MATEMATIK
GU, Chalmers
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

1. Formulate and prove the Floquet representation for the transition matrix of a periodic linear system of ODEs.

   Theorem 2.30 , p. 53. Floquet theorem

   Consider a periodic system $x'(t) = A(t)x(t)$, with period p: $A(t) = A(t+p)$

   Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of the monodromy matrix $\Phi(p,0) : G = \log(\Phi(p,0))$

   There exists a periodic with period p piecewise continuously differentiable function $\Theta(t) : \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular (invertible,

   all eigenvalues are non-zero) for all t, such that

$$\Phi(t,0) = \Theta(t)\exp\left(\frac{t}{p}G\right), \quad \forall t \in \mathbb{R}$$

   Check lecture notes for proof. **(4p)**

2. Formulate and give a proof to Bendixson's criterion for non-existence of periodic solutions to non-linear ODEs in the plane.

   Theorem. Let $x' = f(x)$ with $f : G \to \mathbb{R}^2$, $G \subset \mathbb{R}^2$ be open, $f \in C^1(G)$, and let $D \subset G$ be a **simply connected domain** (domain without "holes" even

   without point holes). It is enough to require that f is locally Lipschitz in G with more knowlege of integration theory.

   Suppose that $\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly positive (or strictly negative) in D, where $f = [f_1, f_2]^T$.

   Then the equation has no periodic solutions with orbits inside D.

   Check lecture notes for proof. **(4p)**

3. Consider the following matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 2 & 6 \\ 4 & -1 & -4 \end{bmatrix}$,

   Write down a general solution to the system of ODEs $x' = Ax$ with this matrix A. Find all initial data such that corresponding solutions are bounded. **(4p)**

   Solution.

   We start with calculation of the characteristic polynomial and finding eigenvalues to A.

$$\det\begin{bmatrix} 1-\lambda & 0 & -1 \\ -6 & 2-\lambda & 6 \\ 4 & -1 & -4-\lambda \end{bmatrix} = (1-\lambda)\det\begin{bmatrix} 2-\lambda & 6 \\ -1 & -4-\lambda \end{bmatrix} - \det\begin{bmatrix} -6 & 2-\lambda \\ 4 & -1 \end{bmatrix}$$

$$\det\begin{bmatrix} 2-\lambda & 6 \\ -1 & -4-\lambda \end{bmatrix} = 2\lambda + \lambda^2 - 2; \ \det\begin{bmatrix} -6 & 2-\lambda \\ 4 & -1 \end{bmatrix} = 4\lambda - 2;$$

$\det \begin{bmatrix} 1-\lambda & 0 & -1 \\ -6 & 2-\lambda & 6 \\ 4 & -1 & -4-\lambda \end{bmatrix} = (1-\lambda)\left(2\lambda+\lambda^2-2\right) - 4\lambda + 2 = 4\lambda - \lambda^2 - \lambda^3 - 2 -$
$4\lambda + 2 = -\lambda^2 - \lambda^3 = -\lambda^2(\lambda+1)$

It gives one simple eigenvalue $\lambda_1 = -1$ and one multiple eigenvalue $\lambda_2 = 0$ of multiplicitet 2.

Eigenvector v_1 to λ_1 satisfies himogeneous system with matrix

$\begin{bmatrix} 2 & 0 & -1 \\ -6 & 3 & 6 \\ 4 & -1 & -3 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

Eigenvector $v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ - one linearly independent vector to a simple eigenvalue.

Eigenvector v_2 to $\lambda_2 = 0$ satisfies homogeneous system with matrix

$\begin{bmatrix} 1 & 0 & -1 \\ -6 & 2 & 6 \\ 4 & -1 & -4 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .Eigenvector $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- one linearly independent vector to a multiple eigenvalue because only one free variable exists in the system.

