June 6, 2020

MATEMATIK	Datum: 2020-06-01	Tid: 08-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
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Lösningsförslag till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Give the definition of the monodromy matrix. Formulate it's main properties. **(2p)**

   Show that if a periodic system of equations $x' = A(t)x(t)$, with period P: $A(t+P) = A(t)$ has a periodic solution with period P, then the monodromy matrix must have an eigenvalue $\lambda = 1$. **(2p)**

   Answers.

   Monodromy matrix is the value of the transfer matrix $\Phi(p, 0)$ for a system with period p.

   Main properties of monodromy matrix are:

   1) $\Phi(t+p, 0) = \Phi(t, 0)\Phi(p, 0)$; 2) $\Phi(np, 0) = (\Phi(p, 0))^n$ and 3) All solutions to the equation $x' = A(t)x$ with p - periodic $A(t)$ are bounded on $\mathbb{R}^+$ if and only if absolute values of all eigenvalues $|\lambda|$ to the monodromy matrix are smaller then 1, and those eigenvalues that have absolute values equal to one are semisimple.

   Check Proposition 2.2, p.47 in the course book about periodic solutions to periodic linear systems.

2. Give a definition of an ω - limit point. State if an ω - limit set necessarily must be:

   a) open, b) closed, c) connected, d) simply connected, e) bounded100f(1p) **(1p)**

   Formulate and prove LaSalle's invariance principle. Specify carefully which mathematical facts are the ground for each step in the proof. **(3p)**

   Answers.

   Check lecture notes about LaSalle's invariance principle.

   a) - wrong, b) yes, c) yes if the orbit is bounded) does not need to be) does not need to be

3. Consider the following system of ODEs: $\dfrac{dz(t)}{dt} = Az(t)$, with $A = \begin{bmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$.

   i) Calculate general solution for this system of equations.

   ii) Find the set of all initial conditions giving unbounded solutions. **(4p)**

   Solution

The characteristic polynomial is $P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 1 & -2 \\ 4 & 1-\lambda & 0 \\ 2 & 1 & -1-\lambda \end{bmatrix}$

$= (-1-\lambda)\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} - \det \begin{bmatrix} 4 & 0 \\ 2 & -1-\lambda \end{bmatrix} - 2\det \begin{bmatrix} 4 & 1-\lambda \\ 2 & 1 \end{bmatrix}$

$(1-\lambda)^2(1+\lambda) - 4(-1-\lambda) - 4(\lambda+1) = \lambda^3 + \lambda^2 - \lambda + 1$

$$\det \begin{bmatrix} -1-\lambda & 1 & -2 \\ 4 & 1-\lambda & 0 \\ 2 & 1 & -1-\lambda \end{bmatrix} = \lambda - \lambda^2 - \lambda^3 + 1 = (1-\lambda)(\lambda+1)^2$$

Eigenvectors are: $v_1 = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_1 = -1$; $v_2 = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 1.$

We have got only one linearly independent eigenvector to the multiple eigevalue $\lambda_1 = -1$. To find a basis to the solution space

we calculate a generalised eigenvector $v_1^{(1)}$ to the eigenvalue λ_1 lineraly independent of v_1.

$v_1^{(1)}$ can be found from the equation$(A - \lambda_2)\, v_1^{(1)} = v_1$

$$(A - \lambda_2) = \begin{bmatrix} -1-(-1) & 1 & -2 \\ 4 & 1-(-1) & 0 \\ 2 & 1 & -1-(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

The extended matrix for the system $(A - \lambda_2)\, v_1^{(1)} = v_1$ is: $\begin{bmatrix} 0 & 1 & -2 & -1 \\ 4 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Gauss elimination $\begin{bmatrix} 0 & 1 & -2 & -1 \\ 4 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & -2 & -1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & -2 & -1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow$
$\begin{bmatrix} 0 & 1 & -2 & -1 \\ 2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Longrightarrow \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ that gives a solution (not unique!)

$v_1^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ that is evidently linearly independent of $v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$(it always must be!). It is also automatically linearly independent of v_2 because v_2 corresponds to another eigenvalue.

$$\begin{bmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

The general solution $x(t) = e^{At}\xi$ to the equation can be written by choosing the initial data x_0 expressed in terms of the basis of eigenvectors and generalized eigenvectors and using for each term the representation for the exponent e^{At} acting on an element $x^{0,j}$ of a particular generalized eigenspace:

$$e^{At}x^{0,j} = \left[\sum_{k=0}^{m_j-1} (A - \lambda_j I)^k \frac{t^k}{k!} \right] e^{\lambda_j t} x^{0,j}$$

where m_j is the multiplicity of the eigenvalue λ_j and $x^{0,j}$ is an element of the corresponding generalized eigenspace.

