MATEMATIK
GU, Chalmers
A.Heintz

Datum: 2024-01-05 Tid: förmiddag 08:30 - 12:30
Hjälpmedel: - Inga
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Tenta i ODE och matematisk modellering, MMG511, MVE162

1. Formulate and prove the theorem about the dimension of the solution space to linear autonomous systems of ODEs. **(4p)**

2. Give definition of a stable equilibrium point to an ODE. Formulate and give a proof to Lyapunov's criterion for stability of equilibrium points to ODEs. **(4p)**

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 3 & 0 & 4 \\ -3 & 1 & -7 \\ -2 & 0 & -3 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are unbounded. **(4p)**

4. Give definition of the monodromy matrix for the linear system $\mathbf{x}' = A(t)\mathbf{x}(t)$ with periodic matrix $A(t+p) = A(t)$.

   Formulate criterion for the boundedness of all solutions to a periodic linear system of ODEs. **(1p)**

   Find the monodromy matrix (scalar here) for the following linear equation with periodic coefficients
   $$x' = \left(a + \sin^3 t\right) x$$
   Find for which real values of parameter a all solutions are bounded. **(3p)**

5. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction.
   $$\begin{cases} x' = & 3y \\ y' = & -x - (4 - x^2)y \end{cases}$$ **(4p)**

6. Show that the following system of ODEs has a periodic solution.
   $$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases}$$ **(4p)**

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points; for Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points.

One must pass both the home assignments and the exam to pass the course.

Total points for the course are calculated as:

$Total = 0.16\, Assignment1 + 0.16\, Assignment2 + 0.68\, Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

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1. Formulate and prove the theorem about the dimension of the solution space to linear systems of ODEs. **(4p)**

Consider

$$x'(t) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R} \tag{1}$$

where A is a constant $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$. In particular we will find solutions to initial value problem (I.V.P.) with initial condition

$$x(\tau) = \xi, \tag{2}$$

We make two simple observations that are valid even for general non-autonomous linear systems with a matrix $A(t)$ that is not constant but is a continuous

function of time on the interval J.

$$x'(t) = A(t)x(t), \quad x(t) \in \mathbb{R}^n, \quad t \in J \tag{3}$$

Lemma. The sets of solutions $\mathcal{S}_{\text{hom}}$ to (1), and to (3) are linear vector spaces. **Proof.** $\mathcal{S}_{\text{hom}}$ includes zero constant vector and is therefore not empty.

By the linearity of the time derivative $x'(t)$ and of the matrix multiplication $A(t)x(t)$, for a pair of solutions $x(t)$ and $y(t)$ their sum $x(t) + y(t)$ and the product $Cx(t)$ with a constant C are also solutions to the same equation:

$$\begin{aligned} (x(t) + y(t))' &= A(t)(x(t) + y(t)) \\ (Cx(t))' &= A(t)(Cx(t)) \end{aligned}$$

Theorem. The dimension of the space $\mathcal{S}_{\text{hom}}$ of solutions. (Proposition 2.7, p.30, L.R. in the case of non-autonomous systems). Let $b_1, ..., b_N$ be a basis in

$\mathbb{R}^N$(or $\mathbb{C}^N$). Then the functions $y_j : \mathbb{R} \to \mathbb{R}^N$(or $\mathbb{C}^N$) defined as solutions to the I.V.P. (1),(2) with $y_j(\tau) = b_j$, $j = 1, ...N$, by

$y_j(t) = \exp(A(t-\tau))b_j$, form a basis for the space $\mathcal{S}_{\text{hom}}$ of solutions to (1). The dimension of the vector space $\mathcal{S}_{\text{hom}}$ of solutions to (1) is equal to N -

the dimension of the system (1). **Idea of the proof.** This property is a consequence of the linearity of the system and the uniqueness of solutions to the

system and is independent of detailed properties of the matrices $A(t)$ and A in (1) and (3).

Proof. Consider a linear combination of $y_j(t)$ equal to zero for some time $\sigma \in \mathbb{R}$: $l(\sigma) = \sum_{j=1}^{N} \alpha_j y_j(\sigma) = 0$. Observe that the trivial constant zero solution $0(t)$ coinsides with l at this time point.