We try to find a generalised eigenvector $v_2^{(1)}$ corresponding to $\lambda_2 = 0$ as a vector satisfiying equation $Av_2^{(1)} = v_2$ with extended matrix:

$\begin{bmatrix} 1 & 0 & -1 & 1 \\ -6 & 2 & 6 & 0 \\ 4 & -1 & -4 & 1 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & 6 \\ 0 & -1 & 0 & -3 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Generalised eigenvector $v_2^{(1)} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$. Vectors v_1, v_2, $v_2^{(1)}$ build a basis of $\mathbb{R}^3$ convenient for calculating a general solution with initial condition $\xi = C_1 v_2 + C_2 v_2 + C_3 v_2^{(1)}$ as

$$\begin{aligned} x(t) &= \exp(At)\left(C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}\right) \\ &= e^{\lambda_1 t} C_1 v_1 + e^{\lambda_2 t} C_2 v_2 + e^{\lambda_2 t} C_3 \left(v_2^{(1)} + t v_2\right) = \\ & \quad e^{-t} C_1 v_1 + C_2 v_2 + C_3 \left(v_2^{(1)} + t v_2\right) \end{aligned}$$

Only solutions with initial conditions $\xi = C_1 v_1 + C_2 v_2$ from the plane spanned by vectors $v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ will be bounded.

4. i) Solve the initial value problem $x'(t) = t\left(1+x^2\right)$, $x(0) = \xi$,with the domain for the equation $J \times G = \mathbb{R}^2$, and find maximal intervals for solutions for arbitrary $\xi \in \mathbb{R}$. **(3p)**

   ii) Draw a sketch for the domain of the transition mapping $\varphi(t, 0, \xi)$ as a function of two variables (t, ξ). **(1p)**

Solution.

The equation has separable variables and is solved correspondingly.

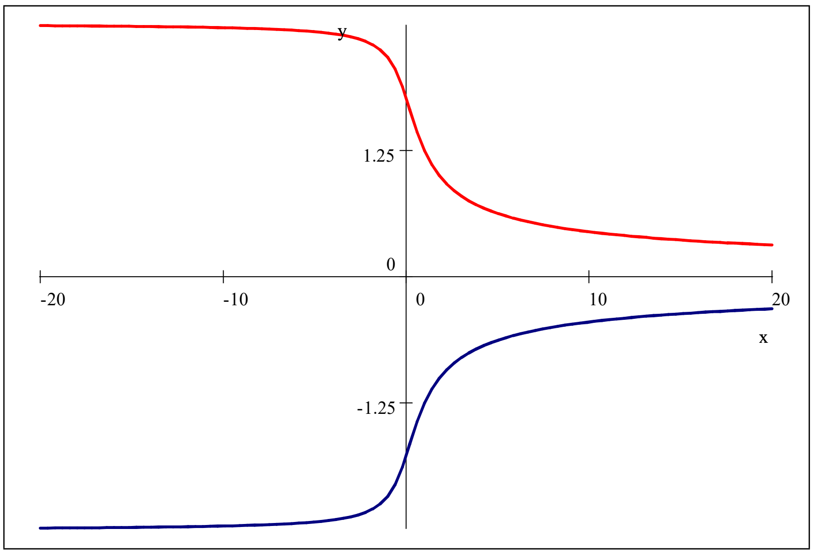
$$
\begin{aligned}
\int \frac{dx}{1+x^2} &= \int t dt \\
arc\tan(x) &= \frac{t^2}{2} + C \\
C + 0 &= \arctan(\xi) \\
x &= \tan(\frac{t^2}{2} + \arctan\xi) \\
-\pi/2 &< \frac{t^2}{2} + \arctan\xi < \pi/2 \\
-\pi &< t^2 + 2\arctan\xi < \pi \\
-\pi - 2\arctan\xi &< t^2 < \pi - 2\arctan\xi \\
0 &\leq t^2 < \pi - 2\arctan\xi \\
-\sqrt{\pi - 2\arctan\xi} &< t < \sqrt{\pi - 2\arctan\xi}
\end{aligned}
$$

Point out that $-\pi - 2\arctan\xi \leq 0$ for all $\xi \in \mathbb{R}$ because $\lim_{\xi\to\infty} \arctan\xi = \pi/2$ and $\lim_{\xi\to-\infty} \arctan\xi = -\pi/2$

$I_{\max}(\xi) = (-\sqrt{\pi - 2\arctan\xi}, \sqrt{\pi - 2\arctan\xi})$. The domain of the transition mapping $\varphi(t, 0, \xi)$ is the set

$$D(\varphi) = \left\{(t,\xi) : \xi \in \mathbb{R}, \quad t \in (-\sqrt{\pi - 2\arctan\xi}, \sqrt{\pi - 2\arctan\xi})\right\}$$

Observe that $\sqrt{\pi - 2\arctan\xi} \to 0$ with $t \to \infty$ and $\sqrt{\pi - 2\arctan\xi} \to \sqrt{2\pi}$ with $t \to -\infty$.