In our particular case the initial data x_0 is expressed as a linear combination $\xi = C_1 v_1 + C_2 v_1^{(1)} + C_3 v_2$.

The the general solution $x(t)$ to the system of equations is a linear combination of linearly independent basis solutions:

$x_1(t) = e^{At} v_1 = e^{\lambda_1 t} v_1$; $x_2(t) = e^{At} v_1^{(1)} = e^{\lambda_1 t}(v_1^{(1)} + t v_1)$; $x_3(t) = e^{At} v_2 = e^{\lambda_2 t} v_2$.

General solution is (point out that other equivalent expressions are possible!)

$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_1 t}(v_1^{(1)} + t v_1) + C_3 e^{\lambda_2 t} v_2 = \\ &= C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right) + C_3 e^{t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} -C_1 e^{-t} - t C_2 e^{-t} \\ 2e^t C_3 + 2C_1 e^{-t} + C_2 e^{-t}(2t+1) \\ e^t C_3 + C_1 e^{-t} + C_2 (t+1) e^{-t} \end{bmatrix} \end{aligned}$$

Initial data ξ corresponding to unbounded solutions correspond to the case when C_3in the representation above is not zero. Geometrically these are points outside the plane through the origin spanned by vectors v_1 and $v_1^{(1)}$. Such ξ for example must not satisfy the linear equation $\xi \cdot (v_1 \times v_1^{(1)}) = 0$ expressing this plane.

4. Solve the initial value problem $x'(t) = -\pi t \left(1 + x^2\right)$, $\quad x(1) = \xi$, with the domain for the equation $J \times G = \mathbb{R}^2$,

and find maximal intervals for solutions. Draw a sketch for the domain of the transfer mapping $x(t) = \varphi(t, 1, \xi)$ as a function of two variables (t, ξ). **(4p)**

$$\begin{aligned} \int \frac{dx}{1+x^2} &= -\pi \int t dt \\ \arctan(x) &= -\frac{\pi}{2} t^2 + C \\ \arctan(\xi) &= -\frac{\pi}{2} + C \\ C &= \arctan(\xi) + \pi/2 \\ \arctan(x) &= -\frac{\pi}{2} t^2 + \arctan(\xi) + \pi/2 \\ x &= \tan\left(-\frac{\pi}{2} t^2 + \pi/2 + \arctan(\xi)\right) \end{aligned}$$

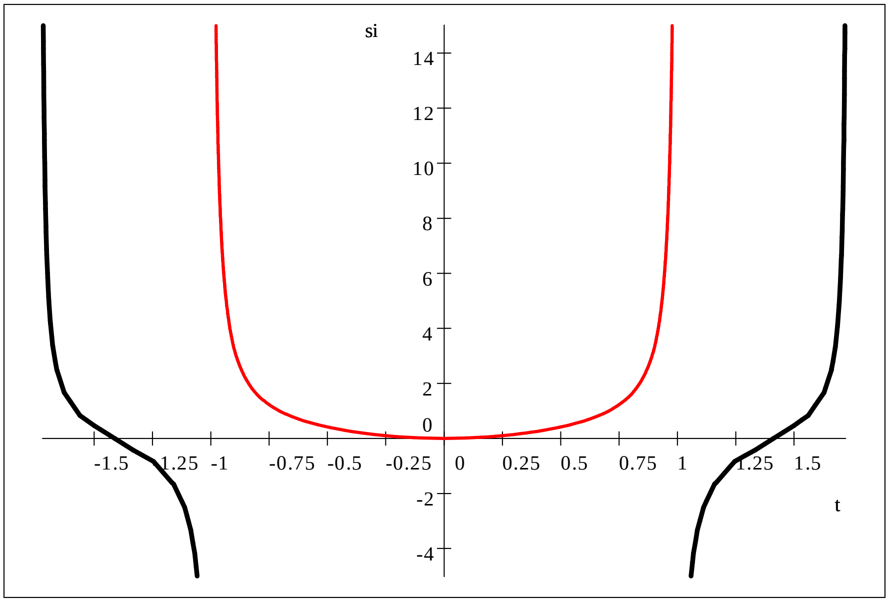
The domain for $\varphi(t, 1, \xi)$ is defined by the requirement that the argument under tan is in the interval $(-\pi/2, \pi/2)$:

$$-\frac{\pi}{2} t^2 + \arctan(\xi) + \pi/2 \in (-\pi/2, \pi/2)$$

Boundaries of this domain are expressed by equations

$$\begin{aligned} t^2 &= 2 + \frac{2}{\pi} \arctan(\xi) \quad (\textit{black} \quad \textit{curve}) \\ t^2 &= \frac{2}{\pi} \arctan(\xi) \quad (\textit{red} \quad \textit{curve}) \end{aligned}$$

The left "hand" of the domain in the picture is not actual for the initial time $\tau = 1$, because correcponding time intervals there do not include $t = 1$.



5. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & -y - x + x^3 \end{cases}$

   Find and investigate stability of all equilibrium points. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

   The system has three equilibrium points: $(-1,0)$, $(0,0)$, $(1,0)$.

   We can investigate their stability first by using linearization. We calculate the Jacobi matrix of the right hand side in the system.

   $J(x,y) = \begin{bmatrix} 0 & 1 \\ -1+3x^2 & -1 \end{bmatrix}$.

$$\begin{aligned} \text{trace}(J(x,y)) &= -1 < 0, \quad \det(J(x,y)) = 1 - 3x^2 \\ \det(J(-1,0)) &= 1 - 3 = -2; \\ \det(J(1,0)) &= -2. \\ \det(J(0,0)) &= 1 > 0, \\ \left[\text{trace}(J(0,0))\right]^2/4 &= 1/4 < \det(J(0,0)) = 1. \end{aligned}$$

   It implies according to the Grobman-Hartman theorem, that the system has saddle points in $(-1,0)$ and $(1,0)$ and an asymptotically stable equilibrium in $(0,0)$.

   To find a domain of attraction for $(0,0)$, we choose the a test function in the form: $V(x,y) = y^2/2 + x^2/2 - x^4/4 = y^2/2 + x^2/2\left(1 - x^2/2\right) > 0$ for $|x| < 4$ that is a sum of the kinetic energy $y^2/2$ and potential energy $x^2/2 - x^4/4$ for this system.

   $V_f = \begin{bmatrix} x - x^3 \\ y \end{bmatrix} \begin{bmatrix} y \\ -y - x + x^3 \end{bmatrix} = y\left(x - x^3\right) + y\left(-x - y + x^3\right) = -y^2 \leq 0.$

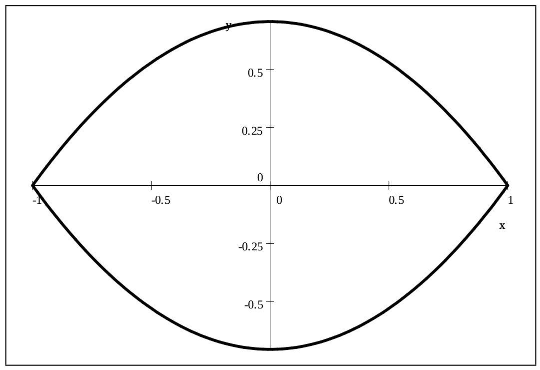
   $V_f^{-1}(0)$ is the x - axis. The largest invariant set in $V_f^{-1}(0)$ consists of three points that are equilibrium points where $y' = 0$. LaSalle's invariance principle states that the maximal invariant set on the set $V_f^{-1}(0)$ attracts trajectories that have compact closure of their orbits. $V_f^{-1}(0)$ is the x - axis and contains three equilibrium points.

   The level set of V crossing equilibrium points $(1,0)$ and $(-1,0)$ is the largest possible level set to V surrounding the asymptotically stable equilibrium point in the origin. This level set has equation corresponding to the value $V(-1,0) = \left(x^2/2 - x^4/4\right)\big|_{x=-1} = 1/4$:

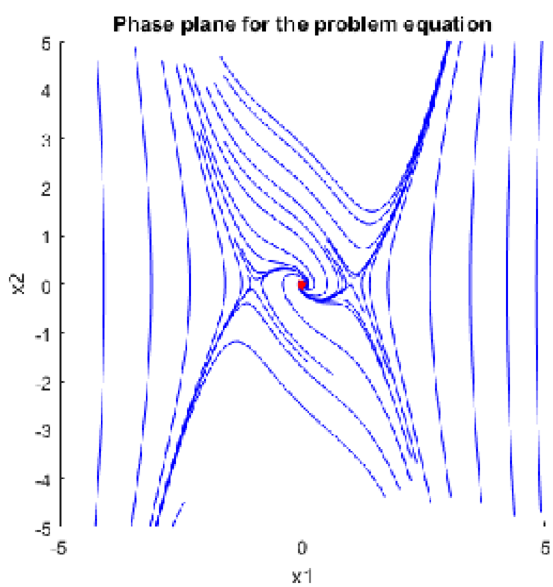
$$V(x,y) = y^2/2 + x^2/2 - x^4/4 = 1/4$$

This cure can be expressed explicitly as $y^2/2 = 1/4 - x^2/2 + x^4/4$. The function $G(x) = 1/4 - x^2/2 + x^4/4$ has the derivative $G'(x) = -x + x^3 = x(x^2-1)$ and therefore has a maximum $G(0) = 1/4$ in $x = 0$ and two local minima $G(\pm 1) = 0$ at $x = \pm 1$.

It implies that the curve $y = \pm\sqrt{1/2 - x^2 + x^4/2}$, $x \in [-1, 1]$



is a closed curve symmetric with respect to coordinate axes such that it bounds an open positively invariant set including only one asymptotically stable equilibrium point in the origin that attracts all trajectories starting inside this set. This set is a domain of attraction for this equilibrium.



6. Consider the following system of ODEs. $\begin{cases} x' = & A - x - \frac{2xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$

   a) Give a sketch of nullclines to this system and find all equilibrium points. **(2p)**

   Find a rectangle in the first quadrant $x > 0$, $y > 0$ bounded by lines parallel to coordinate axes, that is a positively invariant set.

   b) Find conditions on $A > 0$ such that the system has at least one periodic solution. **(2p)**

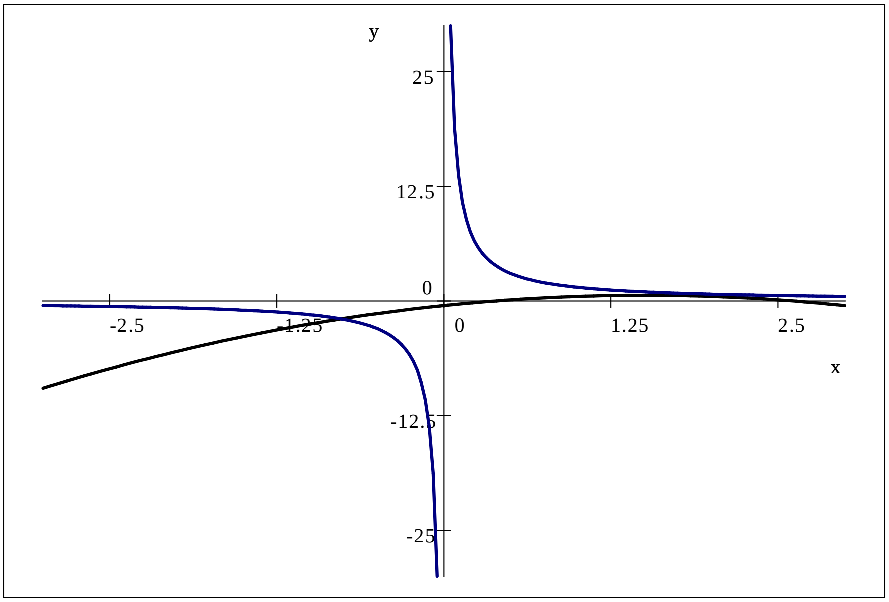
Solution.

a) Nullclines are: for $x' = 0$ it is $A - x - \frac{2xy}{1+x^2} = 0$, or $A - x = \frac{2xy}{1+x^2}$ and for $x \neq 0$,