But by the uniqueness of solutions to (1) it implies that $l(t)$ at arbitrary time must coinside with the trivial zero solution for all times and in particular at time $t = \tau$. Therefore $l(\tau) = \sum_{j=1}^{N} \alpha_j b_j = 0$ (point out that $y_j(\tau) = b_j$). It implies that all coefficients $\alpha_j = 0$ because $b_1, ..., b_N$ are linearly independent

vectors in $\mathbb{R}^N$(or $\mathbb{C}^N$). It implies that $y_1(t), ..., y_N(t)$ are linearly independent for all $t \in \mathbb{R}$ by definition. Arbitrary initial data $x(\tau) = \xi$ in $\mathbb{R}^N$(or $\mathbb{C}^N$) can be

represented as a linear combination of basis vectors $b_1, ..., b_N$: $\xi = \sum_{j=1}^{N} C_j b_j$. The construction of $y_1(t), ..., y_N(t)$ shows that an arbitrary solution to (1)

can be represented as linear combination of $y_1(t), ..., y_N(t)$.

$$
\begin{aligned}
x(t) &= \exp(A(t-\tau))\xi = \exp(A(t-\tau)) \sum_{j=1}^{N} C_j b_j = \\
&= \sum_{j=1}^{N} C_j [\underbrace{\exp(A(t-\tau)) b_j}_{=y_j(t)}] = \sum_{j=1}^{N} C_j y_j(t)
\end{aligned}
$$

Therefore $\{y_1(t), ..., y_N(t)\}$ is the basis in the space of solutions $\mathcal{S}_{\text{hom}}$ and therefore $\mathcal{S}_{\text{hom}}$ has dimension N.$\blacksquare$

2. Formulate and give a proof to Lyapunov's criterion for stability of equilibrium points. **(4p)**

Consider an autonomous system $x' = f(x)$ with $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ open. We suppose that f is a locally Lipschitz continuous function, so the existence and uniqueness of maximal solutions to I.V.P. are valid.

We repeat for convenience definitions of stable and unstable equilibrium points (Equilibrium points are considered here at the origin to make it simpler to

apply the construction with Lyapunov functions) **Comment.** In fact $\mathbb{R}^+ \subset I_\xi$ in this case.

Definition

An equilibrium point $0 \in G$ of the system $x' = f(x)$ is said to be stable if for each $\varepsilon > 0$, there is $\delta > 0$ such that for any ξ taken in the ball

$B(\delta, 0) = \{\xi \in \mathbb{R}^N, \quad |\xi| < \delta\}$ the maximal solution $x(t) = \varphi(t, \xi) : I_\xi \to G$ on the maximal interval I_ξ with initial data $x(0) = \xi$ and $0 \in I_\xi$ will stay in the ball

$B(\varepsilon, 0)$: $\|\varphi(t, \xi)\| < \varepsilon$ for all $t \in I_\xi \cap \mathbb{R}^+$.

Definition

The function $V : U \to \mathbb{R}$, U - open, containing the origin $0 \in U$, is said to be positive definite in U, if $V(0) = 0$ and $V(z) > 0$ for $\forall z \in U$, $z \neq 0$.

Theorem. Lyapunov's theorem on stability. Th.5.2, p.170

Let 0 be an equilibrium point for the system above and there is a **positive definite** continuously differentiable, $C^1(U)$ function $V : U \to \mathbb{R}$, such that

$U \subset G$, $0 \in U$ and $V_f(z) = \nabla V \cdot f(z) \leq 0 \ \forall z \in U$.

Then 0 is a stable equilibrium point.

Remark.

A function V with these properties is usually called the **Lyapunov function of the system.**

Proof.

Take an arbitrary $\varepsilon > 0$ such that $B(\varepsilon, 0) \subset U$ and $\partial B(\varepsilon, 0) \subset U$ for $\partial B(\varepsilon, 0) = S(\varepsilon, 0) = \{z : \|z\| = \varepsilon\}$.

Let

$$\alpha = \min_{z \in S(\varepsilon,0)} V(z)$$

be a minimum of the continuous function V on the boundary of $B(\varepsilon, 0)$, that is the sphere $S(\varepsilon, 0) = \{z : |z| = \varepsilon\}$ and is a compact set (closed and bounded).

