EXAM
MVE165/MMG631
Linear and integer optimization with applications

- **Date:** 2021-06-03
  - **Hours:** 8:30–12:30
- **Examiner:** Ann-Brith Strömberg
- **Aids:** All aids are allowed, but cooperation is not allowed
- **Number of questions:** 5
  - questions are *not* ordered by difficulty
- **Requirements**
  - To pass the exam the student must receive at least eight (8) out of fifteen (15) points (not including bonus points) and at least two passed questions
  - To pass a question requires at least two (2) points out of three (3)
  - For higher grades (i.e., 4, 5, or VG) at most two (2) bonus points can be counted towards the grade

General instructions for the exam

When answering the questions

- use generally valid theory and methodology. All theoretical results and properties used for the solutions should be properly referred to, either from the course literature or from other scientific references, such as scientific textbooks and scientific journal articles;
- state your methodology carefully;
- when reporting numerical calculations, clearly write down a reasonable number of steps so that your understanding can be judged;
- do not use a red pen;
- do not answer more than one question per sheet.

Question 1

A certain type of product is assembled in two stages, A and B, in a factory. For stage A, three machines can be used with a maximum capacity of K_1, K_2, and K_3 units per day, respectively. For stage B, there are two identical machines, each with a maximum capacity of U units per day. The demand for products is D items per day. Due to the placement of the machines in the factory, it is differently complicated to combine the individual machines in stages A and B. Hence, there is a cost assigned to each combination of machines in the two stages, according to the following table.

	machine B_1	machine B_2
machine A_1	1	2
machine A_2	3	2
machine A_3	1	3

(a) **[2p]**

Formulate the problem to fulfill the demand of products, while the capacity of each of the machines is respected, at the lowest possible total cost, as an integer linear optimization problem. Draw an illustration of the model.

(b) **[1p]**

What constraints on the values of the parameters given in the description above (i.e., K_1, K_2, K_3, U, and D) must hold for a feasible solution to exist to the problem stated?

Question 2

(a) **[2p]**

Solve the linear optimization problem to

$$\begin{array}{lrcrcl} \text{maximize} & z = 3x_1 & + & x_2, & & \\ \text{subject to} & x_1 & - & x_2 & \leq & 2, \\ & x_1 & + & 2x_2 & \leq & 9, \\ & x_1, & & x_2 & \geq & 0, \end{array}$$

using the simplex algorithm. Start in the origin of the (x_1, x_2)-space. Illustrate the optimization problem with a figure in the (x_1, x_2)-plane. Show the iterative sequence generated from the simplex algorithm in the figure.

(b) **[1p]**

Add the constraints that x_1 and x_2 must be integers. Use a suitable row from the optimal simplex tableau to generate a Gomory cut. Does the cut define a *facet* of the convex hull of the set of feasible integer solutions? Motivate your answer.

Question 3

Consider the linear optimization problem

$$
\begin{array}{llcccccr}
z^* := \max & 2x_1 & + & 3x_2, & & & & \text{(3.1a)} \\
\text{s.t.} & x_1 & + & x_2 & \leq & 5, & & \text{(3.1b)} \\
 & 2x_1 & + & 5x_2 & \leq & 20, & & \text{(3.1c)} \\
 & x_1, & & x_2 & \geq & 0. & & \text{(3.1d)}
\end{array}
$$

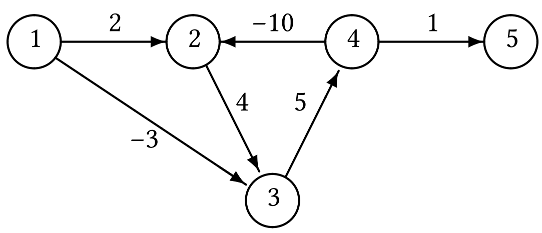
(a) **[1p]**
Compute the shadow price (i.e., the optimal dual variable value) for each of the two inequality constraints (3.1b) and (3.1c). [*Hint: an optimal primal solution can be found graphically.*]

(b) **[1p]**
For what values of the right-hand-side of the constraint (3.1b) is its shadow price valid?

(c) **[1p]**
Assume that a new variable, $x_3 \geq 0$, is included in the problem (3.1), with the constraint coefficients (column) $A_3 = (2, 2)^\top$. For what values of its objective coefficient c_3, does it hold that its optimal value $x_3^* > 0$?

