MATEMATIK	Datum: 2021-05-31	Tid: 8-30 - 12-30
GU, Chalmers	Hjälpmedel: - Alla	
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Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.

Report details of calculatios leading to the result. Just answers without proofs will not be counted.

1. Consider the following system of ODEs: $\dfrac{dx(t)}{dt} = Ax(t)$, with

$$A = \begin{bmatrix} 5 & -1 & -4 \\ -12 & 5 & 12 \\ 10 & -3 & -9 \end{bmatrix}$$

Find the general solution for this system of differential equations. Find all initial conditions that give bounded solutions. **(4p)**

Solution.

Characteristic polynomial to A: $p(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1 = (\lambda + 1)(\lambda - 1)^2$.

A has two eigenvalues $\lambda_1 = -1$ that is simple and $\lambda_2 = 1$ that is multiple. We find corresponding eigenvectors. The eigenvector to $\lambda_1 = -1$ satisfies to the homogeneous system of equations with matrix $A+I = \begin{bmatrix} 6 & -1 & -4 \\ -12 & 6 & 12 \\ 10 & -3 & -8 \end{bmatrix}$, Gaussian elimination leads to $\begin{bmatrix} 6 & -1 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ that has a one dimensional solution space of eigenvectors. We choose one of them as $v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

The eigenvector to $\lambda_2 = -1$ satisfies to the homogeneous system of equations with matrix $A - I = \begin{bmatrix} 4 & -1 & -4 \\ -12 & 4 & 12 \\ 10 & -3 & -10 \end{bmatrix}$, Gaussian elimination leads to $\begin{bmatrix} 4 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ that also has a one dimensional eigenspace. We choose one corresponding eigenvector as $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. To find a basis of solution space it is convenient to find a generalized eigenvector $v_2^{(1)}$ corresponding to $\lambda_2 = 1$.

We try to find is by solving the equation

$$(A - I)v_2^{(1)} = v_2$$

This equation has the extended matrix as $\begin{bmatrix} 4 & -1 & -4 & 1 \\ -12 & 4 & 12 & 0 \\ 10 & -3 & -10 & 1 \end{bmatrix}$, Gaussian elimination leads to the matrix $\begin{bmatrix} 4 & -1 & -4 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and a solution (not unique) to the homogenepus problem in the form $v_2^{(1)} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

that is linearly independent of $\{v_1, v_2\}$.

The general solution to the differential equation $\frac{dx(t)}{dt} = Ax(t)$ with initial data $\xi = C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}$ with arbitrary constants C_1, C_2, C_3 will have the form:

$$x(t) = C_1 e^{-t} v_1 + C_2 e^t v_2 + C_3 \left(e^t v_2^{(1)} + t e^t v_2 \right)$$

Solutions to initial value problems with initial data ξ are bounded in the case when ξ belongs to the line through the origin that is parallel to v_1.■

2. For one particular solution of the system $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$ with a real matrix A, the first component has the form $x_1 = 2t^3 e^t + t^2 + 5t\sin(3t)$.

   Which is the smallest size that the real matrix A can have? **(4p)**

   Solution.

   The structure of the given component solution implies that the Jordan form of the real matrix A must have the eigenvalue $\lambda_1 = 1$ with multiplicity at least 4, the eigenvalue $\lambda_2 = 0$ with multiplicity at least 3, and a pair of complex conjugate complex eigenvalues $\lambda_3 = 3i$ and $\lambda_4 = -3i$ each having multiplicity at least 2.

   The total dimension of corresponding generalized eigenvalues is at least: $\dim = 4+3+2+2 = 11$. It implies that the size of the matrix A must be at least 11×11.■

3. Consider the following periodic system of linear differential equations $x' = A(t)x$ with the following matrix: $A(t) = \begin{bmatrix} -\sin^2(t) & \exp(-\cos^2(t)) \\ \exp(-\sin^2(t)) & a\cos^2(2t) \end{bmatrix}$.

   Find such values of a that this system has at least one unbounded solution $x(t)$. **(4p)**

   Solution.

   There is a theorem based on the Abel's formula stating that if for a periodic linear system $x' = A(t)$ of differential equations with period p

   $$\int_0^p \mathrm{tr}A(s)ds$$

   has positive real part, then the differential equation has at least one unbounded solution such that $\sup\lim_{t\to\infty} \|x(t)\|$.

