## Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Report details of calculatios leading to the result. Just answers without proofs will not be counted.

1. Consider the following system of ODEs: $\frac{d x(t)}{d t}=A x(t)$, with $A=\left[\begin{array}{cc}4 & -1 \\ 1 & 2\end{array}\right]$.

Find the transition matrix for this system of differential equations.

## Solution.

$\Phi(t, 0)=\exp \left(t\left[\begin{array}{cc}4 & -1 \\ 1 & 2\end{array}\right]\right)=\left[\varphi_{1}(t), \varphi_{2}(t)\right]$, where functions $\varphi_{1}(t), \varphi_{2}(t)$ are solutions to the equation $\frac{d x(t)}{d t}=A x(t)$ satisfying initial conditions $\varphi_{1}(0)=\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and $\varphi_{2}(0)=\left[\begin{array}{l}0 \\ 1\end{array}\right]$.
We start with calculating eigenvalues of $A$.
Characteristic polynomial is $p(\lambda)=\lambda^{2}-6 \lambda+9=(\lambda-3)^{2}$. The only eigenvalue is multiple: $\lambda_{1,2}=3$.
Corresponding eigenectors satisfy the system $(A-\lambda I) v=0$ with matrix $\left[\begin{array}{ll}1 & -1 \\ 1 & -1\end{array}\right]$ giving the only linearly independent eigenvector $v=\left[\begin{array}{l}1 \\ 1\end{array}\right]$. The generalized eigenvector can be found from the equation

$$
(A-\lambda I) v^{(1)}=v
$$

with the extended matrix $\left[\begin{array}{lll}1 & -1 & 1 \\ 1 & -1 & 1\end{array}\right]$. It gives the generalized eigenvector $v^{(1)}=\left[\begin{array}{l}1 \\ 0\end{array}\right]$.
The general solution to the differential equation is $x(t)=A x$ with initial condition:

$$
x(0)=C_{1} v+C_{2} v^{(1)}
$$

is

$$
\begin{aligned}
x(t) & =C_{1} e^{3 t}\left[\begin{array}{l}
1 \\
1
\end{array}\right]+C_{2} e^{3 t}\left(\left[\begin{array}{l}
1 \\
0
\end{array}\right]+t\left[\begin{array}{l}
1 \\
1
\end{array}\right]\right)= \\
& =\left[\begin{array}{c}
C_{1} e^{3 t}+C_{2} e^{3 t}(t+1) \\
C_{1} e^{3 t}+C_{2} t e^{3 t}
\end{array}\right]
\end{aligned}
$$

Columns in the transition matrix $\Phi(t, 0)$ satisfy initial conditions: $\varphi_{1}(0)=\left[\begin{array}{l}1 \\ 0\end{array}\right]=v^{(1)}$; $\varphi_{2}(0)=\left[\begin{array}{l}0 \\ 1\end{array}\right]=v-v^{(1)}$.
Therefore $\varphi_{1}(t)=\left[\begin{array}{c}e^{3 t}(t+1) \\ t e^{3 t}\end{array}\right] ; \varphi_{2}(t)=\left[\begin{array}{c}e^{3 t}-e^{3 t}(t+1) \\ e^{3 t}-t e^{3 t}\end{array}\right]=\left[\begin{array}{c}-t e^{3 t} \\ e^{3 t}-t e^{3 t}\end{array}\right]$
Therefore the transition matrix can be expressed as:
$\Phi(t, 0)=\exp \left(t\left[\begin{array}{cc}4 & -1 \\ 1 & 2\end{array}\right]\right)=\left[\begin{array}{cc}e^{3 t}+t e^{3 t} & -t e^{3 t} \\ t e^{3 t} & e^{3 t}-t e^{3 t}\end{array}\right]=e^{3 t}\left[\begin{array}{cc}t+1 & -t \\ t & -t+1\end{array}\right]$

Another way to express $\Phi(t, 0)$ is using te Jordan block corresponding to $A$.
$\Phi(t, 0)=V \exp (J t) V^{-1}=\left[\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right] \exp (3 t)\left[\begin{array}{ll}1 & t \\ 0 & 1\end{array}\right]\left[\begin{array}{cc}0 & 1 \\ 1 & -1\end{array}\right]=$ $\left[\begin{array}{cc}e^{3 t}+t e^{3 t} & t e^{3 t} \\ -t e^{3 t} & e^{3 t}-t e^{3 t}\end{array}\right]$, where : $V=\left[\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right], V^{-1}=\left[\begin{array}{cc}0 & 1 \\ 1 & -1\end{array}\right]$.
2. For one particular solution of the system $\frac{d \mathbf{x}(t)}{d t}=A \mathbf{x}(t)$ the first component has the form $x_{1}=2 t e^{-3 t}+t^{3}+\cos (2 t)$.
Which smallest size can the real matrix $A$ have? Give an example of a matrix that can have such a solution.
(4p)