Question 4

Consider the following network, with distances d_{ij} on the directed arcs



(a) **[1p]**
Provide a linear optimization formulation of the problem to find a shortest path from node 1 to node 5.

(b) **[2p]**
Use the linear optimization dual of your model in (a) to show that the length of the shortest path from node 1 to node 5 is $-\infty$. Motivate all statements by theory from the course.

Question 5

[3p]

Solve the binary knapsack problem

$$\begin{array}{rrl} z^* := & \max & 18x_1 + 14x_2 + 8x_3 + 4x_4\,, \\ & \text{subject to} & 15x_1 + 12x_2 + 7x_3 + 4x_4 \leq 23\,, \\ & & x_1,\quad x_2,\quad x_3,\quad x_4 \in \{0,1\}, \end{array}$$

to optimality using the branch–and–bound method. State carefully the relaxation made and the resulting subproblem(s). Use a depth-first strategy and always search the 1-branch first. You should also indicate the order in which the node problems of the search tree are solved.

For *each node* in the search tree, you shall indicate

(i) the solution to the node subproblem,

(ii) if the branch should be cut in that node, and argue why it should or should not be cut,

(iii) any computed bound(s) on the optimal value z^*, and

(iv) in what part of the search tree each bound computed is valid.

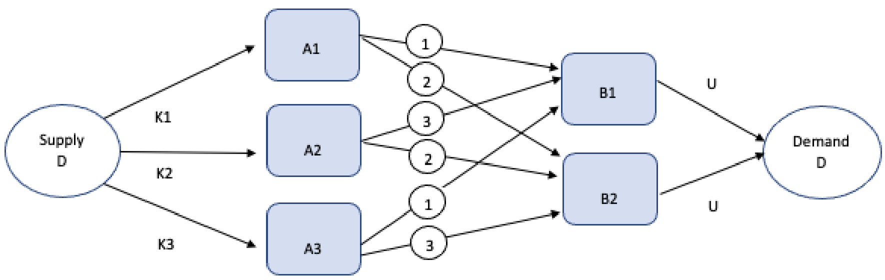
Solution proposals

Note that some of these solutions are quite brief and that more explanations may be needed to pass some of the (part) questions.

Solution to Question 1

(a) Let x_{ij} = the number of products assembled first in machine A_i and then in B_j, $i = 1,2,3$, $j = 1,2$.

$$
\begin{array}{llcccccccl}
\text{minimize } z = & 1x_{11} + 2x_{12} + 3x_{21} + 2x_{22} + 1x_{31} + 3x_{32} \\
\text{subject to} & x_{11} + x_{12} & & & & & & \leq K_1 \\
& & x_{21} + x_{22} & & & & & \leq K_2 \\
& & & x_{31} + x_{32} & & & & \leq K_3 \\
& x_{11} & + x_{21} & + x_{31} & & & & \leq U \\
& x_{12} & + x_{22} & + x_{32} & & & & \leq U \\
& x_{11} + x_{12} + x_{21} + x_{22} + x_{31} + x_{32} & & & & & & \geq D \\
& x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32} & & & & & & \geq 0 \text{ and integer}
\end{array}
$$



(b) The capacity of the machines must be at least as high as the demand for a feasible solution to exists. Thus,

- $K_1 + K_2 + K_3 \geq D$ and
- $2U \geq D$

must hold. (Or rather, since the variables are integer, $\lfloor K_1 \rfloor + \lfloor K_2 \rfloor + \lfloor K_3 \rfloor \geq \lceil D \rceil$ and $2\lfloor U \rfloor \geq \lceil D \rceil$.)

Solution to Question 2

(a) Let $\hat{z} := -z = -\mathbf{c}^\top \mathbf{x}$ and write the problem on standard form:

$$
\begin{array}{lrl}
\text{minimize} & \hat{z} = -3x_1 - x_2, & \\
\text{subject to} & x_1 - x_2 + s_1 & = 2, \\
& x_1 + 2x_2 + s_2 & = 9, \\
& x_1,\ x_2,\ s_1,\ s_2 & \geq 0.
\end{array}
$$

This gives the following simplex tableau (note that we start in the origin of the (x_1, x_2)-space):

	$-\hat{z}$	x_1	x_2	s_1	s_2	$\bar{b}$
$-\hat{z}$	1	−3	−1	0	0	0
s_1	0	1	−1	1	0	2
s_2	0	1	2	0	1	9

with $\mathbf{x}^0 = (0,0)^\top$.