Then $\alpha > 0$ because $V(z) > 0$ outside the equilibrium point 0.

By continuity of the function V and the fact that $V(0) = 0$ one can find a δ, $0 < \delta < \varepsilon$ such that $\forall z \in B(\delta, 0)$ we have $V(z) < \alpha/2$.

On the other hand for any part of the trajectory $x(t) = \varphi(t, \xi)$, inside U the function $V(\varphi(t, \xi))$ is *non-increasing* because

$\frac{d}{dt} V(\varphi(t, \xi)) = (\nabla V \cdot f)(x(t)) \leq 0.$

It implies all trajectories $\varphi(t, \xi)$ with initial points $\xi \in B(\delta, 0)$ satisfy $V(\xi) < \alpha/2$. Therefore $V(\varphi(t, \xi)) < \alpha/2$ and $\varphi(t, \xi)$ cannot reach the sphere $S(\varepsilon, 0)$

where $V(z) \geq \alpha = \min_{z \in S(\varepsilon,0)} V(z)$.

Therefore any such trajectory stays within the ball $B(\varepsilon, 0)$ and by the definition, the equilibrium point in the origin 0 is stable.

It implies also that $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ, because the trajectory stays inside a compact set. ■

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 3 & 0 & 4 \\ -3 & 1 & -7 \\ -2 & 0 & -3 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are unbounded. **(4p)**

Solution.

Characteristic polynomial is:

$\det \begin{bmatrix} 3-\lambda & 0 & 4 \\ -3 & 1-\lambda & -7 \\ -2 & 0 & -3-\lambda \end{bmatrix} = (1-\lambda)\left((3-\lambda)(3-\lambda)+8\right) = (1-\lambda)\left(-9+\lambda^2+8\right) = (1-\lambda)\left(\lambda^2-1\right)$

$\lambda_1 = -1$. Corresponding eigenvector satisfies $[A - \lambda_1 I]\, v_1 = 0$;

$$\begin{bmatrix} 4 & 0 & 4 \\ -3 & 2 & -7 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 4 \\ -3 & 2 & -7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 12 \\ -12 & 8 & -28 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 12 \\ 0 & 8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 12 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvector: $v_1 = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_1 = -1$

$\lambda_2 = 1$. Corresponding eigenvector satisfies $[A - \lambda_2 I]\, v_2 = 0$;

$$\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 8 \\ -6 & 0 & -14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$, \quad v_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 1 \text{ has multiplicity 2 and geometric mupliplicity 1.}$$

A generalized eigenvector $v_2^{(1)}$ satisfies the equation $[A - \lambda_2 I]\, v_2^{(1)} = v_2; \quad \lambda_2 = 1$

$$\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow v_3 = v_2^{(1)} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

The initial data is represented in terms of two linearly independent eigenvectors and one generalized eigenvector: $\xi = C_1 v_1 + C_2 v_2 + C_3 v_3$

Using the formula for general solution, and pointing out that $(A - \lambda_2)\, v_3 = v_2$

$$x(t) = \exp(At)\xi = \sum_i C_i e^{\lambda_i t} \sum_{k=1}^{m(\lambda_i)} \frac{t^k}{k!} (A - \lambda_i)^k = C_1 e^{-t} v_1 + C_2 e^t v_2 + C_3 e^t v_3 + C_3 t\, e^t v_2$$

For all initial points $\xi \in \mathbb{R}^3$ outside the line $\xi = v_1 \mu$ where $\mu \in \mathbb{R}$ the solutions will be unbounded.■

4. Give definition of the monodromy matrix for the linear system $\mathbf{x}' = A(t)\mathbf{x}(t)$ with periodic matrix $A(t + p) = A(t)$.

Formulate criterion for the boundedness of all solutions to a periodic linear system of ODEs. **(1p)**

Find the monodromy matrix (scalar here) for the following linear equation with periodic coefficients

$$x' = \left(a + \sin^3 t\right) x$$

Find for which real values of parameter a all solutions are bounded. **(3p)**

Solution.

The transition matrix function $\Phi(t, \tau)$ is a solution to the matrix equation $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$, $\Phi(\tau, \tau) = I$.