   The period of the given system of equations is $p = \pi/2$ because $\cos^2(2t) = \frac{1+\cos(4t)}{2}$, and $\sin^2(t) = \frac{1-\cos(2t)}{2}$. $\mathrm{tr}A(t) = -\sin^2(t) + a\cos^2(2t)$.

   $\int_0^p \mathrm{tr}A(s)ds = \int_0^{\pi/2} \left(-\sin^2(s) + a\cos^2(2s)\right) ds = \frac{1}{4}\pi a - \frac{1}{4}\pi = \frac{1}{4}\pi\,(a-1)$.

   Therefore for $a > 1$ at least one solution will be unbounded.■

4. Show that the following system of ODEs has a periodic solution.

   $$\begin{cases} x' = -y - x(x^2 + 2y^2 - xy - 1) \\ y' = x - y(x^2 + 2y^2 - xy - 1) \end{cases}$$ **(4p)**

   Solution.

   We choose a test function $V(x,y) = x^2 + y^2$.

   $V_f(x,y) = -2x^2(x^2+2y^2-xy-1) - 2y^2(x^2+2y^2-xy-1) = -2(x^2+y^2)(x^2+2y^2-xy-1)$

   The sign of $V_f(x,y)$ is defined by the sign of the function $x^2 + 2y^2 - xy - 1$.

   Point out that $|2xy| \le x^2+y^2$.Therefore $x^2+2y^2-xy \ge x^2+2y^2-0.5\left(x^2+y^2\right) = 0.5x^2+1.5y^2$ and $x^2+2y^2-xy-1 \ge \frac{x^2+3y^2}{2} - 1$.

   The last expression is positive outside the ellipse $\frac{x^2+3y^2}{2} = 1$ with semiaxes $\sqrt{2}$ and $\sqrt{2/3}$ in x and y directions.

Similar reasons imply that $x^2+2y^2-xy-1 \le x^2+2y^2+0.5\left(x^2+y^2\right)-1 = 1.5x^2+2.5y^2-1 \le 0$ for $\frac{3}{2}x^2+\frac{5}{2}y^2 \le 1$ inside the ellipse with semiaxes $\sqrt{2/3}$ and $\sqrt{2/5}$ in x and y directions.

It implies that the ring B bounded by circles (level sets of the test function V) with radii $\sqrt{2}$ and $\sqrt{2/5}$ is positive definite (trajectories starting inside it do not leave it). The origin is the only equilibrium point and does not belong to the ring B. It is easy to see by analysis of $V_f(x,y)=0$ and right hand sides of the equations.

The Poincare Bendixsons theorem implies that there any positive semi orbit in B must have an ω limit set and this ω limit set can be only a periodic orbit.

5. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction.

$$\begin{cases} x' = & y-\exp(x)+1 \\ y' = & -2x-3\sin(y) \end{cases}$$

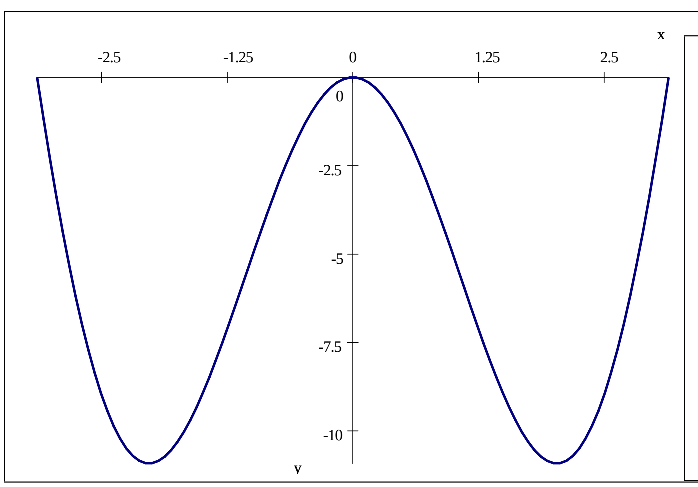
Jacobi matrix in the origin is $A=\begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}$, characteristic polynomial: $\lambda^2+4\lambda+5$, eigenvalues: $-2-i, -2+i$ **(4p)**

Solution.