Iteration 1

Entering variable: $\min\{-3,-1\} = -3 \implies x_1$ enters

Leaving variable: $\min\{\frac{2}{1}, \frac{9}{1}\} = 2 \implies s_1$ leaves

New tableau:

	$-\hat{z}$	x_1	x_2	s_1	s_2	$\bar{b}$
$-\hat{z}$	1	0	-4	3	0	6
x_1	0	1	-1	1	0	2
s_2	0	0	3	-1	1	7

with $\mathbf{x}^1 = (2,0)^\top$.

Iteration 2

Entering variable: $\min\{-4\} = -4 \implies x_2$ enters

Leaving variable: $\min\{\frac{7}{3}\} = \frac{7}{3} \implies s_2$ leaves

New tableau:

	$-\hat{z}$	x_1	x_2	s_1	s_2	$\bar{b}$
$-\hat{z}$	1	0	0	5/3	4/3	46/3
x_1	0	1	0	2/3	1/3	13/3
x_2	0	0	1	$-1/3$	1/3	7/3

with $\mathbf{x}^2 = \frac{1}{3}(13,7)^\top$.

All reduced costs are nonnegative $\implies$ stop!

Optimal solution: $\mathbf{x}^* = \frac{1}{3}(13,7)^\top$ with $\hat{z}^* = -\frac{46}{3}$ $\quad$ ($z^* = \frac{46}{3}$ in the original problem).

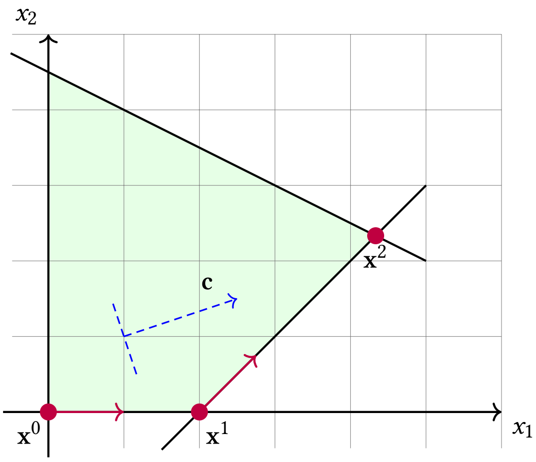


Figure 1: The iterative sequence from the simplex algorithm in the (x_1, x_2)-plane.

(b) **Gomory cut using the x_1-row:**

$\lfloor 1 \rfloor\, x_1 + \left\lfloor \frac{2}{3} \right\rfloor s_1 + \left\lfloor \frac{1}{3} \right\rfloor s_2 \le \left\lfloor 4\frac{1}{3} \right\rfloor$

$\implies x_1 \le 4$

Gomory cut using the x_2-row:

$\lfloor 1 \rfloor\, x_2 + \left\lfloor -\frac{1}{3} \right\rfloor s_1 + \left\lfloor \frac{1}{3} \right\rfloor s_2 \le \left\lfloor 2\frac{1}{3} \right\rfloor$

$$\implies x_2 - s_1 \le 2 \implies [s_1 = 2 - x_1 + x_2] \implies x_2 - 2 + x_1 - x_2 \le 2 \implies x_1 \le 4$$

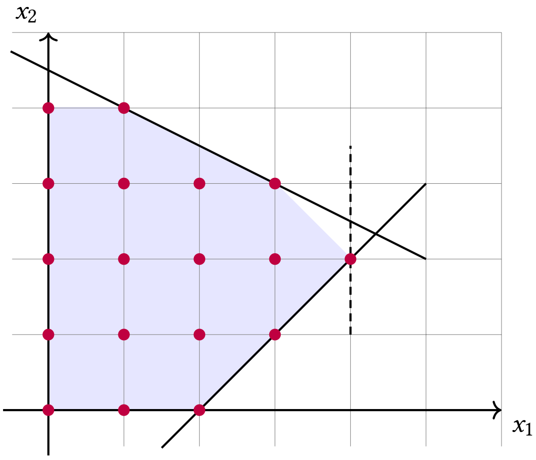


Figure 2: Gomory cut and the convex hull of the set of feasible integer solutions

The convex hull of the set of feasible integer solutions is defined by the blue area in Figure 2, to which *the added cut is not a facet.*

Solution to Question 3

(a) Solving the primal problem graphically gives the optimal solution $\mathbf{x}^* = \frac{1}{3}\begin{pmatrix}5\\10\end{pmatrix}$ with the optimal basis $(x_1, x_2)^\top$ and $\mathbf{B} = \begin{pmatrix}1 & 1\\ 2 & 5\end{pmatrix}$.

