Ordinary Differential Equations and mathematical modelling, MVE162/MMG511
June 2, 2025, 8.30-12.30

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Make sure that your name, personal identity number and anonymization code are *neatly* written on the cover sheet.

Please sort your solutions in the correct order before you hand in!.

Ensure your solutions are written in a clear, concise, and readable manner. When using results from the course, clearly state which ones you are applying and make sure that the conditions of any theorems or lemmas you use are satisfied in the context of the problem. You may use standard results — such as common limits, integrals, derivatives, and Taylor expansions — without citing them.

50 points, 6 problems (No aids permitted except a standard dictionary (english to any language))

Grades (Chalmers): 3 (20-34 points), 4 (35-43 poäng), 5 (44-50 poäng).

Grades (GU): G (20-37 points), VG (38-50 points).

You can write your solutions either in Swedish or English.

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1. Compute $\exp(A)$ for
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$
Hint: Write $A = 2I + N$, where $N^3 = 0$. (7p)

2. Let $a, b : \mathbb{R} \to \mathbb{R}$ be two strictly positive thrice continuously differentiable functions and define $A : \mathbb{R} \to \mathrm{Mat}_2(\mathbb{R})$ by
$$A(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & -a(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Let $\Phi : \mathbb{R} \to \mathrm{Mat}_2(\mathbb{R})$ denote the fundamental matrix, i.e. the solution to the linear differential equation

$$\Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi'(t) = A(t)\Phi(t), \quad t \in \mathbb{R}.$$

a) Show that $\det(\Phi(t)) = 1$ for all $t \in \mathbb{R}$. (5p)

b) Define $s(t) = \mathrm{tr}(\Phi(t))$ for $t \in \mathbb{R}$. Show that

$$s(0) = 2, \quad s'(0) = 0, \quad s''(0) = 2(a(0)^2 + b(0)^2) > 0,$$

and deduce that there exists $t_o > 0$ such that $s(t) > 2$ for all $t \in (0, t_o)$. (7p)

HINT: Consider the Taylor approximation of s of order 2.

c) Show that $\Phi(t)$ has two distinct real eigenvalues (no double roots) for all $t \in (0, t_o)$. *You are free to use the properties of s in b) even if you have not proven them.* (3p)

3. Consider the differential equation

$$x' = -x + y, \quad y' = -x - y + x^2.$$

Find all of the critical points and determine whether they are asymptotically stable, stable or unstable. (8p)

4. Consider the differential equation $x' = f(x)$, where

$$x' = 2y, \quad y' = -2x - (1 - x^2)y.$$

Show that the origin is asymptotically stable. *Hint: It might be useful to study the function $L(x, y) = x^2 + y^2$.* (8p)

5. Consider the differential equation

$$x' = -x^3 + y^2, \quad y' = -2y + x^2.$$

Show that the origin is an asymptotically stable point. (8p)

HINT: Find a center manifold.

6. Consider the differential equation

$$\begin{cases} \dot{x} = y - x(1 - x^2 - y^2) \\ \dot{y} = -x - y(1 - x^2 - y^2) \end{cases}$$

Find a periodic solution which is not a fixed point and determine its period.

HINT: Rewrite the differential equation in polar coordinates. (4p)

4. $L(x,y) \geq 0$, $L(x,y) = 0 \Leftrightarrow x = y = 0$

$\mathring{L}(x,y) = 2x \cdot (2y) + 2y \cdot (-2x - (1-x^2)y) = -2(1-x^2)y^2 \leq 0 \quad \forall |x| \leq 1 \ \forall y$

$\hookrightarrow$ $(0,0)$ stable.

$E = \{(x,y) : \mathring{L}(x,y) = 0\} = \{(x,0) : x \in \mathbb{R}\}$

Pick $(x,0) \in E$, $x \neq 0$ $\rightsquigarrow$ $(x(t), y(t))$ solution with $x(0) = x$, $y(0) = 0$.

$\rightsquigarrow$ $y'(0) = -2x \neq 0$ $\rightsquigarrow$ $y(t) \neq 0$ for small $t > 0$.

$\rightsquigarrow$ $(0,0)$ only invariant set in E $\rightsquigarrow$ (LaSalle's Invariance principle) $(0,0)$ is asymptotically stable

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5. $M[\psi](x) = \psi'(x) \cdot (-x^3 + \psi(x)^2) - (-2\psi(x) + x^2)$

Try $\psi(x) = 0$ $\rightsquigarrow$ $M[\psi](x) = -x^3 - x^2 = O(x^2)$

$\Rightarrow$ $\exists$ center manifold $\varphi(x) = O(x^2)$

Reduced equation: $x' = -x^3 + O(x^2)^2 = -x^3 + O(x^4) = -x^3(1 + O(x))$

$\Rightarrow$ Asymptotic stable.

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6. $r^2 = x^2 + y^2 \Rightarrow 2\dot{r}r = 2x \cdot (y - x(1-r^2)) + 2y \cdot (-x - y(1-r^2)) = -2r^2(1-r^2)$

$\Rightarrow \dot{r} = -r(1-r^2)$ $\rightsquigarrow$ $r = 0$ (critical point), $\underline{r = 1}$.

