MATEMATIK | Datum: 2021-01-04 | Tid: 8-30 - 12-30
GU, Chalmers | Hjälpmedel: - Alla
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Lösningsförsalag för tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.

1. Consider the following system of ODEs: $\dfrac{dz(t)}{dt} = Az(t)$, with $A = \begin{bmatrix} 5 & -1 & -4 \\ -12 & 5 & 12 \\ 10 & -3 & -9 \end{bmatrix}$.,
characteristic polynomial: $X^3 - X^2 - X + 1$
Find a general solution to this system using that one of the eigenvalues to A is $\lambda = 1$.
How one can calculate the fundamental matrix solution using the general solution? **(4p)**
Solution.
Characteristic polynomial of A is $\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -1 & -4 \\ -12 & 5-\lambda & 12 \\ 10 & -3 & -9-\lambda \end{bmatrix} =$
$(5-\lambda)\det\begin{bmatrix} 5-\lambda & 12 \\ -3 & -9-\lambda \end{bmatrix} + \det\begin{bmatrix} -12 & 12 \\ 10 & -9-\lambda \end{bmatrix} + (-4)\det\begin{bmatrix} -12 & 5-\lambda \\ 10 & -3 \end{bmatrix} =$
$(5-\lambda)\left(4\lambda + \lambda^2 - 9\right) + (12\lambda - 12) + (-4)(10\lambda - 14) = \lambda + \lambda^2 - \lambda^3 - 1 = -(\lambda+1)(\lambda-1)^2$
Eigenvalues are $\lambda_{1,2=1}$, $\lambda_3 = -1$.
Eigenvector $\mathbf{v}_1$ to $\lambda_{1,2}$ satyisfies the homogeneous equation $(A - I)\mathbf{v}_1 = 0$ with matrix
$\begin{bmatrix} 4 & -1 & -4 \\ -12 & 4 & 12 \\ 10 & -3 & -10 \end{bmatrix}$, Gaussian elimination gives $\begin{bmatrix} 4 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and one free variable and the only eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We find a generalised eigenvector satisfying the equation
$(A + I)\mathbf{v}_1^{(1)} = \mathbf{v}_1$ with extended matrix $\begin{bmatrix} 4 & -1 & -4 & 1 \\ -12 & 4 & 12 & 0 \\ 10 & -3 & -10 & 1 \end{bmatrix}$, Gaussian elimination gives:
$\begin{bmatrix} 4 & -1 & -4 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,
with one fri variable x_3 that we choose equal to zero giving the genetalized eigenvector
$\mathbf{v}_1^{(1)} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.
The eigenvector $\mathbf{v}_3$ corresponding to $\lambda_3 = -1$ satisfies the equation $(A+I)\mathbf{v}_3 = 0$ with matrix $\begin{bmatrix} 6 & -1 & -4 \\ -12 & 6 & 12 \\ 10 & -3 & -8 \end{bmatrix}$, Gaussian elimination gives: $\begin{bmatrix} 6 & -1 & -4 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ with one free variable x_3 that we choose equal to 2 giving $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$
Vectors $\mathbf{x}_1(t) = e^{At}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1$; $\mathbf{x}_2(t) = e^{At}\mathbf{v}_1^{(1)} = e^{\lambda_1 t}(\mathbf{v}_1^{(1)} + t\mathbf{v}_1)$; $\mathbf{x}_3(t) = e^{At}\mathbf{v}_3 = e^{\lambda_2 t}\mathbf{v}_3$; build a basis of solution space.

$x(t) = C_1 e^t \mathbf{v}_1 + C_2 e^t(\mathbf{v}_1^{(1)} + t\mathbf{v}_1) + C_3 e^{-t}\mathbf{v}_3 =$

$C_1 e^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 e^t \left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) + C_3 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t C_1 + C_3 e^{-t} + e^t C_2 (t+1) \\ 3e^t C_2 - 2C_3 e^{-t} \\ e^t C_1 + te^t C_2 + 2C_3 e^{-t} \end{bmatrix}$

The fundamental solution $\Phi(t) = [\phi_1(t), \phi_2(t), \phi_3(t)]$ with columns $\phi_1(t), \phi_2(t), \phi_3(t)$ is the matrix satisfying the equation $\Phi'(t) = A\Phi(t)$ and the initial condition $\Phi(0) = I = [e_1, e_2, e_3]$ with columns e_1, e_2, e_3.

Build a matrix $X(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)]$ of basis vectors to the solutions space.

Columns $\phi_1(t), \phi_2(t), \phi_3(t)$ in $\Phi(t)$ are expressed with help of the general solution:

$$x(t) = C_1 e^t \mathbf{v}_1 + C_2 e^t(\mathbf{v}_1^{(1)} + t\mathbf{v}_1) + C_3 e^{-t}\mathbf{v}_3$$

by finding constants C_1, C_2, C_3 satisfying initial conditions expressed by linear equations for $\mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$: $X(0)\mathbf{C} = e_1$, $X(0)\mathbf{C} = e_2$, and $X(0)\mathbf{C} = e_3$.