Shadow price = Optimal dual solution: $\mathbf{y}^\top = \mathbf{c}_B^\top \mathbf{B}^{-1} = \frac{1}{3}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}5 & -1\\ -2 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}4 & 1\end{pmatrix}$

(b) Increase the RHS of the constraint (3.1b) by δ: $5 + \delta$.
Compute $\mathbf{B}^{-1}(\mathbf{b} + \binom{\delta}{0}) = \frac{1}{3}\begin{pmatrix}5 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}5+\delta\\ 20\end{pmatrix} = \frac{1}{3}\begin{pmatrix}5+5\delta\\ 10-2\delta\end{pmatrix} \ge \begin{pmatrix}0\\0\end{pmatrix} \iff -1 \le \delta \le 5$

(c) $x_3^* > 0$ if the reduced cost $\bar{c}_3 > 0$:

$$\bar{c}_3 = c_3 - \mathbf{c}_B^\top \mathbf{B}^{-1} A_3 = c_3 - \frac{1}{3}\begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = c_3 - \frac{10}{3} > 0 \iff c_3 > \frac{10}{3}.$$

Solution to Question 4

(a) LP model:

$$\begin{array}{lrrrrrrcr}
\min & 2x_{12} & -3x_{13} & +4x_{23} & +5x_{34} & -10x_{42} & +1x_{45} & & \\
\text{st} & -x_{12} & -x_{13} & & & & & = & -1 \\
& +x_{12} & & -x_{23} & & +x_{42} & & = & 0 \\
& & +x_{13} & +x_{23} & -x_{34} & & & = & 0 \\
& & & & +x_{34} & -x_{42} & -x_{45} & = & 0 \\
& & & & & & +x_{45} & = & 1 \\
& x_{12}, & x_{13}, & x_{23}, & x_{34}, & x_{42}, & x_{45} & \ge & 0
\end{array}$$

(b) LP dual:

$$
\begin{array}{rrrrrrrrl}
\max & -y_1 & & & & +y_5 & & & \\
\text{st} & -y_1 & +y_2 & & & & \leq & 2 & \\
 & -y_1 & & +y_3 & & & \leq & -3 & \\
 & & -y_2 & +y_3 & & & \leq & 4 & \text{(i)} \\
 & & & -y_3 & +y_4 & & \leq & 5 & \text{(ii)} \\
 & & +y_2 & & -y_4 & & \leq & -10 & \text{(iii)} \\
 & & & & -y_4 & +y_5 & \leq & 1 &
\end{array}
$$

Sum the left- and right-hand sides, respectively, of the constraints (i), (ii), and (iii):

$$
\begin{aligned}
& -y_2 + y_3 \leq 4;\ -y_3 + y_4 \leq 5;\ y_2 - y_4 \leq -10 \\
\Longrightarrow \quad & -y_2 + y_3 - y_3 + y_4 + y_2 - y_4 \leq 4 + 5 - 10 \quad \Longleftrightarrow \quad 0 \leq -1,
\end{aligned}
$$

which is a contradiction. Hence, the LP dual has no feasible solution. From duality theory (the course book, Table 6.2 on page 141) then the primal problem, either has no feasible solution, or is unbounded. Since the path 1–3–4–5, corresponding to the solution point given by $x_{13} = x_{34} = x_{45} = 1$; $x_{12} = x_{23} = x_{42} = 0$, is feasible in the model in (a) we conclude that the primal problem is unbounded, such that the objective value tends to $-\infty$ as the values of the variables x_{23}, x_{34}, and x_{42} tend to infinity.

Specifically, any solution given by

$$x_{12} = 0;\ x_{13} = x_{45} = 1;\ x_{34} = 1 + t;\ x_{23} = x_{42} = t$$

is feasible for all $t \geq 0$, with objective value

$$2x_{12} - 3x_{13} + 4x_{23} + 5x_{34} - 10x_{42} + 1x_{45} = 0 - 3 + 4t + 5(1 + t) - 10t + 1 = 3 - t \longrightarrow -\infty$$

as $t \longrightarrow \infty$.