## Solution.

The matrix $A$ of the system of equations must have a multiple eigenvalue -3 of multiplicity at least 2 because the expression $t e^{-3 t}$ appears first in the formula for the exponent of the Jordan block of size 2 for the eigenvalue -3 .
$A$ must have a eigenvalue 0 with multiplicity at least 4 because the term $t^{3}$ appears first for a Jordan block of size 4.
Real matrix $A$ must have two complex conjugate eigenvalues $\pm 2 i$ corresponding to the term $\cos (2 t)$. It makes according to the theorem about the decomposition of the space $\mathbb{R}^{n}$ into the direct sum of generalized eigenspaces, that the dimension of the matrix $A$ must be at least $2+4+2=8$.
For example: $A=\left[\begin{array}{rrrrrrrr}-3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0\end{array}\right]$
3. Consider a periodic system of linear differential equations $x^{\prime}=A(t) x$ with the following matrix $A(t)$ :

$$
A(t)=\left\{\begin{array}{c}
{\left[\begin{array}{cc}
4 & -1 \\
1 & 2
\end{array}\right] ; \quad t \in(0,1]} \\
{\left[\begin{array}{cc}
a & 1 \\
0 & a
\end{array}\right] ; \quad t \in(1,2], \quad a \in \mathbb{R}}
\end{array}\right.
$$

periodically extended on intervals of length 2 .
Formulate the Floquet theorem that gives conditions for boundedness of solutions to periodic linear systems.
Find conditions on the real parameter $a$ such that all solutions to this system of equations will be bounded.
(4p)
Hint. You can use the result from the question 1, where an autonomous system with the $\operatorname{matrix}\left[\begin{array}{cc}4 & -1 \\ 1 & 2\end{array}\right]$ is considered.

## Solution.

Floquet theorem. All solutions to a periodic linear system of differential equations are bounded if and only if all Floquet multipliers, that are eigenvalues to the monodromy matrix,
are smaller or equal to one in absolute value, and those that have absolute value equal to one are semisimple.

We need to calculate the monodromy matrix for the present system of ODE's: that in the case of the period 2 is equal to the value $\Phi(2,0)$ of the transition matrix.

We will use the Chapman- Kolmogorov relation for calculating the value of the monodromy matrix $\Phi(2,0)$ as a product of two simpler transition matrices: one on the interval $[0,1]$ and one on the interval $[1,2]$ :

