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| MATEMATIK | Datum: 2023-08-18 | Tid: förmiddag 08:30 - 12:30 |
| GU, Chalmers | Hjälpmedel: - Inga | |
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**Lösningsförslag till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)**

1. Formulate and prove conditions that for any $n \times n$ complex matrix $A$ imply that $\|\exp(At)\| \to 0$ as $t \to \infty$. **(4p)**

   Check lecture notes.

2. Formulate and give a proof to La Salle's invariance principle. **(4p)**

   Check lecture notes.

3. Consider the following matrix $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -3 & 0 \\ -2 & -3 & -2 \end{bmatrix}$, eigenvalues: $-1$

   (a) Find general solution of the ODE $x' = Ax$ and the canonical Jordan form of the matrix $A$. **(4p)**

   **Solution.**

   Charachteristic polynomial of $A$ is $\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 5 & 1 \\ -1 & -3-\lambda & 0 \\ -2 & -3 & -2-\lambda \end{bmatrix} =$

   $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3$

   The only eigenvalue $\lambda = -1$ has algebraic multiplicity 3.

   Check eigenvectors: Solve the homogeneous system:

   $$(A - \lambda I)V = 0$$

   with matrix $(A - \lambda I) = \begin{bmatrix} 3 & 5 & 1 \\ -1 & -2 & 0 \\ -2 & -3 & -1 \end{bmatrix}$, The reduced echelon form of this matrix after Gauss elimination is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ giving one linearly independent eigenvector $V = C \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$, with arbitrary $C \neq 0$.

   Two linearly independent generalised eigenvectors we try to find form the chain relations

   $$(A - \lambda I)V^{(1)} = V$$

   and

   $$(A - \lambda I)V^{(2)} = V^{(1)}$$

   .

   Corresponding extended matrices for the first linear system is: $\begin{bmatrix} 3 & 5 & 1 & 2C \\ -1 & -2 & 0 & -C \\ -2 & -3 & -1 & -C \end{bmatrix}$, Gaussian elimination gives the reduced echelon form: $\begin{bmatrix} 1 & 0 & 2 & -C \\ 0 & 1 & -1 & C \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and solution in the form $V^{(1)} = C \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$;

The extended matric for the linear system for $V^{(2)}$ is: $\begin{bmatrix} 3 & 5 & 1 & 0 \\ -1 & -2 & 0 & C/2 \\ -2 & -3 & -1 & -C/2 \end{bmatrix}$,the reduced echelon form is: $\begin{bmatrix} 1 & 0 & 2 & \frac{5}{2}C \\ 0 & 1 & -1 & -\frac{3}{2}C \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Choosing free variable as $5C/4$, we get the second linearly independent generalized eigenvector in the form $V^{(2)} = C \begin{bmatrix} 0 \\ -1/4 \\ 5/4 \end{bmatrix}$. Taking for convenience $C = 4$ we arrive to

$V = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}$, $V^{(1)} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$, $V^{(2)} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$.

We use the general formula

$$x(t) = e^{At}x_0 = \sum_{j=1}^{s} \left( \left[ \sum_{k=0}^{m_j - 1} (A - \lambda_j I)^k \frac{t^k}{k!} \right] x^{0,j} e^{\lambda_j t} \right)$$

for solutions with initial data

$$x(0) = x_0 = \sum_{j=1}^{s} x^{0,j}$$

with $x^{0,j} \in E(\lambda_j, A)$ - components of $x_0$ in the generalized eigenspaces $E(\lambda_j, A)$ $= \ker(A - \lambda_j)^{m_j}$ of the matrix $A$. Here $s$ is the number of distinct eigenvalues $\lambda_j$ to $A$ and $m_j$ is the algebraic multiplicity of the eigenvalue $\lambda_j$. We point out that $\mathbb{C}^n = E(\lambda_1, A) \oplus E(\lambda_2, A) \oplus ... \oplus E(\lambda_s, A)$.

Situation simplifies in our case because the matrix $A$ has only one multiple eigenvalue $\lambda = -1$ with the algebraic multiplicity $m = 3$.