Solution to Question 5

Relax the integrality (binary) constraints, such that each node problem becomes an LP problem, specifically a continuous knapsack problem:

$$
\begin{array}{rrl}
z^0_{\text{LP}} = & \max & 18x_1 + 14x_2 + 8x_3 + 4x_4\ , \\
 & \text{subject to} & 15x_1 + 12x_2 + 7x_3 + 4x_4 \leq 23\ , \\
 & & x_1,\quad x_2,\quad x_3,\quad x_4 \in [0, 1].
\end{array}
$$

This relaxed problem is solved to optimality by setting each variable to 1, as long as there is "room left in the knapsack"; the last non-zero variable may get a fractional value. The order in which the variables are chosen is given by decreasing values of the ratios $\frac{c_j}{a_j}$, $j = 1, \ldots, 4$: $\left(\frac{18}{15}, \frac{14}{12}, \frac{8}{7}, \frac{4}{4}\right) \approx (1.2, 1.16, 1.14, 1)$. Hence, the order x_1, x_2, x_3, x_4 is used for solving the continuous knapsack problems.

The search order is given by the node numbering $P_0, P_1, \ldots, P_6$ (see the illustration below). The solutions to the node problems are given by the following:

$$
\begin{array}{llll}
P_0 : & \mathbf{x}_{\mathrm{LP}}^0 = \left(1, \frac{2}{3}, 0, 0\right); & z_{\mathrm{LP}}^0 = 18 \cdot 1 + 14 \cdot \frac{2}{3} = 27 + \frac{1}{3}; & \bar{z}^0 = \lfloor z_{\mathrm{LP}}^0 \rfloor = 27 \\
P_1\,(x_2 = 1) : & \mathbf{x}_{\mathrm{LP}}^1 = \left(\frac{11}{15}, 1, 0, 0\right); & z_{\mathrm{LP}}^1 = \frac{66}{5} + 14 = 27 + \frac{1}{5}; & \bar{z}^1 = \lfloor z_{\mathrm{LP}}^1 \rfloor = 27 \\
P_2\,(x_2 = x_1 = 1) : & \text{infeasible} & & \\
P_3\,(x_2 = 1; x_1 = 0) : & \mathbf{x}_{\mathrm{LP}}^3 = (0, 1, 1, 1)\,; \text{integer;} & z_{\mathrm{LP}}^3 = 14 + 8 + 4 = 26; & \underline{z} = z_{\mathrm{LP}}^3 = 26 \\
P_4\,(x_2 = 0) : & \mathbf{x}_{\mathrm{LP}}^4 = \left(1, 0, 1, \frac{1}{4}\right); & z_{\mathrm{LP}}^4 = 18 + 8 + 4 \cdot \frac{1}{4} = 27; & \bar{z}^4 = \lfloor z_{\mathrm{LP}}^4 \rfloor = 27 \\
P_5\,(x_2 = 0; x_4 = 1) : & \mathbf{x}_{\mathrm{LP}}^5 = \left(1, 0, \frac{\boxed{4}}{7}, 1\right); & z_{\mathrm{LP}}^5 = 18 + 8 \cdot \frac{\boxed{4}}{7} + 4 = \boxed{26} + \frac{\boxed{4}}{7}; & \bar{z}^5 = \lfloor z_{\mathrm{LP}}^5 \rfloor = \boxed{26 =}\ \underline{z} = 26 \\
P_6\,(x_2 = x_4 = 0) : & \mathbf{x}_{\mathrm{LP}}^6 = (1, 0, 1, 0)\,; \text{integer;} & z_{\mathrm{LP}}^6 = 18 + 8 = 26; & \underline{z} = \max\{z_{\mathrm{LP}}^3, z_{\mathrm{LP}}^6\} = 26
\end{array}
$$

The upper bounds z_{LP}^0, z_{LP}^1, z_{LP}^4, and z_{LP}^5 are valid in their respective subtrees.

The lower bound $\underline{z}$ is valid in the whole search tree.

The node P_2 is cut since setting $x_2 = x_1 = 1$ leads to infeasibility in the constraints.

The nodes P_3 and P_6 are cut since the solutions to the corresponding node problems are integral.

The node P_5 is cut, since the upper bound valid in the node (and its subtree) is lower than the computed global lower bound.

There are two optimal solutions, given by $\mathbf{x}_{\mathrm{LP}}^3 = (0, 1, 1, 1)$ and $\mathbf{x}_{\mathrm{LP}}^6 = (1, 0, 1, 0)$, with the optimal value $z_{\mathrm{LP}}^3 = z_{\mathrm{LP}}^6 = 26$.