MATEMATIK	Datum: 2020-08-19	Tid: 14-00 - 18-00
GU, Chalmers	Hjälpmedel: - Alla	
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Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.

1. Consider the following system of ODEs: $\dfrac{dz(t)}{dt} = Az(t)$, with $A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

   Find a general solution to this system. What is a connection between general solution and fundamental matrix solution? **(4p)**

   Solution.

   Characteristic polynomial: $p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2$.

   $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$, eigenvectors: $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = 1, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow \lambda_2 = 2.$

   We find a generalised eigenvector from the equation

   $(A - 2I)\, v_2^{(1)} = v_2$ with extended matrix $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$, Gaussian elimination gives $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

   row echelon form is $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   that gives $v_2^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   Vectors $x_1(t) = e^{At}v_1 = e^{\lambda_1 t}v_1$; $x_2(t) = e^{At}v_2 = e^{\lambda_2 t}v_2$; $x_3(t) = e^{At}v_2^{(1)} = e^{\lambda_2 t}(v_2^{(1)} + tv_2)$ build a basis of solution space.

   Denote $X(t) = [x_1(t), x_2(t), x_3(t)]$.

   General solution is (point out that other equivalent expressions are possible!)

$$
\begin{aligned}
x(t) &= C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + C_3 e^{\lambda_2 t}(v_2^{(1)} + tv_2) = \\
&= C_1 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{2t} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \\
&= \begin{bmatrix} C_2 e^{2t} + C_3 (t+1) e^{2t} \\ e^t C_1 + C_2 e^{2t} + C_3 (t+1) e^{2t} \\ e^t C_1 + C_3 e^{2t} \end{bmatrix}
\end{aligned}
$$

   Fundamental matrix solution is a matrix $\Phi(t)$ that is a solution to the equation $\Phi'(t) = A\Phi(t)$ satisfying the initial condition $\Phi(0) = I$.

   Therefore $\Phi(t)) = X(t)B$ because any solutions is expressed in this form and the matrix B must satisfy the equation $X(0)B = I$.

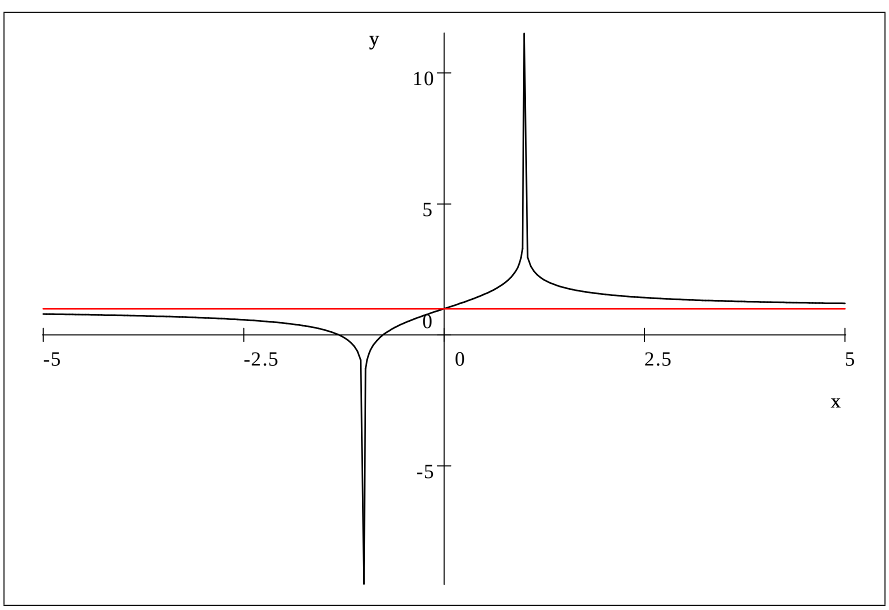
2. Solve the initial value problem $x'(t) = \left(x^2 - 1\right)$, $x(1) = \xi$, with the domain for the equation $J \times G = \mathbb{R}^2$, and find maximal intervals for solutions for arbitrary $\xi \in \mathbb{R}$. **(3p)**

Draw a sketch for the domain of the transfer mapping $x(t) = \varphi(t, 1, \xi)$ as a function of two variables (t, ξ). **(1p)**

Solution.

