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## Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Formulate and give a proof to the theorem about the dimension of the solution space of the system of linear ODEs. Check the formulation and the proof in lecture notes. (4p)
2. Formulate and give a proof to Lyapunov's stability theorem. Check the formulation and the proof in the lecture notes.
3. Consider the following matrix $A=\left[\begin{array}{ccc}-1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & 3 & -3\end{array}\right]$
a) Write down a canonical Jordan form $J$ for the matrix $A$ and find the corresponding matrix $P$ in the relation $J=P^{-1} A P$ using eigenvectors and generalised eigenvectors to $A$ (do not calculate $P^{-1}$ ).
b) Write down a general solution to the system $x^{\prime}=A x$ with this matrix $A$.Find all initial data such that corresponding solutions are bounded.
(4p)
Solution. The characteristic polinomial for the matrix $A=\left[\begin{array}{ccc}-1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & 3 & -3\end{array}\right]$ is $p(\lambda)=$ $\lambda^{3}+3 \lambda^{2}$.
(a) Eigenvalues are $\lambda_{1}=0$ with multiplicity 2 and $\lambda_{2}=-3$.

Eigenvector $v_{1}$ corresponding to $\lambda_{1}$ satisfies the equation $A v_{1}=0$ and can be chosen as $\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]$. The generalized eigenvector $v_{1}^{(1)}$ satisfies the equation $A v_{1}^{(1)}=v_{1}$ and can be chosen as $\left[\begin{array}{c}1 / 3 \\ 4 / 3 \\ 1\end{array}\right]$. The eigenvector $v_{2}$ corresponding to the simple eigenvalue $\lambda_{2}=-3$ satisfies the the homogeneous equation $(A+3 I) v_{2}=0$ with the matrix $(A+3 I)=\left[\begin{array}{ccc}2 & 1 & -1 \\ 2 & 4 & -1 \\ 0 & 3 & 0\end{array}\right]$ and can be chosen as $v_{2}=\left[\begin{array}{l}1 \\ 0 \\ 2\end{array}\right]$.
The canonical Jordan form $J$ for the matrix $A$ has the form $J=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3\end{array}\right]$ with matrix $P=\left[\begin{array}{ccc}0 & 1 / 3 & 1 \\ 1 & 4 / 3 & 0 \\ 1 & 1 & 2\end{array}\right]$ with columns that are eigenvectors and a generalised eigenvector.