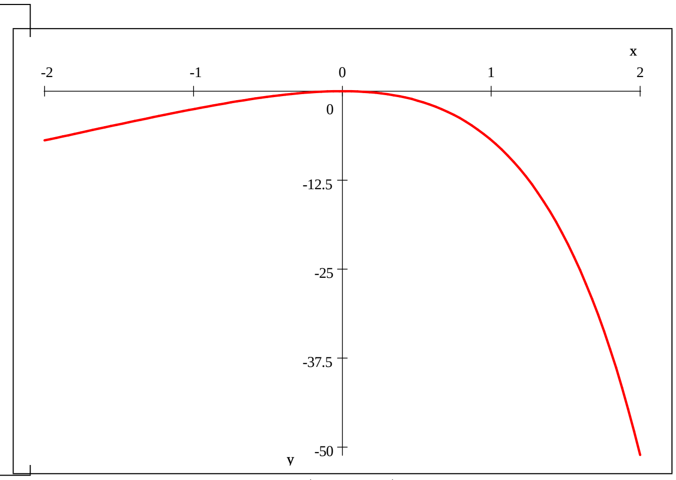
One can try first to check the linearized system if equations $\begin{cases} x' = & -x+y \\ y' = & -2x-3y \end{cases}$ with matrix $A=\begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}$, having characteristic polynomial $p(\lambda)=\lambda^2+4\lambda+5$ and eigenvalues: $\lambda_1=-2-i, \lambda_2=-2+i$. Both eigenvalues have negative real parts. It implies that the equilibrium point in the origin is asymptotically stable.

We try to apply Lyapunovs theory to find a region of attraction for this equilibrium point. Try test function $V(x,y)=2x^2+y^2$. The motivation for this choice is that indefinite terms in the expression for $V_f(x,y)$ would cancel for such test function.

$$V_f(x,y)=4x(y-\exp(x)+1)+2y(-2x-3\sin(y))=-6y\sin y+4x(1-e^x)$$



$-6y\sin y$



$4x(1-e^x)$

Elementray properties of functions exp and sin imply that $-6y\sin y+4x(1-e^x)$ is negative on the set $B=(-\infty,\infty)\times(-\pi,\pi)$ except the origin where $V_f(0,0)=0$.

It implies that the origin is an asymptotically stable equilibrium point with the region of attraction bounded by an that is a level set of $V(x,y)$ (an ellipse) that fits into the set B. The largest such ellipse is $V(x,y)=2x^2+y^2=C$ with $C=\pi^2$.■

6. Consider the differential equation $x' = f(x)$ where f is a Lipschitz function $f : G \to \mathbb{R}^n$ and $G \subset \mathbb{R}^n$ is an open set.

   Consider a point $x_0 \in G$ such that the the closure of the semi orbit $O^+(x_0)$ starting in x_0 is compact and is contained in G. Prove that the $\omega-$ limit set $\Omega(x_0)$ must be a connected set. Hint: use the argument with contradiction and the fact that the $\omega-$ limit set $\Omega(x_0)$ is compact. **(4p)**

   Solution.

   Suppose that the ω - limit set is not connected, namely that there are to disjoint compact sets C_1 and C_2 such that $\Omega(x_0) = C_1 \cup C_2$ and $\text{dist}(C_1, C_2) = \delta > 0$.Introduce two functions $d_i(t) = \text{dist}(\varphi(t, x_0), C_i)$. There are two sequences of times $\left\{t_k^1\right\}_k^\infty$ and $\left\{t_k^2\right\}_k^\infty \to \infty$ with $k \to \infty$, such that $\varphi(t_k^1, x_0)$ converges to an $\omega-$ limit point in C_1 and $\varphi(t_k^2, x_0)$ converges to an $\omega-$ limit point in C_2.

   Therefore there is $K \in N$ such that for all $k > K$

   $$d_1(t_k^1) < \delta/2, \quad d_2(t_k^2) < \delta/2,$$

   Since because of the triangle inequality $d_1(t) + d_2(t) \geq \delta$ for all times t and since the functions $d_1(t)$ and $d_2(t)$ are continuous, there must exist a time t_k between t_k^1 and t_k^2 such that

   $$d_1(t_k) = \delta/2, \quad d_2(t_k) \geq \delta/2,$$

   showing that $\text{dist}(\varphi(t_k, x_0), \Omega(x_0)) \geq \delta/2 > 0$ for all $k > K$.

   On the other hand the sequence $\varphi(t_k, x_0)$ belongs to a compact set $\overline{(O^+(x_0))}$ contained in G and must have a converging subsequence with a limit in $\Omega(x_0)$ that contradicts to our previous conclusion.■

   Max. 24 points; Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\, Assignment1 + 0.16\, Assignment2 + 0.68\, Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.