We express the initial data $x_0$in terms of the basis of $\mathbb{C}^n$ in terms of eigenvectors $v_j$ and generalized eigenvectors

$$x_0 = C_0 V + C_1 V^{(1)} + C_2 V^{(2)}$$

It implies the formula for general solution:

$$x(t) = e^{At}x_0 = \left[ \sum_{k=0}^{2} (A - \lambda I)^k \frac{t^k}{k!} \right] \left[ C_0 V + C_1 V^{(1)} + C_2 V^{(2)} \right] e^{\lambda t}$$

Chain relations between the generalized eigenvectors that we constructed, implies that several terms in this sum are canceled or simplified.

$$\begin{aligned} x(t) &= e^{At}x_0 = C_0 V e^{\lambda t} + \\ & C_1 e^{\lambda t}\left(V^{(1)} + tV\right) \\ & C_2 e^{\lambda t}\left(V^{(2)} + tV^{(1)} + \frac{t^2}{2}V\right) \end{aligned}$$

We do not need to substitute numbers to the final formula. ∎

4. Solve the initial value problem

$$x' = t\,x^3, \qquad x\,(1) = 1$$

and find maximal interval for the solution. **(2p)**

Can one conclude which maximal interval have solutions to the equation $x' = t^3 \sin(x)x$ without solving it? **(2p)**

**Solution.** It is the equation with separable variables.

$$\begin{aligned} \frac{dx}{dt} &= tx^3 \\ \int \frac{dx}{x^3} &= \int tdt \\ \frac{-1}{2x^2} &= \frac{t^2}{2} - C \\ C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2} = 1 \\ \frac{-1}{2x^2} &= \frac{t^2-2}{2}; \quad \frac{1}{x^2} = 2 - t^2 \\ x^2 &= \frac{1}{(2-t^2)} \\ x &= \frac{1}{\sqrt{2-t^2}}; \quad x(1) = 1; \quad t \in (-\sqrt{2}, \sqrt{2}) \end{aligned}$$

Checking the solution: $\frac{d}{dt}\left(\frac{1}{\sqrt{2-t^2}}\right) = \frac{t}{2\sqrt{2-t^2}-t^2\sqrt{2-t^2}} = (2-t^2)^{-1}\left(\sqrt{2-t^2}\right)^{-1} t = \left(\sqrt{2-t^2}\right)^{-3} t = tx^3$.

The maximal interval for this solution is $I = (-\sqrt{2}, \sqrt{2})$ and is open in accordance with the general theory.

The equation $\dot{x} = t^3 \sin(x)x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$ , $R > 0$ the estimate $|t^3 \sin(x)x| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies according to a theorem in the course, that the maximal existence interval for all solutions to this equation is $\mathbb{R}$. ∎

5. Formulate the criterium for instability of an equilibrium point using test functions.

Show that the equilibrium point in the origin of the following ODE is unstable:

$$\begin{cases} x' = x^2 - y^2 \\ y' = -2xy \end{cases}$$ **(4p)**

Hint: try the test function $V(x,y) = axy^2 + bx^3$ with appropriate constants $a$ and $b$.

**Solution.**

$\nabla V(x,y) = \begin{bmatrix} ay^2+3bx^2 \\ 2axy \end{bmatrix}$; $\begin{bmatrix} ay^2+3bx^2 \\ 2axy \end{bmatrix} \begin{bmatrix} x^2-y^2 \\ -2xy \end{bmatrix} =$

$ax^2y^2 - ay^4 + 3bx^4 - 3bx^2y^2 - 4ax^2y^2 = -ay^4 + 3bx^4 + x^2y^2(-3a-3b)|_{a=-1} =$

$y^4 + 3bx^4 + x^2y^2(3-3b))|_{b=1/3} = y^4 + x^4 + 2x^2y^2 = (x^2+y^2)^2 > 0.$

Therefore for $a = -1$ and $b = 1/3$ we get $V(x,y) = -xy^2 + 1/3x^3$ and $V_f(x,y) > 0$, for $(x,y) \neq (0,0)$.

