MATEMATIK | Datum: 2022-08-17 | Tid: 14-00 - 18-00
GU, Chalmers | Hjälpmedel: - Inga |
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Tenta i ODE och matematisk modellering, MMG511, MVE162

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate the theorem about boundedness and zero limits of solutions to linear autonomous systems of ODE's. Give a proof to the sufficiency conditions in the theorem. **(4p)**

   Check the formulation and the proof in lecture notes.

2. Formulate and prove Bendixsons criterium for the non-existence of periodic solutions to ODE's in the plane. **(4p)**

   Check the formulation and the proof in lecture notes.

3. Consider the following system of ODEs: $\dfrac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$, with the constant matrix

$$A = \begin{bmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

   i) Write down Jordans canonical form of the matrix A.

   ii) Find general solution to the system of ODEs.Find all initial conditions giving bounded solutions. **(4p)**

   Solution.

   Matrix A is block triangular with diagonal blocks $B = \begin{bmatrix} 3 & -4 \\ 4 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$. Eigenvalues to the matrix A are eigenvalues to B and C, because the characteristic polynomial to A

$$p_A(\lambda) = \det(A - I) = \det(B - \lambda I)\det(C - \lambda I) = \left(\lambda^2 + 2\lambda + 1\right)\left(\lambda^2 - 2\lambda + 1\right) = (\lambda+1)^2(\lambda-1)^2$$

   Therefore the matrix A has two distinct eigenvalues: $\lambda_1 = 1$ of multiplicity 2 and $\lambda_2 = -1$ of multiplicity 2.

   Eigenvectors to $\lambda_1 = 1$ are found from the system of homogeneous linear equations with matrix

$$A - I = \begin{bmatrix} 2 & -4 & 0 & 2 \\ 4 & -6 & -2 & 4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

   that gives one dimensional eigenspace of vectors $v_1 = C_1[1,1,1,1]$, $C_1 \neq 0$.

   A generalised eigenvector $v_1^{(1)}$ can be found from the inhomogeneous system of equations $(A - I)\, v_1^{(1)} = v_1$ with extended matrix

$$\begin{bmatrix} 2 & -4 & 0 & 2 & C_1 \\ 4 & -6 & -2 & 4 & C_1 \\ 0 & 0 & 2 & -2 & C_1 \\ 0 & 0 & 2 & -2 & C_1 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 2 & -4 & 0 & 2 & C_1 \\ 0 & 2 & -2 & 0 & -C_1 \\ 0 & 0 & 2 & -2 & C_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{Gauss}{\Longrightarrow}$$

$$\overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & -2 & 0 & 1 & C_1/2 \\ 0 & 1 & -1 & 0 & -C_1/2 \\ 0 & 0 & 1 & -1 & C_1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & -1 & C_1/2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & C_1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

giving for example $v_1^{(1)} = [C_1/2, 0, C_2/2, 0]^T$. It is convenient to choose $C_1 = 2$.

Eigenvectors to $\lambda_2 = -1$ are found from the system of homogeneous linear equations with matrix

$$A - I = \begin{bmatrix} 4 & -4 & 0 & 2 \\ 4 & -4 & -2 & 4 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

giving one dimensional space of eigenvectors in the form $v_2 = C_2[1, 1, 0, 0]^T$.

Generalised eigenvectors $v_2^{(1)}$ are found form the system $(A - \lambda_2 I)v_2^{(1)} = v_2$ and have the form $v_2^{(1)} = C_2[0, -1/4, 0, 0]^T$. It is convenient to choose $C_2 = 4$. We have got the basis of eigenvectors and generalised eigenvectors to $\mathbb{R}^4$ as:

$$v_1 = \begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix}, \quad v_1^{(1)} = \begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}, \quad v_2 = \begin{matrix} 4 \\ 4 \\ 0 \\ 0 \end{matrix}, \quad v_1^{(1)} = \begin{matrix} 0 \\ -1 \\ 0 \\ 0 \end{matrix}$$

Corresponding Jordan matrix has the form:

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Point out that we do not find generalised eigenvectors explicitely for stating this.