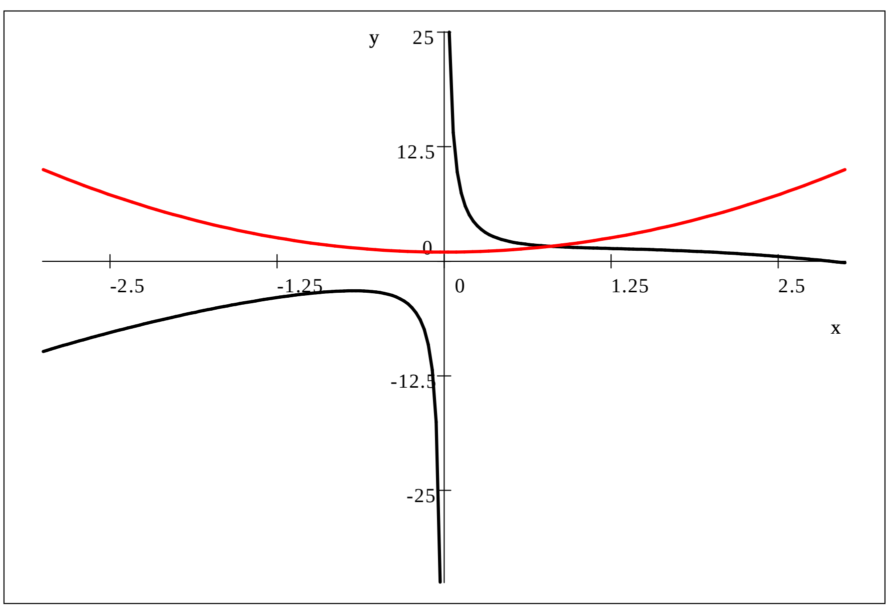
$$y = (A - x)\left(1 + x^2\right)\frac{1}{2x} = \frac{1}{2}Ax - \frac{1}{2}x^2 - \frac{1}{2} + \frac{1}{2}\frac{A}{x}$$

It is a sum of a parabola $y = \frac{1}{2}Ax - \frac{1}{2}x^2 - \frac{1}{2}$ and hyperbola $y = \frac{1}{2}\frac{A}{x}$ looking as hyperbola for small $|x|$ and as parabola for large $|x|$.

For example $A = 3$: $y = \frac{3}{2}x - \frac{1}{2}x^2 - \frac{1}{2} = \left(-\frac{1}{2}\right)\left(x^2 - 3x + 1\right)$ and $y = \frac{3}{2}\frac{1}{x}$



For $A = 3$ the curve $y = \frac{3}{2}x - \frac{1}{2}x^2 - \frac{1}{2} + \frac{1}{x}$ and looks as (black line):



nullclines for $y' = 0$ are $x = 0$ and $y = 1 + x^2$ (red line on the picture).

There is only one equilibrium point (x_*, y_*) for all $A > 0$ that is calculated by first solving the second equation:

$$\begin{aligned} x\left(1 - \frac{y}{1+x^2}\right) &= 0 \\ y &= 1 + x^2 \end{aligned}$$

where $x = 0$ cannot satisfy the system. We substitute the result in the first equation:

$$\begin{aligned} A - x - 2x\left(\frac{y}{1+x^2}\right) &= 0; \\ A - x - 2x &= 0; \\ x_* &= \frac{A}{3} \\ y_* &= 1 + \left(\frac{A}{3}\right)^2 \end{aligned}$$

Observe that the first quadrant is a positively invariant set. For $y = 0$ we have $\dot{x} = A > 0$ and for $y = 0$ and $x > 0$ we have $y' = x > 0$.

Observe also that $\dot{y} < 0$ for $y > 1 + x^2$ and $x > 0$; $\dot{x} < 0$ for $x > A$ and $y > 0$.

It implies that the rectangle $[0, A] \times \left[0, 1 + A^2\right]$ is a positively invariant compact set.

b) We investigate the stability of the equilibrium point $(x_*, y_*) = \left(\frac{A}{3}, 1 + \left(\frac{A}{3}\right)^2\right)$

Elements of the Jacobi matrix in an arbitrary point (x, y) are calculated as:

$\frac{\partial}{\partial x}\left(A - x - \frac{2xy}{1+x^2}\right) = 4x^2\frac{y}{2x^2+x^4+1} - 2\frac{y}{x^2+1} - 1 = 4x^2\frac{y}{(x^2+1)^2} - 2\frac{y}{x^2+1} - 1$

$\frac{\partial}{\partial y}\left(A - x - \frac{2xy}{1+x^2}\right) = -2\frac{x}{x^2+1} = -2\frac{x}{x^2+1}$

$\frac{\partial}{\partial x}\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = 2x^2\frac{y}{2x^2+x^4+1} - \frac{y}{x^2+1} + 1 = 2x^2\frac{y}{(x^2+1)^2} - \frac{y}{x^2+1} + 1$