5. Investigate stability of the equilibrium point in the origin of the following ODE and find it's possible domain of attraction.

$$\begin{cases} x' = -x + y - yx^2 \\ y' = -y - x + xy^2 \end{cases}$$ **(4p)**

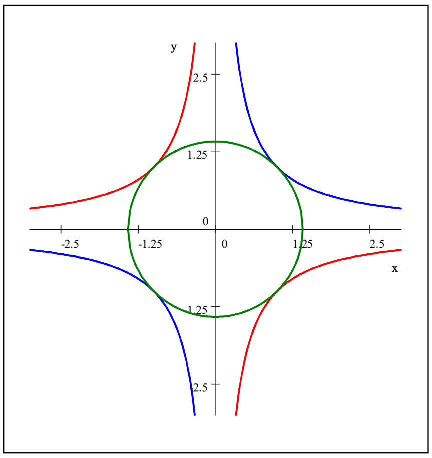
Solution.

The Jacobi matrix of the right hand side of the equation in the origin is the matrix $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, eigenvalues: $-1-i, -1+i$. $\operatorname{tr}(A) = -2 < 0$, $\det(A) = 2 > 0$, the characteristic polynomial is $p(\lambda) = \lambda^2 + 2\lambda + 2$. Point out also that $(tr(A))^2/4 = 1 < \det(A) = 2$.

It implies accordning to the Grobman-Hartman theorem, that the origin is an asymptotically stable equilibrium both for the linearized system $x' = Ax$ and for the original non-linear system of ODEs. To find a domain of attraction of this equilibrium to the non-linear system, we try to find a test function that would give us a boundary of a compact positive invariant set such that all trajectories starting there tend to the origin as $t \to \infty$. Try the simple test function $V(x, y) = \frac{1}{2}(x^2 + y^2)$.

$$\begin{aligned} V_f(x,y) &= \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -x + y - yx^2 \\ -y - x + xy^2 \end{bmatrix} = -x^2 + xy - x^3y - y^2 - yx + xy^3 \\ &= -x^2(1+xy) - y^2(1-xy) \end{aligned}$$

Observe that $V_f(x, y) \leq 0$ for $1 + xy \geq 0$ and $1 - xy \geq 0$, that is equivalent to $-1 \leq xy$ and $xy \leq 1$. First inequality is satisfied for points between branches of the (red) hyperbola $y = -1/x$ (see the picture). The second inequality is satisfied for points between branches of the (blue) hyperbola $y = 1/x$. Level sets of the test function V are circles $(x^2 + y^2) = C$ with positive constants C. The maximal such circle that fits into the area between these two hyperbolas where the inequality $V_f(x, y) \leq 0$ is satisfied, is one with radius $r = \sqrt{2}$one: $C = 2$.It touches the hyperbolas in four points: $(\pm1, \pm1)$.



All trajectories starting in this circle stay in it and must have an ω limit set there. $V_f^{-1}(0)$ consists of only the origin. According to the LaSalles invariance principle all solutions starting inside the circle $x^2 + y^2 < 2$ must tend to the origin, that is an asymptotically stable equilibrium point. Therefore the domain $x^2 + y^2 < 2$ is the domain of attraction for the equilibrium point in the origin.

The origin is not globally asymptotically stable, because there are other equilibrium points, one in each quadrant.

6. Formulate Poincare-Bendixson's theorem. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x + y - x\,(x^2 - xy + y^2) \\ y' = -x + y - y\,(x^2 - xy + y^2) \end{cases}$$ **(4p)**

Hint. The Cauchy inequality $|ab| \leq 0.5\,(a^2 + b^2)$ can be useful for analysis here.

Solution.