$$
\Phi(2,0)=\Phi(2,1) \Phi(1,0)
$$

We will use the expression for the transition matrix on the interval $[0,1]$ derived i fact in the question 1:
$\Phi(t, 0)=\exp \left(t\left[\begin{array}{cc}4 & -1 \\ 1 & 2\end{array}\right]\right)=\left[\begin{array}{cc}e^{3 t}+t e^{3 t} & -t e^{3 t} \\ t e^{3 t} & e^{3 t}-t e^{3 t}\end{array}\right]$ for $t \in[0,1]$.
$\Phi(1,0)=\left.\left[\begin{array}{cc}e^{3 t}+t e^{3 t} & -t e^{3 t} \\ t e^{3 t} & e^{3 t}-t e^{3 t}\end{array}\right]\right|_{t=1}=\left[\begin{array}{cc}2 e^{3} & -e^{3} \\ e^{3} & 0\end{array}\right]$
The value of the transition matrix $\exp \left(t\left[\begin{array}{ll}a & 1 \\ 0 & a\end{array}\right]\right)$ for the autonomous system with matrix $\left[\begin{array}{ll}a & 1 \\ 0 & a\end{array}\right]:$
$\exp \left(t\left[\begin{array}{ll}a & 1 \\ 0 & a\end{array}\right]\right)=\left[\begin{array}{cc}e^{a t} & t e^{a t} \\ 0 & e^{a t}\end{array}\right]$ gives the value of the transition matrix to our periodic system on the interval [1, 2]:
$\Phi(t, \tau)=\left[\begin{array}{cc}e^{a(t-\tau)} & (t-\tau) e^{a(t-\tau)} \\ 0 & e^{a(t-\tau)}\end{array}\right]$ for $t, \tau \in[1,2]$
We use the Chapman- Kolmogorov relation for calculating the value of the monodromy matrix $\Phi(2,0)$ as a product of two transition matrices: one on the interval $[0,1]$ and one on the interval [1,2].
$\Phi(2,0)=\Phi(2,1) \Phi(1,0)=\left[\begin{array}{cc}e^{a} & e^{a} \\ 0 & e^{a}\end{array}\right]\left[\begin{array}{cc}2 e^{3} & -e^{3} \\ e^{3} & 0\end{array}\right]=\left[\begin{array}{cc}3 e^{a} e^{3} & -e^{a} e^{3} \\ e^{a} e^{3} & 0\end{array}\right]=e^{3+a}\left[\begin{array}{cc}3 & -1 \\ 1 & 0\end{array}\right]$
The matrix $\left[\begin{array}{cc}3 & -1 \\ 1 & 0\end{array}\right]$ has eigenvalues $\left(\frac{3}{2}-\frac{1}{2} \sqrt{5}\right),\left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right)$
Therefore the monodromy matrix eigenvalues (also called Floquet multipliers): $\lambda_{1}=e^{3+a}\left(\frac{3}{2}-\frac{1}{2} \sqrt{5}\right)$, and $\lambda_{2}=e^{3+a}\left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right)$ are both positive because $\sqrt{5}<3$ and are simple.
The find values of $a$ making that the largest eigenvalue (Floquet multiplier) $\lambda_{2}=e^{3+a}\left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right)$ is smaller or equal to 1 in absolute value that implies boundedness of all solutions:

$$
e^{3+a}\left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right) \leq 1
$$

We calculate the monotone function $\ln$ of the left and right hand sides of the inequality.

$$
\begin{gathered}
(3+a)+\ln \left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right) \leq 0 \\
a \leq-3-\ln \left(\frac{1}{2} \sqrt{5}+\frac{3}{2}\right)
\end{gathered}
$$

It gives the condition on $a$ implying boundedness of all solutions to the differential equation.
4. Consider the differential equation $x^{\prime}=f(x)$ where $f$ is a Lipschitz function $f: G \rightarrow \mathbb{R}^{n}$ and $G \subset \mathbb{R}^{n}$ is an open set.

Show that the transfer mapping $\varphi(t, \xi)$ corresponding to this equation is locally Lipscitz with respect to $\xi$.
Hint. Use the integral form of the initial value problem and apply Grönwall's inequality. (4p)
Solution. Choose a ball $B(a, R) \subset G$ and two arbitrary points $\xi$ ad $\zeta$ in the ball $B(a, R / 2)$ of half size with the same center $a$.
Write the initial value problem in the integral form:

$$
x(t)=\xi+\int_{0}^{t} f(x(s) d s
$$

The same equation in terms of the transfer mapping $\varphi(t, \xi)$ looks as

$$
\varphi(t, \xi)=\xi+\int_{0}^{t} f(\varphi(s, \xi)) d s
$$

The solution $\varphi(t, \xi)$ to this equation exists according to Picard Lindelöf theorem on some possibly small time interval $\left[0, T_{\xi}\right]$ dependent on the $\sup _{x \in B(a, R)}\|f(x)\|$ in the ball $B(a, R) \in$ $G$.
We consider another value of the transfer mapping $\varphi(t, \zeta)$ satisfying

$$
\varphi(t, \zeta)=\zeta+\int_{0}^{t} f(\varphi(s, \zeta)) d s
$$

The solution $\varphi(t, \zeta)$ to this equation exists according to the Picard Lindelöf theorem on another possibly small time interval $\left[0, T_{\zeta}\right]$ dependent on the $\sup _{x \in B(a, R)}\|f(x)\|$ in the ball $B(a, R) \in G$.
We choose $T=\min \left(T_{\xi}, T_{\zeta}\right)$ so that both solutions exist on the time interval $[0, T]$.
The difference $\varphi(t, \xi)-\varphi(t, \zeta)$ satisfies the integral equation for $t \in[0, T]$

$$
\varphi(t, \xi)-\varphi(t, \zeta)=(\xi-\zeta)+\int_{0}^{t} f(\varphi(s, \xi))-f(\varphi(s, \zeta)) d s
$$