$\tan(\theta) = y/x \Rightarrow \dot{\theta}(1 + \tan^2(\theta)) = \dfrac{x \cdot \dot{y} - \dot{x} y}{x^2} = \dfrac{x(-x - y(1-r^2)) - y(y - x(1-r^2))}{x^2}$

$= -\dfrac{(x^2+y^2)}{x^2} = -(1 + \tan^2\theta)$

$\Rightarrow \dot{\theta} = -1$

$\Rightarrow$ $(x(t), y(t)) = (\cos t, -\sin t)$ $\quad$ $\underline{2\pi\text{-periodic}}$ solution.

Check: $\left.\begin{array}{l} \dot{x}(t) = -\sin t = y(t) - 0 \\ \dot{y}(t) = -\cos t = -x(t) - 0 \end{array}\right\}$ ok!

Solutions Exam 1/2025 (MVE162/MMG511)

1. $A = 2I + N$, $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $N^3 = 0$.

$(2I)\cdot N = N \cdot (2I)$

$$\Rightarrow \exp(A) = \exp(2I)\cdot \exp(N) = e^2 \cdot \left(I + N + \tfrac{1}{2}N^2\right) = e^2 \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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2. a) Liouville's formula: $\det(\Phi(t)) = e^{\int_0^t \operatorname{tr}(A(s))\,ds} = \{\operatorname{tr}(A(s)) = 0 \ \forall s\} = 1$.

b) $s(t) = \operatorname{tr}(\phi(t)) \Rightarrow s'(t) = \operatorname{tr}(\phi'(t)) = \operatorname{tr}(A(t)\phi(t))$

$\Rightarrow s''(t) = \operatorname{tr}(A'(t)\phi(t) + A(t)\phi'(t)) = \operatorname{tr}(A'(t)\phi(t) + A(t)^2\phi(t))$

$\Rightarrow s'''(t) = \operatorname{tr}\left(A''(t)\phi(t) + A'(t)A(t)\phi(t) + (A'(t)A(t) + A(t)A'(t))\phi(t) + A(t)^2\phi'(t)\right)$

$\rightsquigarrow s(0) = \operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$, $s'(0) = \operatorname{tr}(A(0)) = 0$, $s''(0) = \operatorname{tr}(A'(0)) + \operatorname{tr}(A(0)^2) = \operatorname{tr}(A(0)^2)$

<u>Note</u>: $A(0)^2 = \begin{pmatrix} a(0) & b(0) \\ b(0) & -a(0) \end{pmatrix}\begin{pmatrix} a(0) & b(0) \\ b(0) & -a(0) \end{pmatrix} = \begin{pmatrix} a(0)^2 + b(0)^2 & * \\ * & a(0)^2 + b(0)^2 \end{pmatrix}$

$\Rightarrow s''(0) = 2\left(a(0)^2 + b(0)^2\right) =: 2c > 0$ (since $a(0), b(0) > 0$)

Since a, b 2-times continuously diff, s''' continuous, hence $|s'''(t)| \leq M \ \forall |t| \leq 1$

<u>Taylor's formula</u>: $\left|s(t) - \underbrace{\left(s(0) + s'(0)t + s''(0)\tfrac{t^2}{2}\right)}_{= 2 + c\cdot t^2/2}\right| \leq M t^3/6 \quad \forall |t| \leq 1$

$\Rightarrow s(t) \geq 2 + \underbrace{ct^2/2 - Mt^3/6}_{= \frac{t^2}{2}(c - Mt/3) > 0 \quad \forall t \in (0, 3c/M)} > 2 \quad \forall t \in (0, 3c/M)$.

c) Characteristic polynomial for $\phi(t)$: $\lambda^2 - s(t)\lambda + 1 = 0$

$\Rightarrow \lambda_\pm = \frac{s(t)}{2} \pm \left(\frac{s(t)^2 - 4}{2}\right)^{1/2}$, $\left(\frac{s(t)^2 - 4}{2}\right) > 0 \quad \forall t \in (0, 3c/M) \rightsquigarrow$ no double root.

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3. <u>Critical points</u>: $x = y$, $-2x + x^2 = x(x-2) = 0 \Rightarrow p = (0,0)$, $q = (2,2)$

<u>Vector field</u>: $f(x,y) = (-x + y, -x - y + x^2)$

$\Rightarrow Df(x,y) = \begin{pmatrix} -1 & 1 \\ -1+2x & -1 \end{pmatrix} \Rightarrow Df(p) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $Df(q) = \begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$

Eigenvalues $= -1 < 0$
$\rightsquigarrow$ Asymptotically stable

$\lambda^2 + 2\lambda - 2 = 0$

$\lambda_\pm = -1 \pm \sqrt{1+2}$

$= -1 \pm \sqrt{3}$

$\lambda_+ = -1 + \sqrt{3} > 0$

$\rightsquigarrow$ Unstable