The general solution to the system of differential equations $x'(t) = Ax(t)$ with arbitrary initial conditions

$$x(0) = c_1 v_1 + c_2 v_1^{(1)} + c_3 v_2 + c_4 v_2^{(1)}$$

has the form:

$$x(t) = c_1 e^t v_1 + c_2 e^t \left(v_1^{(1)} + t\, v_1 \right) + c_3 e^{-t} v_2 + c_4 e^{-t} \left(v_2^{(1)} + t\, v_2 \right)$$

Solutions are bounded for initial conditions in the form $x(0) = c_3 v_2 + c_4 v_2^{(1)}$.

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4. Investigate stability of the origin and find in case if it exists, a domain of attraction for the following system of ODE by using an appropriate Lyapunov function.

$$\begin{cases} x' = y \\ y' = -y + y^3 - x^5 \end{cases} \qquad \textbf{(4p)}$$

Solution.

We choose a test function $V(x,y) = x^6 + ay^2$ with unknown positive coefficient a because there are terms x^5 in the second equation and y both in the first and in the second equation. We calculate

$$\nabla V \cdot f = \begin{bmatrix} 6x^5 \\ 2ay \end{bmatrix} \cdot \begin{bmatrix} y \\ -y + y^3 - x^5 \end{bmatrix} = 6x^5y - 2ay^2 + 2ay^4 - 2ayx^5$$

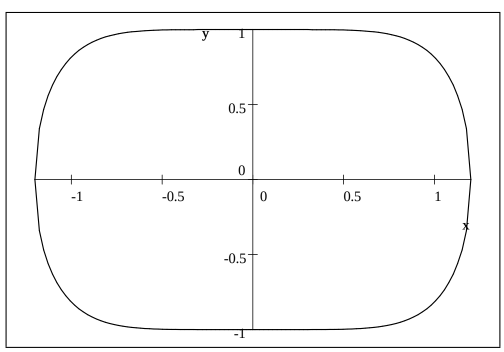
and observe that with the choice $a = 3$ and $V(x,y) = x^6 + 3y^2$ we get:

$$\nabla V \cdot f = 6x^5y - 6y^2 + 6y^4 - 6yx^5 = -6y^2\left(1 - y^2\right) \le 0$$

for $|y| \le 1$. Therefore the stationary point in the origin is stable by Lyapunov's theorem.

To decide if it is asymptotically stable or not we check the set of points (x,y) where $\nabla V \cdot f = 0$ These are points on the x - axis $y = 0$.

We observe that trajectories starting on the x - axis have velocities in y direction $y' = -x^5$ that are zero only in the origin $(0,0)$. Therefore all trajectories starting on the x - axis leave it except the trajectory starting in the origin that is a stationary point. Therefore there are no complete orbits on the x axis except the origin and therefore the origin is asymptotically stable by a corollary to the la'Salle principle. Level sets of of the Lyapunovs function $V(x,y) = x^6 + 3y^2$ are ellipse like closed curves symmetric with respect to coordinate axes. The "largest" such level set inside the stripe $|y| \le 1$ must, because of the symmetry, go through the point $(0,1)$ and is $V(0,1) = 3$. Therefore a domain of attraction of the origin can be identified using this Lyapunov function as the domain inside this level set: $S = \left\{(x,y) : x^6 + 3y^2 < 3\right\}$, see the picture:



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5. Show that the following system of ODEs has at least one periodic solution.

$$\begin{cases} x' = y \\ y' = -4x - y\ln\left(4x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution.