Alternatively one can carry out the same calculations in matrix form collecting the lenear equation above into the matrix equation $X(0)\mathbb{C} = I$ and expressing $\Phi(t)$ as $\Phi(t) = X(t)\mathbb{C}$.■

2. Solve the initial value problem $x'(t) = (x - x^2)$, $x(0) = \xi$,with the domain for the equation $J \times G = \mathbb{R}^2$, and find maximal intervals for solutions for arbitrary $\xi \in \mathbb{R}$. **(4p)**

Solution.

$\frac{dx}{dt} = (x - x^2)$; $\frac{dx}{x-x^2} = dt$; $\int \frac{dx}{x-x^2} = t + C$; $\int \frac{dx}{x-x^2} = \int \frac{dx}{x(1-x)} = \int \left(\frac{1}{x} - \frac{1}{1-x}\right) dx = \ln|x| - \ln|x-1|$.

$\ln\left|\frac{x}{x-1}\right| = t + C$; $\left(\frac{x}{x-1}\right) = K\exp(t)$, $K = \pm\exp(C)$

$x = (x-1)K\exp(t)$; $x(-1 + K\exp(t)) = K\exp(t)$;

$$x(t) = \frac{K\exp(t)}{(K\exp(t) - 1)};$$

Express K as a function of initial data ξ.

$$\xi = \frac{K}{(K-1)}; K = \frac{\xi}{\xi - 1}$$

and

$$x(t) = \frac{\xi\exp(t)}{(\xi\exp(t) - \xi + 1)}$$

The derivation of this solution is valid for $\xi \neq 0$ and $\xi \neq 1$, because the integrals above do not exist at this values of x. On the other hand two equilibrium solutions exist: $x(t) = 1$, and $x(t) = 0$ that are eventually included in the final formula. It gives us the transfer mapping $\varphi(t, 0, \xi)$ in the form:

$$\varphi(t, 0, \xi) = \frac{\xi\exp(t)}{(\xi\exp(t) - \xi + 1)}$$

We must exclude from the domain of $\varphi(t, 0, \xi)$ points where $(\xi(\exp(t) - 1) + 1) = 0$, namely $\xi = \frac{-1}{\exp(t)-1}$ or $\exp(t) = (\xi - 1)/\xi$ or $t = \ln((\xi - 1)/\xi)$.

For $\xi \in [0, 1]$ the maximal interval is $\mathbb{R}$. For $\xi \in (1, \infty)$ the maximal interval is $(t_{\min}, \infty)$ with $t_{\min} = \ln\left(\frac{(\xi-1)}{\xi}\right) < 0$. For $\xi \in (-\infty, 0)$ the maximal interval is $(-\infty, t_{\max})$ with $t_{\max} = \ln\left(\frac{(\xi-1)}{\xi}\right) > 0$.

3. Consider the following system of ODEs. $\begin{cases} x' = & 1 - xy \\ y' = & x - y^3 \end{cases}$

Find all equilibrium points and investigate their stability. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

Solution.

Equilibrium points are $(1,1)$ and $(-1,-1)$ can be found by substitution. $x = y^3$, $1 = xy = y^4$.

Jacoby matrix of te right hand side is $J(x,y) = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}$; $J(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$; $J(-1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. $\det(J(1,1)) = 4$, $\mathrm{sp}(J(1,1)) = -4$. Therefore the equilibrium point $(1,1)$ is asymptotically stable.

$\det(J(-1,-1)) = -4$. Therefore the linearized around $(-1,-1)$ system has a sadle point and the equilibrium point $(-1,-1)$ is unstable.

We shift the origin of te coordinate system into the point $(1,1)$ by introducing new variables $u = x - 1, v = y - 1$.

$$\begin{cases} u' = & -u - v - uv \\ v' = & u - 3v - 3v^2 - v^3 \end{cases}$$

Consider a test function $E(u,v) = \frac{1}{2}\left(u^2 + v^2\right)$

$$\begin{aligned} \frac{d}{dt}E(u(t),v(t)) &= \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} -u - v - uv \\ u - 3v - 3v^2 - v^3 \end{bmatrix} = \\ &= -u^2 - uv - u^2 v + uv - 3v^2 - 3v^3 - v^4 = \\ &= -u^2(1 - v) - 3v^2 \underbrace{\left(1 + v + v^2\right)}_{>0} < 0 \end{aligned}$$

$$if \quad v < 1, \quad (u,v) \neq (0,0)$$

The largest circle in (u,v) plane satisfying the condition $v \leq 1$ has radius 1. Therefore the circle of radius 1 around the point $(1,1)$ is the domain of attraction for the equilibrium $(1,1)$ of the original system of ODEs.■

4. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & -x - y\left[\ln\left(x^2 + 4y^2\right)\right] \end{cases}$.

Show that this system has a non-trivial periodic solution. **(4p)**

Solution.