Point out that $V(x,y) > 0$ along the line $x < 0$, $y = -x$, because $V(x,y) = -x^3 + 1/3x^3 = -2/3x^3 > 0$ for $x < 0$. It implies that there are points arbitrarily close to the origin where $V(x,y) > 0$. Therefore the equilibrium point in the origin is unstable.■

6. Formulate the Poincare-Bendixson theorem. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = y \\ y' = -y(x^2+y^2-1) - x^3 \end{cases}$$ **(4p)**

**Solution.**

The Poincare Bendixson theorem states that if a differential equation $x' = f(x)$ in plane $R^2$ has an $\omega$ limit set that does not contain any equilibrium points. Then this $\omega$ - limit set must be an orbit of a periodic solution.

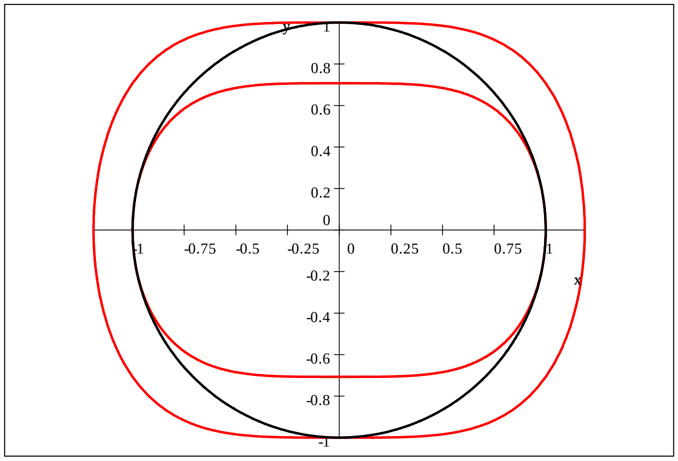
The only equilibrium pont of the system is the origin. The system of equation represents Newton's law $x'' = -x'\left(x^2 + (x')^2 - 1\right) + x^3$ with a friction force $-x'\left(x^2 + (x')^2 - 1\right)$ and a potential force $-x^3$. We choose a test function as a sum of kinetic energy $\frac{1}{2}y^2$ and potential energy $\int x^3 dx$: $V(x,y) = \frac{1}{2}y^2 + \int x^3 dx = \frac{1}{2}y^2 + \frac{1}{4}x^4$ so that indefinite terms in the expression for $V_f$ cancel.

$V_f(x,y) = \nabla V \cdot f(x,y) = \begin{bmatrix} x^3 \\ y \end{bmatrix} \begin{bmatrix} y \\ -y(x^2+y^2-1) - x^3 \end{bmatrix} = x^3y - x^3y - y^2(x^2+y^2-1) = -y^2(x^2+y^2-1)$

We observe that $V_f(x,y) \geq 0$ inside the circle $x^2+y^2 = 1$ and $V_f(x,y) \leq 0$ outside the circle $x^2+y^2 = 1$.

It makes that one can choose close curves $V(x,y) = C_1$ and $V(x,y) = C_2$ flattened along the $y$ - direction so, that the first one lays inside the circle $x^2+y^2 = 1$ and the second one lays outside it. It is easy to see that the optimal choice is $C_1 = 1/4$, so that the level set $V(x,y) = 1/4$ goes through the points$(\pm 1, 0)$. Another optimal choice is $C_2 = 1/2$, so that the level set $V(x,y) = 1/2$ goes through the points$(0, \pm 1)$.

$\frac{1}{2}y^2 + \frac{1}{4}x^4 = 1/4$




The ring between these two level sets is a compact positive invariant set for the system and it does not contain any equilibrium points.

This compact positive invariant set must contain at least one $\omega$ - limit set for the ODE.

The Poincare-Bendixson theorem implies that this $\omega$ - limit set must be an orbit of a periodic solution.$\blacksquare$

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course are calculated as:

$Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total amount of points for the course.

*Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will collect all necessary points.*