We try to use the simplest test function $V(x, y) = \frac{1}{2}\,(x^2 + y^2)$

$V_f(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x + y - x\,(x^2 - xy + y^2) \\ -x + y - y\,(x^2 - xy + y^2) \end{bmatrix} = x^2 + xy - xy + y^2 - x^4 - y^4 - x^2y^2 + x^3y + xy^3 - x^2y^2 =$

$= x^2 + y^2 - x^4 - y^4 + xy^3 + x^3y - 2x^2y^2 = x^2(1 - x^2 - y^2 + xy) + y^2(1 - x^2 - y^2 + xy)$

It is easy to see that the ellipse $x^2 - xy + y^2 = 1$ separates areas where $V_f(x, y) \leq 0$ and $V_f(x, y) \geq 0$. to find level sets of V (circles), where $V_f(x, y) \leq 0$ and $V_f(x, y) \geq 0$ we simplify the last expression for V_f using the Cauchy inequality.

Point out that $|xy| \leq \frac{1}{2}\,(x^2 + y^2)$.

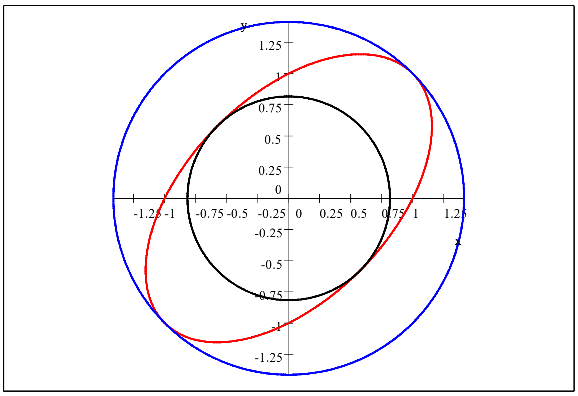
It implies that

$$x^2(1 - x^2 - y^2 - \frac{1}{2}\,(x^2 + y^2)) + y^2(1 - x^2 - y^2 - \frac{1}{2}\,(x^2 + y^2)) \leq V_f(x, y)$$
$$\leq x^2(1 - x^2 - y^2 + \frac{1}{2}\,(x^2 + y^2)) + y^2(1 - x^2 - y^2 + \frac{1}{2}\,(x^2 + y^2))$$

$$x^2(1 - \frac{3}{2}\,(x^2 + y^2)) + y^2(1 - \frac{3}{2}\,(x^2 + y^2)) \leq V_f(x, y) \leq x^2(1 - \frac{1}{2}\,(x^2 + y^2)) + y^2(1 - \frac{1}{2}\,(x^2 + y^2))$$

It implies that $0 \leq V_f(x, y)$ in the area where $(x^2 + y^2) \leq \frac{2}{3}$ and that $V_f(x, y) \leq 0$ in the area where $2 \leq (x^2 + y^2)$ .We conclude that the compact annulus set C with $C = \left\{(x, y) : \frac{2}{3} \leq x^2 + y^2 \leq 2\right\}$ is a positively invariant set for our system of ODEs.

We draw circles $x^2 + y^2 = 2$ and $x^2 + y^2 = \frac{2}{3}$ together with the (red) ellipse $x^2 - xy + y^2 = 1$ that separates areas where $V_f(x, y) \leq 0$ and $V_f(x, y) \geq 0$:



We need to check equilibrium points of the system that satisfy the system of equations
$$\begin{cases} x + y - x\,(x^2 - xy + y^2) = 0 \\ -x + y - y\,(x^2 - xy + y^2) = 0 \end{cases}$$
Multiply the first equation by y, the second equation by x and substract from each other. It will give us an equation

$xy + y^2 + x^2 - xy = 0$ and $y^2 + x^2 = 0$, that implies that the only equilibrium point is the origin.

Another way to prove it is to observe that $V_f(x) = 0$ in equilibrium points.

$V_f(x) = x^2(1 - x^2 - y^2 + xy) + y^2(1 - x^2 - y^2 + xy) = 0 = (x^2 + y^2)\,(1 - x^2 - y^2 + xy)$

It implies that $1 = x^2 - y^2 + xy$ or $(x^2 + y^2) = 0$. If $1 = x^2 - y^2 + xy = 1$ the equation for equilibrium points implies $y = 0$ and $-x = 0$. With the same conclusion.

We conclude that each trajectory $\varphi(t, \xi)$ with $\xi \in C$, starting in C must have a limit set in C, which according to the Poincare - Bendixson theorem must be a periodic orbit, and therefore the system of equations has at least one periodic solution with orbit in C.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course are calculated as:

$Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will collect all necessary points.