The general solution $x(t)=e^{A t} x_{0}$ to the equation can be written by choosing the initial data $x_{0}$ expressed in terms of the basis of eigenvectors and generalized eigenvectors: $x_{0}=C_{1} v_{1}+C_{2} v_{1}^{(1)}+C_{3} v_{2}$ and using for each term the representation for the exponent $e^{A t}$ acting on an element $x^{0, j}$ of a particular generalized eigenspace:
$e^{A t} x^{0, j}=\left[\sum_{k=0}^{m_{j}-1}\left(A-\lambda_{j} I\right)^{k} \frac{t^{k}}{k!}\right] e^{\lambda_{j} t} x^{0, j}$
where $m_{j}$ is the algebraic multiplicity of the eigenvalue $\lambda_{j}$ and $x^{0, j}$ is an element of the corresponding generalized eigenspace. It implies that the general solution is the linear combination of expressions of this type and is expressed in our particular case as
$x(t)=C_{1} v_{1}+C_{2} v_{1}^{(1)}+t C_{2} v_{1}+C_{3} e^{-3 t} v_{2}$
because $m_{1}=2, m_{2}=1$, and $\left(A-\lambda_{1} I\right) v_{1}^{(1)}=v_{1}$. The solutions will be bounded for all $x_{0}$ in the form $x_{0}=C_{1} v_{1}+C_{3} v_{2}$.
4. The following system of equations describes the evolution of variables $x$ and $y$ representing scaled numbers of two competing species.
$\left\{\begin{array}{l}x^{\prime}=x(2-x-y) \\ y^{\prime}=y(3-2 x-y)\end{array}\right.$
Explain by analysing system's equilibrium points, and nullclines, how these equations make it mathematically possible but extremely unlikely for both species to survive together.
(4p)
Solution. There is only one possible equilibrium point: $x=1, y=1$ with both components non-zero. It is the only possible point for both species to survive together.
We try to analyse stability of this point using linearization. The variational matrix is $A(x, y)=\left.\left[\begin{array}{cc}2-2 x-y & -x \\ -2 y & 3-2 x-2 y\end{array}\right]\right|_{(x, y)=(1,1)}=\left[\begin{array}{cc}-1 & -1 \\ -2 & -1\end{array}\right]$,
The characteristic equation is $\lambda^{2}+2 \lambda-1=0$.
It's eigenvectors and eigenvalues in the point $(x, y)=(1,1)$ are: $v^{1}=\left\{\left[\begin{array}{c}\frac{1}{2} \sqrt{2} \\ 1\end{array}\right]\right\} \leftrightarrow$ $\lambda^{1}=-\sqrt{2}-1<0 ; v^{2}=\left[\begin{array}{c}-\frac{1}{2} \sqrt{2} \\ 1\end{array}\right] \leftrightarrow \lambda_{2}=\sqrt{2}-1>0 ;$
The linearized system has a saddle point in the origin, that is hyperbolic because both eigenvalues have a nonzero real part. General solution to the linearized system is $r=C_{2} e^{-(\sqrt{2}+1) t} v^{1}+C_{2} e^{(\sqrt{2}-1) t} v^{2}$. The only initial data giving solutions tending to the origin with $t \rightarrow \infty$ are those on the line $r=C_{1} v^{1}$.
The Grobman-Hartmann theorem states that in the neighbourhood of the point $(1,1)$ the phase portrait of the original nonlinear system is homeomorfic to the one of the linear system $\vec{r}^{\prime}=\left[\begin{array}{ll}-1 & -1 \\ -2 & -1\end{array}\right] \vec{r}$. Therefore there is an orbit through the equilibrium point $(1,1)$ such that evolution of the original system starting on this orbit leads to the equilibrium point. All other trajectories escape this equilibrium point as they do for the linearized equation.
The nulclines to the system that are $x=0, x+y=2$ that are $x$ - nullclines and $y=0,2 x+y=3$, that are $y$ - nullclines.


The analysis of directions of velocities in the domains in the first quadrant that are bounded by the nullclines shows that almost all trajectories tend to two equilibrium points $(0,3)$ and $(2,0)$ leading to extinction of one of the species, except two particular orbits discused above and tending to the non-stable saddle point $(1,1)$.
5. Investigate the stability of the origin and find it's possible domain of stability for the following system of ODE by using an appropriate Lyapunov function.

$$
\left\{\begin{array}{l}
x^{\prime}=y  \tag{4p}\\
y^{\prime}=-y+y^{3}-x^{5}
\end{array}\right.
$$

## Solution.

We choose a test function $V(x, y)=x^{6}+a y^{2}$ with unknown positive coefficient $a$ because there are terms $x^{5}$ in the second equation and $y$ both in the first and in the second equation. We calculate
$\nabla V \cdot f=\left[\begin{array}{l}6 x^{5} \\ 2 a y\end{array}\right] \cdot\left[\begin{array}{l}y \\ -y+y^{3}-x^{5}\end{array}\right]=6 x^{5} y-2 a y^{2}+2 a y^{4}-2 a y x^{5}$
and observe that with the choice $a=3$ and $V(x, y)=x^{6}+3 y^{2}$ we get:
$\nabla V \cdot f=6 x^{5} y-6 y^{2}+6 y^{4}-6 y x^{5}=-6 y^{2}\left(1-y^{2}\right) \leq 0$
for $|y| \leq 1$. Therefore the stationary point in the origin is stable by Lyapunov's theorem.