Point out first that for $\xi = \pm 1$ solutions are constant $x(t) = \pm 1$ and have infinite maximal interval $I_\xi = (-\infty, \infty)$.

$$\begin{aligned}
\frac{dx}{x^2-1} &= dt \\
\int \frac{dx}{x^2-1} &= \int dt \\
\frac{1}{2}\ln(|x-1|) - \frac{1}{2}\ln(|x+1|) &= t + C \\
\frac{1}{2}\ln\left(\left|\frac{x-1}{x+1}\right|\right) &= (t+C) \\
C &= \frac{1}{2}\ln\left(\left|\frac{\xi-1}{\xi+1}\right|\right) - 1 \\
\left(\frac{x-1}{x+1}\right) &= \exp(2(t+C)) \\
(x-1) &= (x+1)\exp(t+C) \\
x(1-\exp(2(t+C))) &= \exp(2(t+C)) + 1 \\
x &= \frac{\exp(2(t+C))+1}{(1-\exp(2(t+C)))} \\
\text{Boundary_for_ max _interval: } 1 &= \exp(2(t+C)) \\
t &= -C \\
t &= 1 - \frac{1}{2}\ln\left(\left|\frac{\xi-1}{\xi+1}\right|\right)
\end{aligned}$$



For $\xi > 0$ the maximal interval is $(-\infty,\ 1 - \frac{1}{2}\ln\left(\left|\frac{\xi-1}{\xi+1}\right|\right)$, For $\xi < 0$ the maximal interval is

$(1-\frac{1}{2}\ln\left(\left|\frac{\xi-1}{\xi+1}\right|\right),\infty)$, for $\xi=\pm 1$ maximal interval is (-∞,∞) and solution is a constant (equilibrium point) equal to $\xi=\pm 1$.

3. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & -y-2x-x^2 \end{cases}$

   Find and investigate stability of all equilibrium points. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

   Solution.

   The system has two equilibrium points: $(0,0)$, $(-2,0)$. We investigate first the stability of these equilibriums using linearization.

   The Jacoby matrix of the right hand side in the system is

$$A(x,y)=\begin{bmatrix} 0 & 1 \\ -2-2x & -1 \end{bmatrix}; \qquad A(0,0)=\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}; \qquad A(-2,0)=\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix};$$

   The characteristic polynomial of the 2×2 matrix A is $\lambda^2-\lambda\operatorname{tr}(A)+\det A$

   $\det A(-2,0)=-2$ Therefore the equilibrium $(-2,0)$ is a saddle point and is unstable.

   $\det A(0,0)=2$, $\operatorname{tr}A(0,0)=-1<0$ therefore the equilibrium $(0,0)$ is asymptotically stable.

   Point out that $\frac{(\operatorname{tr}A(0,0))^2}{4}=\frac{1}{4}<\det(A(0,0))=2$. Therefore the origin is a stable focus, with spiral type of phase portrait around it.

   We try now to find a domain of attraction for the asymptotically stable equilibrium in the origin.

   Consider a test function in the form

$$V(x,y)=\frac{y^2}{2}+\int\left(2x+x^2\right)dx=\frac{y^2}{2}+x^2+\frac{x^3}{3}$$

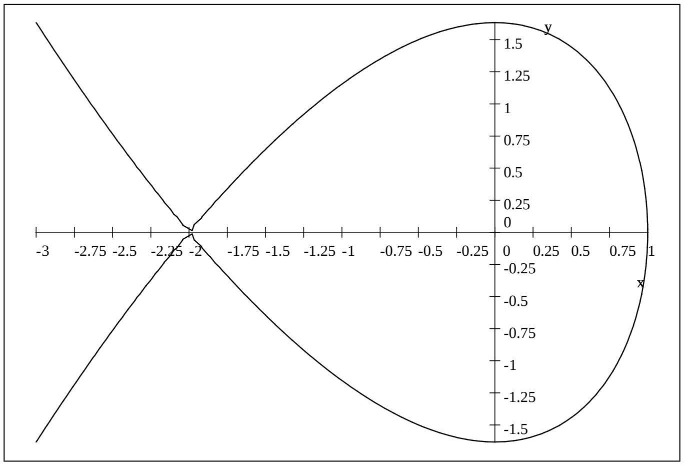
   This test function is positive definite in the half plane $x^2+\frac{x^3}{3}=x^2(1+\frac{x}{3})>0$, or $x>-3$.

