{4}{5}$, $x_2 = \frac{8}{5}$ is optimal whenever all the reduced costs of the non-basic variables are non-positive.

The required relations are thus given by the inequalities $-2c_1 + c_2 \leq 0$ and $c_1 - 3c_2 \leq 0$, or equivalently, $\frac{1}{2}c_2 \leq c_1 \leq 3c_2$.

(b) The point $x_1 = \frac{4}{5}$, $x_2 = \frac{8}{5}$ is the only optimal point when the reduced costs are both negative, i.e., when $-2c_1 + c_2 < 0$ and $c_1 - 3c_2 < 0$ hold, or equivalently, $\frac{1}{2}c_2 < c_1 < 3c_2$. E.g., $c_1 = c_2 = 1$ fulfil these strict inequalities.

(c) The optimal values of the (basic) variables x_1 and x_2 can be expressed as

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}\mathbf{b} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 4 + \delta \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 4 + \delta \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8 - 4 - \delta \\ -4 + 12 + 3\delta \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 - \delta \\ 8 + 3\delta \end{pmatrix}.$$

The basis is infeasible when either x_1 or x_2 attains a negative value, i.e., if either of the strict inequalities $4 - \delta < 0$ and $8 + 3\delta < 0$ holds. Hence, it is infeasible whenever $\delta > 4$ or $\delta < -\frac{8}{3}$ holds.

Solution to Question 5

(a)

$$
\begin{array}{lrrrrrrcl}
\max & & & & & & v & & \\
\text{s.t.} & -x_{12} & -x_{13} & & & & +v & = & 0 \\
& +x_{12} & & -x_{23} & -x_{24} & & & = & 0 \\
& & +x_{13} & +x_{23} & & -x_{34} & & = & 0 \\
& & & & +x_{24} & +x_{34} & -v & = & 0 \\
& x_{12} & & & & & & \leq & 2 \\
& & x_{13} & & & & & \leq & 3 \\
& & & x_{23} & & & & \leq & 4 \\
& & & & x_{24} & & & \leq & 10 \\
& & & & & x_{34} & & \leq & 5 \\
& x_{12}, & x_{13}, & x_{23}, & x_{24}, & x_{34} & & \geq & 0
\end{array}
$$

(b)

$$
\begin{array}{lrrrrrrrrrcl}
\min & & & & & +2\gamma_{12} & +\, 3\gamma_{13} & +\, 4\gamma_{23} & +\, 10\gamma_{24} & +\, 5\gamma_{34} & & \\
\text{s.t.} & -\pi_1 & +\pi_2 & & & +\gamma_{12} & & & & & \geq & 0 \\
& -\pi_1 & & +\pi_3 & & & +\gamma_{13} & & & & \geq & 0 \\
& & -\pi_2 & +\pi_3 & & & & +\gamma_{23} & & & \geq & 0 \\
& & -\pi_2 & & +\pi_4 & & & & +\gamma_{24} & & \geq & 0 \\
& & & -\pi_3 & +\pi_4 & & & & & +\gamma_{34} & \geq & 0 \\
& +\pi_1 & & & -\pi_4 & & & & & & = & 1 \\
& & & & & \gamma_{12}, & \gamma_{13}, & \gamma_{23}, & \gamma_{24}, & \gamma_{34} & \geq & 0
\end{array}
$$

The dual problem seeks the *minimum capacity cut*, where a cut is defined by a partition of the nodes such that the origin (here, node 1) and the destination (here, node 4) are on either side of the cut. The capacity of the cut is defined as the sum of link capacities for the links passing from the origin side to the destination side of the cut. The minimum cut can be interpreted as the smallest "bottleneck" of the network.