To decide if it asymptotically stable or not we check the set of points $(x, y)$ where $\nabla V \cdot f=0$ These are points on the $x$ - axis: $y=0$.
We observe that trajectories starting on the $x$ - axis have velocities in the $y$-direction $y^{\prime}=-x^{5}$ that are zero only in the origin $(0,0)$. Therefore all trajectories starting on the $x$ - axis leave it except the trajectory starting in the origin that is a stationary point. Therefore there are no complete orbits on the $x$ axis except the origin and the origin is asymptotically stable by a corollary to the Krasovsky - la'Salle principle. Level sets of of the Lyapunovs function $V(x, y)=x^{6}+3 y^{2}$ are ellipse like closed curves symmetric with respect to coordinate axes. The "largest" such level set inside the stripe $|y| \leq 1$ must, because of the symmetry, go through the point $(0,1)$ and is $V(0,1)=3$. Therefore the domain of attraction for the equilibrium in the origin can be identified as the domain inside the level set $V(x, y)=3$ :

$$
S=\left\{(x, y): x^{6}+3 y^{2}<3\right\}
$$

| $\begin{array}{ll} \mathrm{y} & 1 \\ & \\ & 0.5 \\ & \\ & 0 \end{array}$ |  |
| :---: | :---: |
|  -1 -0.5  <br>     <br>    -0.5 <br>     <br>     <br>     |  |

6. Show that the following system of ODE-s has a periodic solution.

$$
\left\{\begin{array}{l}
x^{\prime}=4 x+y-x\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)  \tag{4p}\\
y^{\prime}=-x+4 y-y\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)
\end{array}\right.
$$

Hint. The Cauchy inequality $|a b| \leq 0.5\left(a^{2}+b^{2}\right)$ can be useful for analysis here.
Solution. We like to apply the Poincare-Bendixson theorem to prove that the system has a periodic solution by showing that some of it's trajectories must have a periodic orbit as an $\omega$-limit set. To show it we find a positively - invariant set that does not include equilibrium points. By the Poincare-Bendixson theorem all trajectories starting in this positively invariant set will have an $\omega$ limit set that is a periodic orbit.
We consider the test function $V(x, y)=\frac{1}{2}\left(x^{2}+y^{2}\right)$ and try to find two such circles (level sets to $V(x, y)$ ) that they bound a positively invariant set.

$$
\begin{aligned}
& V_{f}(x, y)=\left[\begin{array}{l}
4 x+y-x\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right) \\
-x+4 y-y\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]= \\
& x\left(4 x+y-x\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right)+y\left(-x+4 y-y\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right)= \\
& {\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right]\left(x^{2}+y^{2}\right) .}
\end{aligned}
$$

We see that $V_{f}(x, y)<0$ for $4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)<0$ and $V_{f}(x, y)>0$ for $4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)>0$.
The curve $4=\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)$ is an ellipse (red curve on the picture) because the expression $\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)$ is positive definite by the Cauchy inequality $|x y| \leq$ $0.5\left(x^{2}+y^{2}\right):$
$5 x^{2}-2 \sqrt{3} x y+7 y^{2} \geq 5 x^{2}-2 \sqrt{3}\left(x^{2}+y^{2}\right) 0.5+7 y^{2}=x^{2}(5-\sqrt{3})+y^{2}(7-\sqrt{3})>0$, $(x, y) \neq 0$.

One can also observe it by investigating eigenvalues of the matrics corresponding this quadratic form:
$Q(x, y)=5 x^{2}-2 \sqrt{3} x y+7 y^{2}=\left[\begin{array}{ll}x & y\end{array}\right]\left[\begin{array}{cc}5 & -\sqrt{3} \\ -\sqrt{3} & 7\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right] \cdot\left[\begin{array}{cc}5 & -\sqrt{3} \\ -\sqrt{3} & 7\end{array}\right]$.
The matrix $\left[\begin{array}{cc}5 & -\sqrt{3} \\ -\sqrt{3} & 7\end{array}\right]$ has eigenvalues: $4,8>0$, eigenvectors are orthogonal vectors $\left[\begin{array}{c}\sqrt{3} / 2 \\ 0.5\end{array}\right]$ and $\left[\begin{array}{c}-0.5 \\ \sqrt{3} / 2\end{array}\right]$ that define the orientation of the ellips.
This ellipse separates the area where $V_{f}(x, y)<0$ and trajectories of the system go inside circles, that are level sets of $V(x, y)$ from the area where $V_{f}(x, y)>0$ and trajectories of the system go outside circles that are level sets of $V(x, y)$.