We apply the euclidean norm and the triangle inequality to the right and left hand sides of the last equation.

$$
\|\varphi(t, \xi)-\varphi(t, \zeta)\| \leq\|\xi-\zeta\|+\int_{0}^{t}\|f(\varphi(s, \xi))-f(\varphi(s, \zeta))\| d s
$$

The Lipschitz property of $f(t, x)$ :

$$
\|f(\varphi(s, \xi))-f(\varphi(s, \zeta))\| \leq L\|\varphi(t, \xi)-\varphi(t, \zeta)\|
$$

implies that

$$
\|\varphi(t, \xi)-\varphi(t, \zeta)\| \leq\|\xi-\zeta\|+L \int_{0}^{t}\|\varphi(s, \xi)-\varphi(s, \zeta)\| d s
$$

Grönwall's inequality

$$
\|x(t)\| \leq\|x(\tau)\| \exp (L(t-\tau))
$$

follows from the integral inequality:

$$
\|x(t)\| \leq\|x(\tau)\|+L \int_{\tau}^{t}\|x(\sigma)\| d \sigma
$$

In our case it implies that

$$
\|\varphi(t, \xi)-\varphi(t, \zeta)\| \leq\|\xi-\zeta\| \exp (L t)
$$

For any bounded interval of time $t \in[0, T]$ it implies that the transfer mapping $\varphi(t, \xi)$ is a Lipschitz function with respect to $\xi$ with Lipschitz constant $\exp (L T)$.
5. Show that the following system of ODEs has a periodic solution.

$$
\left\{\begin{array}{l}
x^{\prime}=-y-x\left(x^{2}+y^{2}-x y-1\right)  \tag{4p}\\
y^{\prime}=2 x-y\left(x^{2}+y^{2}-x y-1\right)
\end{array}\right.
$$

Solution.
Point out that the origin is an equilibrium point.
Consider the test function in the form $V(x, y)=2 x^{2}+y^{2}$ that lets to cancel indefinite terms in $V_{f}$.
$. V_{f}(x, y)=4 x\left(-y-x\left(x^{2}+y^{2}-x y-1\right)\right)+2 y\left(2 x-y\left(x^{2}+y^{2}-x y-1\right)\right)=$
$-2\left(2 x^{2}+y^{2}\right)\left(x^{2}+y^{2}-x y-1\right)$
The sign of $V_{f}(x, y)$ depends only on the sign of the expression $x^{2}+y^{2}-x y-1$.
Applying the inequality $|x y| \leq\left(x^{2}+y^{2}\right) \frac{1}{2}$ we conclude that

$$
\left(x^{2}+y^{2}\right) \frac{1}{2} \leq\left(x^{2}+y^{2}-x y\right) \leq\left(x^{2}+y^{2}\right) \frac{3}{2}
$$

Therefore for $\left(x^{2}+y^{2}\right) \frac{1}{2}>1$ we have $V_{f}(x, y)<0$ and for $\left(x^{2}+y^{2}\right) \frac{3}{2}<1$ we have $V_{f}(x, y)>0$.
We choose an ellipse $V(x, y)=A$ such that it is contained in the set were $V_{f}(x, y)<0$ outside the circle $\left(x^{2}+y^{2}\right)=2$ (red on the picture). The smallest such ellipse is $2 x^{2}+y^{2}=4$.
We choose an ellipse $V_{f}(x, y)=B$ such that it is contained in the set where $V_{f}(x, y)>0$ inside the circle $\left(x^{2}+y^{2}\right)=\frac{2}{3}$ (red on the picture). The largest such ellipse is $2 x^{2}+y^{2}=\frac{2}{3}$.


These two ellipses define a positive invariant set for the system that does not contain the origin, that is an equilibrium point.

We check that there are no other equilibrium points except the origin. We point out that equality

$$
V_{f}(x, y)=0
$$

must be valid in equilibrium points. $V_{f}(x, y)=-2\left(2 x^{2}+y^{2}\right)\left(x^{2}+y^{2}-x y-1\right)=0$ if and only if $\left(x^{2}+y^{2}-x y-1\right)=0$. In such a case $x^{\prime}=-y$ and $y^{\prime}=2 x$ and the right hand side of the equation can be zero only in the origin.
We conclude that the set $\frac{2}{3} \leq 2 x^{2}+y^{2} \leq 4$ satisfies conditions in the The Poincare Bendixson theorem and therefore must contain at least one periodic orbit.
6. Consider the following system of ODEs. Consider the following system of ODEs:

$$
\begin{aligned}
x^{\prime} & =y^{3} \\
y^{\prime} & =-y-x+x^{2}
\end{aligned}
$$

