MATEMATIK GU, Chalmers A.Heintz

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Tid: 08-30 - 12-30 Tel.: 5329

Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Formulate and give a proof to the theorem by Floquet about boundedness and zero limits of solutions to periodic linear systems of ODE's. **(4p)**

2. Formulate and prove Lyapunov's theorem on stability of equilibrium points. **(4p)**

3. Consider the following system of ODEs: $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$, with the constant matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$.

   i) Find general solution to the system.

   ii) Find all initial conditions giving bounded solutions. **(4p)**

Solution. For the given matrix A, the characteristic polynomial is

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)(-1-\lambda)+4 = 3\lambda^2-\lambda^3-4+4 = \lambda^3-3\lambda^2$$

The equation $p(\lambda) = \lambda^3 - 3\lambda^2 = 0$ gives eigenvalues that are $\lambda_1 = 0$ (with multiplicity 2) and $\lambda_2 = 3$ (simple).

Eigenvectors v^1 and v^2 satisfy equations $(A - 0I)\,v^1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$, and $(A - 3I)v^2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$,or equivalently$\begin{cases} 2x+y=0 \\ 2y+4z=0 \\ x-z=0 \end{cases}$, with solution $[x = z, y = -2z]$, and $\begin{cases} -x+y=0 \\ -y+4z=0 \\ x-4z=0 \end{cases}$, with solution: $[x = 4z, y = 4z]$. We see that the eigenvalue $\lambda_1 = 0$ has only

a one-dimensional set of eigenvectors. We specify one linearly independent eigenvector corresponding to $\lambda_1 = 0$ by choosing $z = 1$: $v^1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and the eigenvalue $\lambda_2 = 3$ has also only one linearly independent eigenvector corresponding to $\lambda_2 = 3 : v^2 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$,(this we could expect because this eigenvalue is simple). We try to calculate a generalized eigenvector $v^{1,(1)}$ corresponding to $\lambda_1 = 0$ by solving the equation

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ or } \begin{cases} 2x+y=1 \\ 2y+4z=-2 \\ x-z=1 \end{cases}, \text{ Solution is: } [x = z+1, y = -2z-1]$$

We choose a generalized eigenvector as $v^{1,(1)} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ taking $z = 1$.

Representing arbitrary initial vector $\vec{r}_0 = C_1 v^1 + C_2 v^{1,(1)} + C_3 v^2$ where $C_1 v^1 + C_2 v^{1,(1)} \in M(0, A)$ and $C_3 v^2 \in M(3, A)$, we get an expression for a general solution as

$$\begin{aligned} \vec{r}(t) &= \left[\sum_{k=0}^{1} \frac{t^k}{k!} (A - 0I)^k \left(C_1 v^1 + C_2 v^{1,(1)} \right) \right] + C_3 e^{3t} v^2 = \\ &= C_1 v^1 + C_2 \left(v^{1,(1)} + t v^1 \right) + C_3 e^{3t} v^2 = \\ &= C_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2+t \\ -3-2t \\ 1+t \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}. \end{aligned}$$

It is easy to observe that solutions are bounded if and only if $r_0 = C_1 v^1$.

4. Consider the following system of ODEs. Find all equilibrium points, investigate their stability and find a possible domain of attraction.

$$\begin{cases} x' = y \\ y' = -y - 6x - 3x^2 \end{cases}.$$

Hint: A Lyapunov function for the equilibrium in the origin can be found as a sum of kinetic and potential energy for the equivalent second order equation $x'' = -x' - 6x - 3x^2$, **(4p)**

Solution.

There are two equilibrium points $(0,0)$ and $(-2,0)$. The Jacobian of the right hand side in the ODE is $J(x,y) = \begin{bmatrix} 0 & 1 \\ -6(1+x) & -1 \end{bmatrix}$.

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -6 & -1 \end{bmatrix}$, $\mathrm{Tr}(J(0,0) = -1 < 0$; $\det(J(0,0)) = 6 > 0$; $(\mathrm{Tr}(J(0,0))^2/4 < \det(J(0,0))$ because $1/4 < 6$. Therefore the origin is an asymptotically stable spiral in linear approximation and is an attractor for the original non-linear equation.

$J(-2,0) = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}$, $\det(J(0,0)) = -6 < 0$. Therefore the equilibrium in the point $(-2,0)$ is a saddle point and is unstable (but not repeller).

We try to find a region of attraction for the asymptotically stable equilibrium in $(0,0)$.

We choose a test function around the origin $(0,0)$ as a sum of kinetic energy and potential energy: $V(x,y) = 1/2y^2 + \int (6x + 3x^2)dx = 1/2y^2 + 3x^2 + x^3$.