Consider the test function $E(x,y) = \frac{1}{2}\left(4x^2 + y^2\right)$

$$\frac{d}{dt}E(u(t),v(t)) = \begin{bmatrix} 4x \\ y \end{bmatrix} \begin{bmatrix} y \\ -4x - y\left[\ln\left(4x^2 + y^2\right)\right] \end{bmatrix} = -y^2\left[\ln\left(4x^2 + y^2\right)\right] \begin{cases} \ge 0 & 4x^2 + y^2 < 1 \\ \le 0 & 4x^2 + y^2 > 1 \end{cases}$$

The boundary curve $4x^2 + y^2 = 1$ is the ellipse with halv axes $1/2$ and 1 with the center in the origin.

Therefore any ellipse $4x^2+y^2 = const$ with $const < 1$ is never entered by a trajectory starting outside it.

Similarly any ellipse $4x^2 + y^2 = const$ with $const > 1$ is never left by a trajectory.

Such two ellipses build an annulus that is a positively invariant set for this system of ODEs.

It implies that the annulus $1/4 \le 4x^2 + y^2 \le 2$ is a positively invariant set. This annulus contains no equilibrium points, because the origin is the only equilibrium point. Therefore by Poincare - Bendixson theorem this annulus must contain at least one periodic orbit.

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6. Consider the following initial value problem: $y' = \cos(y)t^2$; $y(1) = 2$.

   a) Reduce the initial value problem to an integral equation and give a general description of iterations approximating the solution as in the proof to the existence and uniqueness theorem by Picard and Lindelöf. **(2p)**

   b) Find the length $d > 0$ of a time interval $[1, 1 + d]$ such that these approximations converge to the solution of the initial value problem on this interval. **(2p)**

   Solution.

   The integral form of the I.V.P. is

$$y(t) = 2 + \int_1^t \cos(y(s))s^2 ds$$

Picard approximations are calculated according to the recursive formula

$$\begin{aligned} y^{(0)}(t) &= 2 \\ y^{(k+1)}(t) &= 2 + \int_1^t \cos(y^{(k)}(s))s^2 ds = 2 + B(y^{(k)}) \end{aligned}$$

The norm of the operator B in the space of continuous functions $C([1, 1 + d])$on the interval $[1, 1+d]$ with the norm $\|y\|_{C([1,1+d])} = \sup\limits_{t\in[1,1+d]} |y(t)|$can be estimated using intermediate value theorem:

$$\cos(y(s)) - \cos(x(s)) = (y(s) - x(s)) (-\sin(\xi))$$

where ξ is some point between $y(s)$ and $x(s)$.

$$\begin{aligned} \|B(y) - B(x)\|_{C([1,1+d])} &\le \sup_{t\in[1,1+d]} \left| \int_1^t |\cos(y(s)) - \cos(x(s))| \, s^2 ds \right| \le \\ &\le \left(\sup_{s\in[1,1+d]} |y(s) - x(s)| \right) \left(\sup_{\xi\in\mathbb{R}} |\sin(\xi)| \right) \int_1^{1+d} s^2 ds \le \\ &\le \left(d + d^2 + \frac{1}{3}d^3 \right) \|y - x\|_{C([1,1+d])} \end{aligned}$$

The expression $f(d) = d + d^2 + \frac{1}{3}d^3$ is monotone for $d > 0$ and is strictly smaller then 1 if $d < 0.5$:

$$0.5 + (0.5)^2 + (1/3)(0.5)^3 = 0.791\,67$$

Therefore

$$\|B(y) - B(x)\|_{C([1,1+d])} \le L \|y - x\|_{C([1,1+d])}$$

with $L < 1$ if $d < 0.5$ and the operator B is a contraction in this case. The same reasoning implies that for $d < 0.5$ the operator B maps a unit ball around the constant function 2 in $C([1, 1+d])$ into itself. It implies according to the Banach contraction theorem convergence of Picard iterations to the solution of this initial value problem.

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Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\, Assignment1 + 0.16\, Assignment2 + 0.68\, Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.