MATEMATIK	Datum: 2023-01-05	Tid: 8-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
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Tenta i ODE och matematisk modellering, MMG511, MVE162

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the theorem about "infinite" extensibility of solutions for ODEs with a linear bound on the right hand side. **(4p)**

2. Formulate and give a proof to the theorem about stability of equilibrium points to autonomous ODEs by Lyapunovs functions. **(4p)**

3. Consider the following system of ODEs: $\dfrac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$, with the constant matrix

$$A = \begin{bmatrix} -1 & 6 & -6 & 10 \\ -3 & 8 & -5 & 7 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & -5 \end{bmatrix}.$$

(i) write down Jordans canonical form of the matrix A.

(ii) write down the general form of the solution for this equation. **(4p)**

Solution.

The matrix A is bloch triangular and it's eigenvalues are eigenvalues of diagonal blocks $A_1 = \begin{bmatrix} -1 & 6 \\ -3 & 8 \end{bmatrix}$ with eigenvalues: $\lambda_1 = 2, \lambda_2 = 5$, and $A_2 = \begin{bmatrix} -1 & -4 \\ 1 & -5 \end{bmatrix}$ with multiple eigenvalue $\lambda_3 = -3$.

Eigenvectors are: $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow 2$,satisfyting the homogeneous system with matrix $\begin{bmatrix} -3 & 6 & -6 & 10 \\ -3 & 6 & -5 & 7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

$\mathrm{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow 5$,satisfyting the homogeneous system with matrix $\begin{bmatrix} -6 & 6 & -6 & 10 \\ -3 & 3 & -5 & 7 \\ 0 & 0 & -6 & -4 \\ 0 & 0 & 1 & -10 \end{bmatrix}$

and v_3 satisfyting the homogeneous system with matrix $\begin{bmatrix} 2 & 6 & -6 & 10 \\ -3 & 11 & -5 & 7 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} 2 & 6 & -6 & 10 \\ 0 & 20 & -14 & 22 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Gaussian - Jordan elimination: $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{10} \\ 0 & 1 & 0 & -\frac{3}{10} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, giving the only linearly independent eigenvector $v_3 = \begin{bmatrix} 1 \\ 3 \\ 20 \\ 10 \end{bmatrix}$. There is a generalised eigenvector $v_3^{(1)}$ linearly independent of v_3 and satisfying the equation $(A+3I)^2\ v_3^{(1)} = 0$ or $(A+3I)\ v_3^{(1)} = v_3$.

$$\begin{bmatrix} 2 & 6 & -6 & 10 & 1 \\ -3 & 11 & -5 & 7 & 3 \\ 0 & 0 & 2 & -4 & 20 \\ 0 & 0 & 1 & -2 & 10 \end{bmatrix}, \Longrightarrow \text{Gauss elimination gives row echelon form: } \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{10} & \frac{353}{40} \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{289}{40} \\ 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and a generaslize deigenvector (not required) $v_3^{(1)} = \begin{bmatrix} \frac{353}{40} \\ \frac{289}{40} \\ 10 \\ 0 \end{bmatrix}$.

The Jordans canonical form is $J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

The general solution of the system for initial data $\mathbf{x}(0) = C_1 v_1 + C_2 v_2 + C_3 v_3 + C_4 v_3^{(1)}$ has the structure

$\mathbf{x}(0) = C_1 v_1 e^{2t} + C_2 v_2 e^{5t} + C_3 v_3 e^{-3t} + C_4 e^{-3t} \left(v_3^{(1)} + t\,(A + 3I)\, v_3^{(1)} \right)$.

4. Show that the origin is an unstable equilibrium point for the following system of ODEs

$\begin{cases} x' = y \\ y' = y - x^3 \end{cases}$, by using a test function in the form: $V(x, y) = ax^4 + bx^2 + cxy + dy^2$. **(4p)**

Solution.

We use the theorem stating that if in a neighbourhood of the equilibrium in the origin it is valid for a positive definite function $V(x, y)$ that $V_f(x, y) = \nabla V \cdot f(x, y)$ is also positive definit, then the origin is an unstable equilibrium.

Thy the test function in the form $V(x, y) = ax^4 + bx^2 + cxy + dy^2$.

$$V_f = \begin{bmatrix} y \\ y - x^3 \end{bmatrix} \cdot \begin{bmatrix} 4ax^3 + 2bx + cy \\ cx + 2dy \end{bmatrix} = 2bxy + cxy + cy^2 + 2dy^2 + 4ax^3y - cx^4 - 2dx^3y$$

We like to have $V_f \geq 0$. We eliminate indefinite terms with x^3y by choosing $d = 2a$ and $d = 1$. We make term $-cx^4$ positive definite by choosing $c = -1$ and eliminate terms with xy by choosing $c = -1, b = 1/2$.

$V_f(x, y) = y^2 + x^4 > 0$ for $(x, y) \neq 0$. $V(0, 0) = 0$.

We check that chosen $V(x, y) = 1/2x^4 + 1/2x^2 - xy + y^2$ is positive definite using the Cauchy inequality:

$|xy| \leq 1/2(x^2 + y^2)$.

It implies that $V(x, y) = 1/2x^4 + 1/2x^2 - xy + y^2 \geq 1/2x^4 + 1/2x^2 - 1/2(x^2 + y^2) + y^2$

$= 1/2x^4 + 1/2y^2 > 0$ for $(x, y) \neq (0, 0)$.

It implies by the theorem about instability above that the origin is an unstable equilibrium point.