Consider the test function $E(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$

$\frac{d}{dt}E(u(t),v(t)) = \begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} y \\ -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{bmatrix} = -y^2\left[\ln\left(x^2+4y^2\right)\right]\begin{cases} \geq 0 & x^2 + 4y^2 < 1 \\ \leq 0 & x^2 + 4y^2 > 0 \end{cases}$

The boundary curve $x^2 + 4y^2 = 1$ is the ellipse with halv axes 1 and $1/2$ with center in the origin.

Therefore any circle with the center in the origin inside this ellipse is never entered by a trajectory.

Similarly any circle with the center in the origin autside this ellipse is never left by a trajectory.

Such two circles build an annulus that is a positively invariant set for this system of ODEs.

For example an annulus $1/4 \leq x^2 + y^2 \leq 1$ satisfies this conditions. This annulus contains no equilibrium points, because the origin is the only equilibrium point. Therefore by Poincare - Bendixson theorem this annulus must contain at least one periodic orbit.■

5. For one particular solution of the system $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$ with a real matrix A, the first component has the form $x_1 = t^2 + t\sin(t)$.

   Which smallest size can the real matrix A have? **(4p)**

   Solution.

   The term $t\sin(t)$ in the solution is a sign that the Jordan form of the matrix A has a Jordan block corresponding to the eigenvalue $\lambda_1 = i$ that has multiplicity at least 2, for example $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ or multiplicity 3 : $\begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$ etc. On the other hand te matrix A is real and therefore it's characteristic plolynomial has real coefficients and therefore all complex eigenvalues must appear as conjugate pairs: the matrix A must have the eigenvalue $\lambda_2 = -i$ havingthe same multiplicity as λ_1, at least 2 and with corresponding Jordan block $\begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}$. The presence of the term t^2 in one component of a solution shows that the matrix A must have te eigenvalue $\lambda = 0$ with multiplicity at least 3 with correspoding Jordan block $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   All these observations imply that the real matrix A must have dimensions at least 7×7, because the sum of dimensions of sizes of Jordan blocks is at least $2 + 2 + 3 = 7$.■

6. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right), \quad a > 0 \end{cases}$.

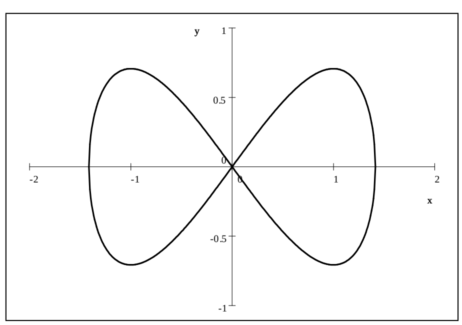
   Find all systems equilibrium points. Show using the test function $H = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right)$ and La Salles invariance principle, that the level set $H(x, y) = 0$ includes ω - limit sets of this system for all points in the plane except a finite number. Sketch these ω - limit sets. **(4p)**

   Solution.

   The system has three equilibrium points, all on the $x-$axis: $(-1, 0)$, $(0, 0)$, $(1, 0)$. The level set $H(x, y) = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right) = 0$ has the shape of ∞ with the center in the origin. One can see it by solving by expressing y in terms of x:

$$y = \pm |x| \sqrt{1 - \frac{1}{2}x^2}$$

   The ∞ figure is symmetrical with respect to x - axis and cuts it in points $\pm\sqrt{2}$. The formula above implies that $H(x, y) > 0$ outside of the ∞ figure, and $H(x, y) < 0$ inside of the ∞ figure.



   We calculate how the H function changes along trajectories.

$$H_f(x, y) = \frac{d}{dt}H(x(t), y(t)) = \begin{bmatrix} -x + x^3 \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right) \end{bmatrix} =$$

$$\underbrace{-xy + x^3y + xy - x^3y}_{0} - ay^2\left(y^2 - x^2 + \frac{1}{2}x^4\right)$$

We point out that $\frac{d}{dt}H(x(t),y(t))=0$ on the level set $H(x,y)=0$ (the ∞ figure) and on the x - axis. It means that trajectories are tangential to the level set $H(x,y)=0$. Therefore ∞ - figure is an invariant set for the system and consists of three orbits: the equilibrium in the origin (that is a saddle point, easily seen by linerization) and two closed branches of the ∞ figure correspoding to $x>0$ and $x<0$ in the expression $y=\pm|x|\sqrt{1-\frac{1}{2}x^2}$.

$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) < 0$ outside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t),y(t))=0$.

$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) > 0$ inside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t),y(t))=0$.

By La Salles invariance principle all trajectories are attracted to the largest invariat set inside the set $H_f^{-1}(0)$,were $H_f(x,y)=0$. This set consists of the union of the ∞ figure and the x - axis. There are no invariant sets on the x - axis except three equilibrium points $(-1,0)$, $(0,0)$, $(1,0)$.

It implies that for all points in the plain except equilibrium points, and points on the ∞ figure, $H(x(t),y(t))$ tends to zero along trajectories. The ω - limit sets for these points consist of one of the branches of the ∞ figure (for points inside it) or of the whole ∞ figure - for points outside it. The origin is the ω - limit set for all points on the ∞ figure. Equilibrium points are ω - limit sets of themselfs.

Max. 24 points; Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.