$V(x,y)$ is positive definite in some finite domain around the origin.

$$V_f(x,y) = \nabla V \cdot f(x,y) = y\left(-y - 6x - 3x^2\right) + 6x\,(y) + 3x^2\,(y) = -y^2 \leq 0.$$

$V_f(x,y) = 0$ only for $y = 0$. The line $y = 0$ includes just two orbits that are invariant sets: both are stationary points: in the origin $(0,0)$ and in the point $(-2,0)$.

$y' = -6x - 3x^2$ for $y = 0$ and any trajectory leaves the line $y = 0$ in case it crosses it at some point $x \neq 0$ and $x \neq -2$. Therefore the Lasalle's invariance principle also implies that the origin is asymptotycally stable.

One can identify a region of attraction for the origin by choosing the set $D = \{(x,y) : V(x,y) < l\}$ bounded by a level set $V(x,y) = l$ so that it does not include the unstable equilibrium point $(-2,0)$.

The LaSalle's invariance principle will imply that all trajectories starting in such a region would be attracted to the origin.

We find the largest possible region of attraction of such kind by choosing $l = V(-2,0) = 4$ that is the value of V in the point that we like to exclude. The set

$$D = \{(x,y) : V(x,y) < 4\}$$

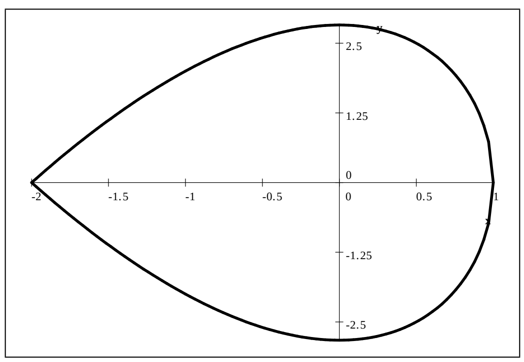
has the boundary $V(x,y) = 4$ that goes exactly through the point $(-2,0)$.

Solving the equation $1/2y^2 + 3x^2 + x^3 = 4$ with respect to y we find that D is bounded by two lines

$$y = \pm\sqrt{8 - 6x^2 - 2x^3}$$

symmetric with respect to the x - axis. By construction the expression $8 - 6x^2 - 2x^3$under the root is equal to zero at the point $x = -2$. Another root of this expression $x = 1$ is easy to guess. It makes it easy to factorise this expression: $8 - 6x^2 - 2x^3 = (-2)(x-1)(x+2)^2$. The expression under the root is positive on the interval $(-2,1)$. It is enough to identify the region of attraction as the domain D above.

The derivative of this expression: $\frac{d}{dx}\left(8 - 6x^2 - 2x^3\right) = -12x - 6x^2 = (-6)\,x\,(x+2)$ changes the sign in $x = 0$ from plus to minus. It means that $\sqrt{8 - 6x^2 - 2x^3}$ rises on the interval $[-2,0)$ and decreases on the interval $(0,1)$. It implies the shape of the upper part of the boundary of D with one maximum in $x = 0$ and the geometry of the region of attraction D in the sketch below.



5. Show that the following system of ODEs has at least one periodic solution.

$$\begin{cases} x' = y - x\left(x^2 + y^2 - 2x - 3\right) \\ y' = -x - y\left(x^2 + y^2 - 2x - 3\right) \end{cases} \qquad \textbf{(4p)}$$

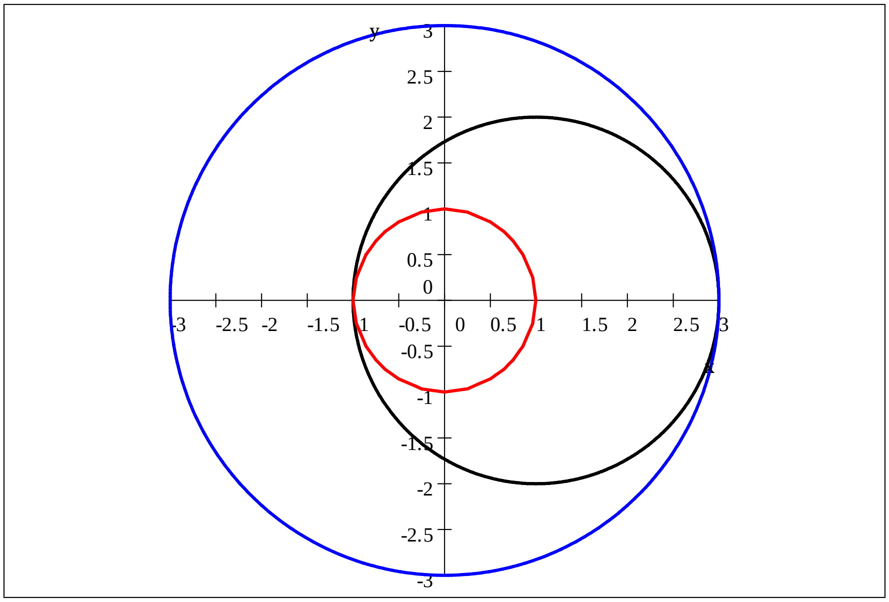
Solution. We are going to construct a positively invariant set that does not include any stationary point and conclude using Poincare Bendixson theorem. A positively invariant set is constructed by using a simple text function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$ and considering it's derivative along trajectories:

