Ordinary Differential Equations and mathematical modelling, MVE162/MMG511
August 22, 2025, 8.30-12.30

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Make sure that your name, personal identity number and anonymization code are *neatly* written on the cover sheet.

Please sort your solutions in the correct order before you hand in!.

Ensure your solutions are written in a clear, concise, and readable manner. When using results from the course, clearly state which ones you are applying and make sure that the conditions of any theorems or lemmas you use are satisfied in the context of the problem. You may use standard results — such as common limits, integrals, derivatives, and Taylor expansions — without citing them.

50 points, 6 problems (No aids permitted except a standard dictionary (english to any language))

Grades (Chalmers): 3 (20-34 points), 4 (35-43 poäng), 5 (44-50 poäng).

Grades (GU): G (20-37 points), VG (38-50 points).

You can write your solutions either in Swedish or English.

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1. Let $\mathrm{Mat}_2(\mathbb{R})$ denote the real vector space of real 2×2-matrices, equipped with the norm
$$N(A) = \mathrm{tr}(AA^{\mathrm{tr}})^{1/2}, \quad A \in \mathrm{Mat}_2(\mathbb{R}),$$
where A^{tr} denotes the transpose of A. Let
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$
Find a positive real number μ such that the limit
$$L := \lim_{t \to +\infty} e^{-\mu t} N(\exp(tA))$$
exists and is positive (and finite). (10p)

2. Let $\Phi : \mathbb{R} \to \mathrm{Mat}_2(\mathbb{R})$ denote the solution of the linear system

$$\Phi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Phi'(t) = \begin{pmatrix} \cos(t) & \frac{1}{1+t^4} \\ -\frac{1}{e^t+t^2} & \cos(t) \end{pmatrix} \Phi(t), \quad t > 0.$$

Show that there exists $t_o > 0$ such that

$$\det(\Phi(t_o)) = e^2,$$

and determine the smallest such t_o. (10p)

3. Determine whether the origin is stable/asymptotically stable or unstable for the differential equation $x' = f(x)$, where

$$f(x) = (x_2 \sin(x_1) + x_1 \sin(x_2), \ln(1 + x_2) + \sin(x_2^2)).$$

(8p)

4. Consider the differential equation

$$x' = -y(1 + xy), \quad y' = 2x.$$

Show that $(0, 0)$ is an asymptotically stable critical point. HINT: *The function* $V(x, y) = 2x^2 + y^2$ *might be useful to study.* (10p)

5. Examine the stability of the origin for the system

$$x' = x^2 - 4xy + y^2, \quad y' = -10y + x^2y^2 + x^5.$$

(8p)

6. Consider the differential equation

$$\begin{cases} \dot{x} = y - x(1 - x^2 - y^2) \\ \dot{y} = -x - y(1 - x^2 - y^2) \end{cases}$$

Find a periodic solution which is not a fixed point and determine its period.

HINT: Rewrite the differential equation in polar coordinates. (4p)

1.) $\underbrace{A \text{ symmetric}, \det A = 1, \operatorname{tr} A = 3}$ $\Rightarrow$ $A = P^{-1} D P$, $P^{-1} = P^{tr}$, $D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & 1/\lambda_+ \end{pmatrix}$

$$\lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 1} = \frac{3 \pm \sqrt{5}}{2} \qquad (\lambda_+ \lambda_- = 1) \quad \lambda_+ > \lambda_-$$

$\rightsquigarrow \exp(tA) = P^{tr} \cdot \exp(tD) \cdot P$

$\rightsquigarrow N(\exp(tA))^2 = \operatorname{tr}(P^{tr} \exp(tD) \underbrace{P P^t}_{= I} \exp(tD) P) = \operatorname{tr}(\underbrace{PP^{tr}}_{= I} \cdot \exp(2tD))$

$= e^{2t\lambda_+} + e^{2t/\lambda_+}$

$\rightsquigarrow e^{-t\lambda_+} N(\exp(tA)) = e^{-t\lambda_+} (e^{2t\lambda_+} + e^{2t/\lambda_+})^{1/2}$

$= (1 + e^{2t \underbrace{(1/\lambda_+ - \lambda_+)}_{<0}})^{1/2} \longrightarrow 1, \quad t \to +\infty.$

$\rightsquigarrow$ $\mu = \lambda_+ = \frac{3+\sqrt{5}}{2}$, $L = 1$

2.) Liouville's formula: $\det \phi(t) = \underbrace{\det(\phi(0))}_{= 1} \cdot \exp\Big(\underbrace{\int_0^t \operatorname{tr}(A(s))\, ds}_{= 2 \cdot \int_0^t \cos(s)\, ds \; = 2\sin(t)}\Big)$

$\underbrace{e^{2\sin(t_0)} = e^2}_{\text{smallest } t_0 > 0} \Rightarrow t_0 = \pi/2$

$(\Leftrightarrow \sin(t_0) = 1) \rightsquigarrow t_0 = \pi/2$

3.) $Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow$ Eigenvals $= 0$ and 1 pos.

$\Rightarrow$ (0,0) unstable!

4.) $\nabla V(x,y) \cdot f(x,y) = (4x, 2y) \cdot (-y(1+xy), 2x)$

$= -4xy(1+xy) + 4xy = -4(xy)^2 \leq 0$

$M = \{(x,y) : \nabla V(x,y) \cdot f(x,y) = 0\} = \{(x,y) : xy = 0\}$

$= \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$

sps. solution starts at $(x_0, 0) \rightsquigarrow y'(0) = 2x_0 \neq 0 \Rightarrow y(t) \neq 0 \quad t \approx 0$

$(0, y_0) \rightsquigarrow x'(0) = -y_0 \neq 0 \Rightarrow x(t) \neq 0 \quad t \approx 0$

$\Rightarrow \{(0,0)\}$ only invariant subset of M

Lasalle $\Rightarrow$ (0,0) asymptotically stable.

5.)

$$x' = \overset{=\,A\,\leq\,0}{0} \cdot x + f(x,y) \qquad f(x,y) = x^2 - 4xy + y^2$$

$$y' = \underbrace{-10}_{=\,B\,<\,0}\, y + g(x,y) \qquad g(x,y) = x^2 y^2 + x^5$$

satisfy: $f(0,0) = g(0,0) = 0$

$\nabla f(0,0) = \nabla g(0,0) = 0$

$$M[\psi](x) = \nabla \psi(x) \cdot \big(0 \cdot x + f(x, \psi(x))\big) - \big(-10 \cdot \psi(x) + g(x, \psi(x))\big)$$

Test: $\psi(x) = 0 \Rightarrow M[\psi](x) = O(x^5)$

$\Rightarrow \exists$ centre-manifold φ s.t. $\varphi(x) = O(x^5)$

$$\Rightarrow x' = x^2 - 4x \cdot \varphi(x) + \varphi(x)^2 = x^2 + O(x^6) = \underbrace{x^2}_{>0}\big(1 + O(x^4)\big)$$

$\Rightarrow$ $(0,0)$ unstable

6.) Same as Exam 1