5. Show that the following system of ODEs has at least one periodic solution.

$$\begin{cases} x' = y \\ y' = -x + y\left(1 - 3x^2 - 2y^2\right) \end{cases} \qquad \textbf{(4p)}$$

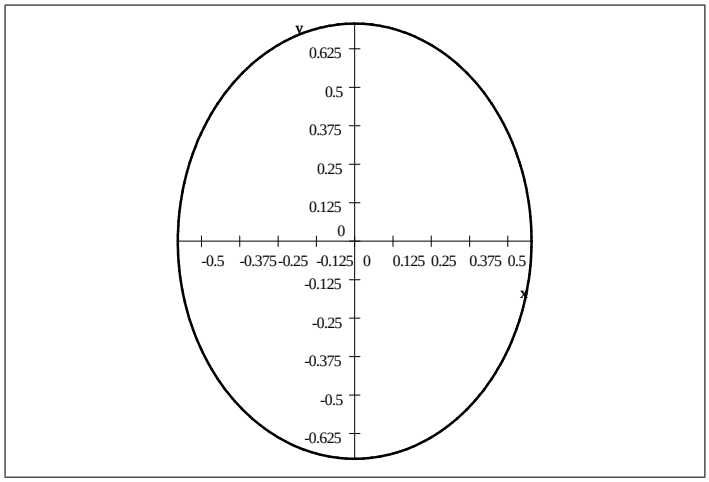
Solution.

We observe that the origin is the equilibrium point. We try the test function $V(x, y) = x^2 + y^2$.

$$V_f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} y \\ -x + y\left(1 - 3x^2 - 2y^2\right) \end{bmatrix} = 2xy - 2xy + 2y^2\left(1 - 3x^2 - 2y^2\right) = 2y^2\left(1 - 3x^2 - 2y^2\right)$$

We observe that $V_f(x,y) \le 0$ for $3x^2+2y^2>1$ and $V_f(x,y) \ge 0$ for $3x^2+2y^2>1$.

The curve $3x^2+2y^2=1$ is an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with halvaxes $a=1/\sqrt{3}$ and $b=1/\sqrt{2}$. See the picture.



We observe that the ellipse includes the circle $V(x,y)=x^2+y^2=\left(\frac{1}{2}\right)^2$ and is itself included into the circle $V(x,y)=x^2+y^2=1$.

It implies that the ring $1/4 \le x^2+y^2 \le 1$ is a positively invariant set for the system of differential equations, because $V_f(x,y) \le 0$ on the circle $V(x,y)=x^2+y^2=1$ and $V_f(x,y) \ge 0$ on the circle $1/4=x^2+y^2$.

The origin is the only equilibrium point, that is easy to see from the ODE. These two facts and the Poincare - Bendixson theory imply that the ring $1/4 \le x^2+y^2 \le 1$ must contain at least one periodic orbit.■

6. Consider a population of cells with two types of cells: cells with two chromosomes and cells with 4 chromosomes. Corresponding numbers of cells are denoted by D and U. The evolution of the population is described by the system of ODEs:

$\begin{cases} D' = (\lambda-\mu)D \\ U' = \mu D + \nu U \end{cases}$ with λ, μ, $\nu > 0$. Show that in the case if $\lambda > \mu+\nu$, the proportion of the cells of type D in the population stabilises with $t \to \infty$ to a percentage independent of the initial numbers D and U. **(4p)**

Solution.

We write the system in matrix form:

$\overrightarrow{r}' = A\overrightarrow{r}$ with $A = \begin{bmatrix} \lambda-\mu & 0 \\ \mu & \nu \end{bmatrix}$.

Eigenvalues to A are $\lambda_1 = \lambda-\mu$, and $\lambda_2=\nu$. Corresponding eigenvectors are chosen as solutions to the homogeneous systems with matrices $\begin{bmatrix} 0 & 0 \\ \mu & \nu+\mu-\lambda \end{bmatrix}$ giving eigenvector $\overrightarrow{v}_1 = \begin{bmatrix} \nu+\mu-\lambda \\ -\mu \end{bmatrix}$, and $\begin{bmatrix} \lambda-\mu & 0 \\ \mu & 0 \end{bmatrix}$ with eigenvector $\overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Arbitrary solution with initial data $\overrightarrow{r}(0) = C_1\overrightarrow{v}_1 + C_2\overrightarrow{v}_2$ is given by formula:

$$\overrightarrow{r}(t) = \begin{bmatrix} D(t) \\ U(t) \end{bmatrix} = C_1\overrightarrow{v}_1 e^{(\lambda-\mu)t} + C_2\overrightarrow{v}_2 e^{\nu t} =$$

$$\begin{bmatrix} C_1(\nu+\mu-\lambda)e^{(\lambda-\mu)t} + 0 \\ C_1(-\mu)e^{(\lambda-\mu)t} + C_2 e^{\nu t} \end{bmatrix}$$

The relation between $D(t)$ and $U(t)$ is:

$$\begin{aligned}\frac{D(t)}{U(t)} &= \frac{C_1\left(\nu+\mu-\lambda\right)e^{(\lambda-\mu)t}}{C_1\left(-\mu\right)e^{(\lambda-\mu)t}+C_2e^{\nu t}} = \\ &= \frac{(\nu+\mu-\lambda)}{(-\mu)+\frac{C_2}{C_1}e^{(\nu+\mu-\lambda)t}} \underset{t\to\infty}{\longrightarrow} \frac{(\lambda-\nu-\mu)}{\mu}\end{aligned}$$

in the case if $\lambda > \nu + \mu$. The relation $\frac{(\lambda-\nu-\mu)}{\mu}$ is independent of initial data C_1 and C_2. ■

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.