$$V_f(x,y) = -\left(x^2 + y^2\right)\left(x^2 + y^2 - 2x - 3\right) = -\left(x^2 + y^2\right)\left((x-1)^2 + y^2 - 4\right)$$

The expression $\left((x-1)^2 + y^2 - 4\right)$ is negative inside the circle

$$(x-1)^2 + y^2 = 4$$

with the center in the point $x = 1$, $y = 0$ radius 2 and is positive outside this circle.



It implies that the $V_f(x,y) < 0$ inside the large circle $x^2 + y^2 = 9$ including the circle $(x-1)^2 + y^2 = 4$ and $V_f(x,y) > 0$ inside the circle $x^2 + y^2 = 1$ that is completely inside the circle $(x-1)^2 + y^2 = 4$. Therefore the ring-shaped set $1 \leq x^2 + y^2 \leq 9$ is a positively invariant set for the given system.

Stationary points of the system satisfy the system of equations $\begin{cases} 0 = y - x\left(x^2 + y^2 - 2x - 3\right) \\ 0 = -x - y\left(x^2 + y^2 - 2x - 3\right) \end{cases}$ and therefore satisfy the equation $V_f(x,y) = -\left(x^2 + y^2\right)\left(x^2 + y^2 - 2x - 3\right) = 0$. Solutions to this equation are the origin and also points satisfying the equation $\left(x^2 + y^2 - 2x - 3\right) = 0$. In the second case we see from the equations that stationary points (x,y) must also belong to the origin. Therefore the only stationary point of the system is the origin that does not belong to the constructed positively invariant set $\left\{(x,y) : \quad 1 \leq x^2 + y^2 \leq 9\right\}$.

Therefore by Poincare -Bendixsons theorem the system must have at least on periodic orbit inside this set.

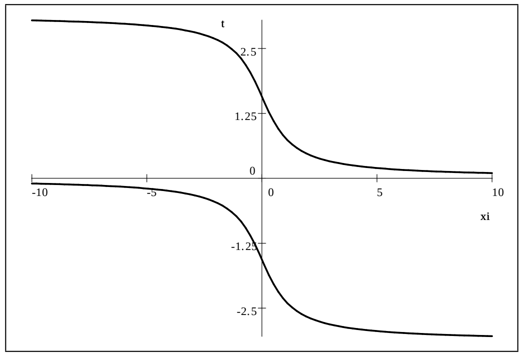
6. Consider the equation
$$x' = 1 + x^2$$

Find the maximal interval $I_{\max}(\xi)$ for arbitrary initial data $x(0) = \xi$. Describe corresponding transfer mapping $\varphi(t,\xi)$ and give a sketch of it's domain of definition D in the plane of ξ and t. **(4p)**

Solution.
$$\frac{dx}{1+x^2} = dt$$

$$\begin{aligned}
\arctan(x) &= t + C, \\
x(0) &= \xi. \\
C &= \arctan(\xi); \\
\arctan(x) &= t + \arctan(\xi), \\
x(t) &= \varphi(t,\xi) = \tan\left(t + \arctan(\xi)\right) \\
I_{\max}(\xi) &= \left(-\pi/2 - \arctan(\xi), \pi/2 - \arctan(\xi)\right) \\
\arctan(\xi) &\to \pi/2, \quad \xi \to \infty \\
\arctan(\xi) &\to -\pi/2, \quad \xi \to -\infty
\end{aligned}$$

The domain of $\varphi(t,\xi)$ is the open set in the plane (ξ,t) between lines $t=-\pi/2-\arctan(\xi)$ and $t=\pi/2-\arctan(\xi)$ consisting of maximal intervals $I_{\max}(\xi)$ for each $\xi\in\mathbb{R}$. See the picture:



Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.