Finding two circles $x^{2}+y^{2}=R^{2}$ and $x^{2}+y^{2}=r^{2}, R>r>0$ such that the first one is completely outside the ellipse $4=\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)$ and the second one is completely inside the ellipse, will give us the desired ring shaped positively invariant set: $r^{2}<x^{2}+y^{2}<R^{2}$. It is intuitively evident that such $R$ - large enough and $r$ small enough exist.
Then we must check that the ring shaped positively invariant set does not contain any equilibrium points. In any equilibrium point we must have $V_{f}(x, y)=0$. It implies that $\left(x^{2}+y^{2}\right)\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right]=0$ that gives us that an equilibrium point must be in the origin, that is outside our positively invariant set, or on the ellipse $4=\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right) . \quad$ We observe from the ODE, that on this ellipse $x^{\prime}=y$ and $y^{\prime}=-x$. Therefore equilibrium points can be only the origin $(x, y)=(0,0)$. It is outside the ellipse and outside the positively invariant set.
Therefore all trajectories starting in the positively invariant set $r^{2}<x^{2}+y^{2}<R^{2}$ must have an $\omega$ - limit set inside it and this limit set must be a periodic orbit by the Poincare-Bendixson theorem. Therefore the system must have at least one periodic orbit inside the positively invariant set.
We can also find some explicit estimates for $R$ and $r$.
We consider the expression $\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right]$ and try to find a circle $x^{2}+y^{2}=$ $R^{2}$ such that $\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right]<0$ on it.
$\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right] \leq\left[4-5 x^{2}+2 \sqrt{3}|x y|-7 y^{2}\right] \leq\left[4-5 x^{2}+\sqrt{3}\left(x^{2}+y^{2}\right)-7 y^{2}\right]$
$\left[4-5 x^{2}+\sqrt{3}\left(x^{2}+y^{2}\right)-7 y^{2}\right]=4-(5-\sqrt{3}) x^{2}-(7-\sqrt{3}) y^{2} \leq 4-(5-\sqrt{3}) x^{2}-$ $(5-\sqrt{3}) y^{2} \leq 0$.
Therefore for $x^{2}+y^{2}=R^{2} \geq 4 /(5-\sqrt{3})$ the desired inequality $V_{f}(x, y) \leq 0$ is valid. We found an outer boundary of the ring shaped positively invariant set.
$R \geq 2$ for example would work.
We consider the expression $\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right]$ and try to find a circle $x^{2}+y^{2}=$ $r^{2}$ such that $\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right] \geq 0$ on this circle.

$$
\begin{aligned}
& {\left[4-\left(5 x^{2}-2 \sqrt{3} x y+7 y^{2}\right)\right] \geq\left[4-\left(5 x^{2}+2 \sqrt{3}|x y|+7 y^{2}\right)\right] \geq} \\
& {\left[4-\left(5 x^{2}+2 \sqrt{3}|x y|+7 y^{2}\right)\right] \geq\left[4-(5+\sqrt{3}) x^{2}-(7+\sqrt{3}) y^{2}\right] \geq\left[4-(7+\sqrt{3}) x^{2}-(7+\sqrt{3}) y^{2}\right]} \\
& 0
\end{aligned}
$$

Therefore for $x^{2}+y^{2}=r^{2}<4 /(7+\sqrt{3})$ the desired inequality $V_{f}(x, y) \geq 0$ is valid. We have found the internal boundary for the ring shaped positively invariant set that finally is defined by $\left\{4 /(7+\sqrt{3})<x^{2}+y^{2}<4 /(5-\sqrt{3})\right\}$. Check the picture of the ellips and two circles that we found.


Max. 24 points;
Threshold for marks: for GU: VG: 19 points; G: 12 points. For Chalmers: 5: 21 points; 4: 17 points; 3: 12 points;

One must pass both the home assignments and the exam to pass the course.
Total points for the course are calculated as:
Total $=0.16$ Assignment $1+0.16$ Assignment $2+0.68$ Exam - that is the average of the points for the home assignments (32\%) and for this exam ( $68 \%$ ). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will collect all necessary points.