KMake a sketch of nullclines and find equilibrium points.
Investigate stability properties of equilibrium points and find a possible region of attraction. (4p)

## Solution

Nullclines are a parabola $y=x(-1+x)$ through points 0 and 1 on the $x$ - axis (is the $y$ nullcline), and the $x$ - axis (it is the $x$ - nullcline). Equilibrium points are $(0,0),(1,0)$. We choose the test function

$$
V(x, y)=\frac{1}{4} y^{4}+\frac{1}{2} x^{2}-\frac{1}{3} x^{3}
$$

that is a linear combination of primitive functions for the first component of velocity and the primitive function of $x$-dependent part of the second component of velocity.

$$
\begin{aligned}
V_{f}(x, y) & =y^{3}\left(x-x^{2}\right)+\left(-y-x+x^{2}\right) y^{3}=-y^{4} \leq 0 \\
V_{f}^{-1}(0) & =\{x-\text { axis }\}
\end{aligned}
$$

It implies that the origin is a stable equilibrium. La Salle's invariance principle implies that it is even asymptotically stable. It follows from two facts. The set $W$ bounded by the closed curve that is a part of the level set of $V$ :

$$
\frac{1}{4} y^{4}+\frac{1}{2} x^{2}-\frac{1}{3} x^{3}=1 / 6
$$

going through the equilibrium point $(1,0)$ is a positive invariant set because $V_{f}(x, y) \leq 0$. We check the suggested geometry of $W$.

Checking the derivative $x(x-1)$ of the function $Y(x)=1 / 6+\frac{1}{3} x^{3}-\frac{1}{2} x^{2}$ we see that it is positive for $x<0$ and negative for $x \in(0,1)$. It implies that $Y$ as maximum at $x=1 / 6$ and decreases to 0 on the interval $(0,1]$. On the interval $(-\infty, 0) Y(x)$ increases from $-\infty$ to $1 / 6$ and attains the value 0 at some point $x_{l e f t}=-1 / 2<0$.


It makes that the boundary of the set $W$ consists of two graphs, symmetric with respect to the $x$ - axis:

$$
y= \pm \sqrt[4]{4\left(1 / 6+\frac{1}{3} x^{3}-\frac{1}{2} x^{2}\right)}, \quad x \in\left[x_{l e f t}, 1\right]
$$



It is easy to see by checking the right hand side of the differential equation on the $x$ - axis, that the maximal invariant set inside $W$ consists of only one point $(0,0)$. It implies that this equilibrium point is an attractor and is asymptotically stable.

The region of attraction of $(0,0)$ is the set $W$.
The equilibrium point $(1,0)$ is unstable. It follows from the geometry of nullclines and corresponding directions of the velocity field around the equilibrium point ( 1,0 ). Velocities point out from the equilibrium in two of four angles formed by the nullclines in a neighbourhood of the equilibrium point $(1,0)$.Trajectories that start with $x(0)>1$ and $y(0)>0$ between the $y$-nullcline $y=x^{2}-x$ and the $x$ - axis, that is $y-$ nullcline, are enclosed in this area. On the other hand velocities in this region have $x^{\prime}>0$ ad $y^{\prime}>0$. It makes that a these trajectories leave a cirle of radius $1 / 2$ around the point $(1,0)$. Similar argument works for the figure bounded by the nullcline $y=x^{2}-x$ and the nullcline $-x$ - axis for $x \in(0.5,1)$.

Another even shorter argument leads to the same conclusion by the observation that the region of attraction identified for the equilibrium $(0,0)$ has the equilibrium point $(1,0)$ on it's boundary. It makes that arbitrarily close to the equilibrium point $(1,0)$ there are points $\xi$ that give solution $x(t)=\varphi(1, \xi)$ tending to the origin $(0,0) /$ and leaving a fixed neighbourhood of the equilibrium point ( 1,0 ).

1. Max. 24 points; Threshold for marks: for GU: VG: 19 points; G: 12 points. For Chalmers: 5: 21 points; 4: 17 points; 3: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as Total $=0.16$ Assignment $1+0.16$ Assignment $2+0.68$ Exam - that is the average of the points for the home assignments ( $32 \%$ ) and for this exam $(68 \%)$. The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.