The monodromy matrix is the value $\Phi(p, 0)$ where p is the period of the matrix $A(t)$. In the scalar case we can find $\Phi(t, \tau)$ explicitely.

We calculate the primitive function of the coefficient in the equation to write down an explicit expression for $\Phi(t, \tau)$.

$P(t) = \int \left(a + \sin^3 t\right) dt = at - \frac{3}{4}\cos t + \frac{1}{12}\cos 3t = at + \int (1 - \cos^2(t)) \sin t dt = at - \int (1 - \cos^2(t))\, d\,(\cos(t)) = at - \cos(t) + \frac{1}{3}\cos^3(t)$.

The transition matrix function (scalar for the scalar equation) is in our case $\Phi(t, \tau) = \exp(P(t) - P(\tau))$.

$x(t) = \exp(P(t) - P(\tau)) x_0$ is a solution of the given equation with initial data $x(\tau) = x_0$.

The period of the coefficient in the ODE is $p = 2\pi$.

The monodromy matrix (scalar here) is the value of the transition matrix function with starting time $\tau = 0$ and at time t equal to the period p:

$\Phi(p,0) = \exp(P(2\pi) - P(0)) = \exp\left(2\pi a - 1 + \frac{1}{3} + 1 - 1/3\right) = \exp\left(2\pi a\right)$

All solutions will be bounded if the Floquet multiplicator $\exp(2\pi a) \leq 1$. It is valid if and only if $a \leq 0$.■

5. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction.

$$\begin{cases} x' = & 3y \\ y' = & -x - (4 - x^2)y \end{cases} \qquad \textbf{(4p)}$$

Solution. We choose a Lyapunov function in the form $V(x,y) = x^2 + ay^2$. The choice $a = 3$ is optimal, it gives $V_f(x,y) = 6xy - 6xy - 2y^2(4 - x^2) = -2y^2(4 - x^2) \leq 0$ in the stripe where $|x| \leq 2$. Therefore the origin is a stable equilibrium point.

We apply the LaSalle invariance principle in the domain $|x| < 2$ to prove that the origin is also an asymptotically stable equilibrium.

We observe that $V_f^{-1}(0) = \{x\text{−axis}\}$ because $V_f(x,y) = 0$ if and only if $y = 0$. Checking the right hand side in the system for $y = 0$ implies $y' = -x$ that is zero only in the origin. It implies that the origin is the maximal invariant subset in $V_f^{-1}(0) = \{x\text{−axis}\}$. Therefore the origin is asymptotically stable. The domain of attraction is the set bounded by a level set of $V(x,y) = C = x^2 + 3y^2$ (an ellipse) that is inside the stripe $|x| \leq 2$. The largest such ellipse goes through points $x = \pm 2$, $y = 0$. Therefore $C = 4$ and the domain of attraction for the equilibrium in the origin consists of points (x,y) satisfying the inequality $x^2 + 3y^2 < 4$. All trajectories starting in this domain tend to the origin with $t \to \infty$.■

6. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x,y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vector function $F(x,y)$ we conclude that

$$\nabla V(x,y) = \nabla\left(2x^2 + y^2\right) = \begin{matrix} 4x \\ 2y \end{matrix};$$

$$\nabla V \cdot F(x,y) = \begin{bmatrix} 4x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} x - 2y - x\left(2x^2 + y^2\right) \\ 4x + y - y\left(2x^2 + y^2\right) \end{bmatrix} = 4x^2 - 8xy + 8yx + 2y^2 - 8x^4 - 2y^4 - 4x^2y^2 - 4y^2x^2 = 4x^2 + 2y^2 - 8x^4 - 2y^4 - 8x^2y^2;$$

$$\nabla V \cdot F(x,y) = 2\left(2x^2 + y^2\right)\left(1 - \left(2x^2 + y^2\right)\right)$$

It implies that the elliptic shaped ring: $R = \{(x,y) : 0.5 \leq (2x^2 + y^2) \leq 2\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$. The origin is the only equilibrium point of the system. One can see it by observing that $V(x,y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot F(x,y) = 0$ only if $(x,y) = (0,0)$ or if $(2x^2 + y)$ =1.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x,y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x,y) = (0,0)$.

By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.■

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