   $\nabla V=\nabla\left(\frac{y^2}{2}+x^2+\frac{x^3}{3}\right)=\begin{bmatrix} 2x+x^2 \\ y \end{bmatrix}$; $V_f(x,y)=\nabla V\cdot f=\begin{bmatrix} 2x+x^2 \\ y \end{bmatrix}\cdot\begin{bmatrix} y \\ -y-2x-x^2 \end{bmatrix}=$
   $2xy-2yx-yy-y(x)^2+x^2y=-y^2\le 0$

   According to La Salles invariance principle orbits are attracted to the maximal invarinat set in a positively invariant set.

   We chose a level set of V that goes through the unstable equilibrium point $(-2,0)$ where $V(-2,0)=4-8/3=\frac{4}{3}$

   $\frac{y^2}{2}+x^2+\frac{x^3}{3}=4/3$:



The closed curve $y = \pm\sqrt{2}\sqrt{4/3 - x^2 - \frac{x^3}{3}}$ is the boundary of a positively invarinat set that includes the maximal invariant set consisting of only one point, the origin. It implies that the origin is asymptotically stable and this set is the domain of attraction for the origin.

The polynomial $p(x) = 4/3 - x^2 - \frac{x^3}{3}$ has derivative $-2x - x^2$ that is positive on $(-2, 0)$ and is negative on $(0, 1)$.

4. Consider the following system of ODEs. $\begin{cases} x' = & -y - \ln(r)x \\ y' = & x - \ln(r)y \end{cases}$, where $r^2 = x^2 + y^2$.

Show that the system has exactly one periodic solution. (**2p** for showing that the system has at least one periodic solution) **(4p)**

Solution.

Express the system in polar coordinates ρ, θ.

Consider the system

$$\begin{aligned} x' &= -y + f(r)x \\ y' &= x + f(r)y \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$. We will try to find an explicit expression for the corresponding flow by introducing polar coordinates $x = \cos(\theta)r$, $y = \sin(\theta)r$. We differentiate $r(t)$ using expressions for r and for x', y' in the equation, and arrive to following formulas:

$$\begin{aligned} \left(r^2\right)' &= 2rr' = (x^2 + y^2)' = 2xx' + 2yy' \\ &= 2x(-y + f(r)x) + 2y\,(x + f(r)y) = 2f(r)\left(x^2 + y^2\right) = 2f(r)r^2 \end{aligned}$$

Therefore:

$$r' = f(r)r$$

The equation for the polar angle θ can be derived by differentiating $\tan(\theta(t))$:

$$\begin{aligned} (\tan(\theta))' &= \theta'\left(\frac{1}{\cos^2(\theta)}\right) = \left(\frac{y}{x}\right)' = \frac{y'x - x'y}{x^2} \\ &= \frac{x^2 + f(r)xy - \left(-y^2 + f(r)xy\right)}{x^2} = \frac{x^2 + y^2}{x^2} = \frac{1}{\cos^2\theta} \end{aligned}$$

Therefore

$$\theta' = 1$$

It implies that each zero of the function $f(r)$ corresponds to a periodic solution. In our particular case $f(r) = -\ln(r) = 0$ when $r = 1$. It gives the only periodic orbit as a circle $r = 1$.Trajectories go counterclockwise with constant speed around it.

Checking the sign of $r'(r) = -\ln(r)r$ we observe that any ring $0 < a < r < b$ with $b > 1$ is a positive invariant set for the system that includes no equilibrium points. Therefore any trajectory starting in this ring must have an ω - limit inside it. The Poincare-Bendixson theorem implies that this ω limit set must be a periodic orbit and that this ring must include at least one periodic orbit.

5. Consider the system of equations $x' = f(t,x)$ where $f(t,x) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous with respect to $(t,x) \in \mathbb{R}^{n+1}$, and is locally Lipschitz continuous

   with respect to x - variable in $\mathbb{R}^n$. Show that any solution to an initial value problem for such a system is unique. **(4p)**

   Check the proof in the course book at the page 118.

6. Formulate conditions for boundedness of all solutions to the autonomous system of linear ODEs $x' = Ax$ with constant matrix A and give a sketch of the proof. **(4p)**

   Check the formulation and the proof in Lecture notes for the course.

   Max. 24 points; Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.