$\frac{\partial}{\partial y}\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = -\frac{x}{x^2+1} = -\frac{x}{x^2+1}$

The Jacobi matrix is:

$$\begin{aligned} J(x,y) &= \left[\left.\begin{array}{cc} \frac{4x^2y}{(x^2+1)^2} - 2\frac{y}{x^2+1} - 1 & -2\frac{x}{x^2+1} \\ \frac{2x^2y}{(x^2+1)^2} - \frac{y}{x^2+1} + 1 & -\frac{x}{x^2+1} \end{array}\right]\right| = \\ &\quad \left.\left[\begin{array}{cc} 2y\frac{(x^2-1)}{(x^2+1)^2} - 1 & -2\frac{x}{x^2+1} \\ y\frac{(x^2-1)}{(x^2+1)^2} + 1 & -\frac{x}{x^2+1} \end{array}\right]\right|_{x=A/3} \\ &= \left[\begin{array}{cc} \frac{4(A/3)^2}{((A/3)^2+1)} - 3 & -2\frac{(A/3)}{(A/3)^2+1} \\ \frac{2(A/3)^2}{((A/3)^2+1)} & -\frac{(A/3)}{(A/3)^2+1} \end{array}\right] \\ trace\left(J(\frac{A}{3}, 1 + \frac{A^2}{9})\right) &= \left[\begin{array}{cc} \left(A^2+9\right)^{-1}\left(A^2 - 27\right) & (-6)\left(A^2+9\right)^{-1}A \\ 2\left(A^2+9\right)^{-1}A^2 & (-3)\left(A^2+9\right)^{-1}A \end{array}\right] \end{aligned}$$

$$trace\left(J(\frac{A}{3}, 1 + \frac{A^2}{9})\right) = \left(A^2+9\right)^{-1}\left(A^2 - 3A - 27\right)$$

One can also analyze the trace first simplifying it without explicitely expressing the equilibrium point:

$$\begin{aligned}
\operatorname{trace}[J(x_*,y_*)] &= \frac{4x_*^2}{(x_*^2+1)} - 3 - \frac{x_*}{x_*^2+1} = \\
&= \frac{x_*^2 - x_* - 3}{(x_*^2+1)}; \\
x_* &= \frac{A}{3}, \quad y_* = 1 + \left(\frac{A}{3}\right)^2
\end{aligned}$$

$\operatorname{trace}(A) > 0$ if and only if $x_*^2 - x_* - 3 > 0$, or $3^{-2}\left(A^2 - 3A - 27\right) > 0$.

The polynomial $A^2 - 3A - 27$ has roots $\lambda_1 = \frac{3}{2}\sqrt{13} + \frac{3}{2} > 0$ and $\lambda_2 = \frac{3}{2} - \frac{3}{2}\sqrt{13} < 0$.

Therefore $\operatorname{trace}(J) > 0$ for $A > \frac{3}{2}\sqrt{13} + \frac{3}{2}$.

We check also $\det(J(x_*,y_*)) = \det\begin{bmatrix} \frac{4x_*^2}{(x_*^2+1)} - 3 & -2\frac{x_*}{x_*^2+1} \\ \frac{2x_*^2}{(x_*^2+1)} & -\frac{x_*}{x_*^2+1} \end{bmatrix} = \left(\frac{4(A/3)^2}{((A/3)^2+1)} - 3\right)\left(-\frac{A/3}{(A/3)^2+1}\right) - \left(-2\frac{A/3}{(A/3)^2+1}\right)\left(\frac{2(A/3)^2}{((A/3)^2+1)}\right)$.

We calculate each term separately:

$\left(\frac{4(A/3)^2}{((A/3)^2+1)} - 3\right)\left(-\frac{A/3}{(A/3)^2+1}\right) = \frac{A}{\frac{1}{9}A^2+1} - \frac{4}{27}\frac{A^3}{\left(\frac{1}{9}A^2+1\right)^2} = (-3)\left(A^2+9\right)^{-2}\left(A^2-27\right)A$

$\left(-2\frac{A/3}{(A/3)^2+1}\right)\left(\frac{2(A/3)^2}{((A/3)^2+1)}\right) = -\frac{4}{27}\frac{A^3}{\left(\frac{1}{9}A^2+1\right)^2} = (-12)\left(A^2+9\right)^{-2}A^3$

$\det(J(x_*,y_*)) = (-3)\left(A^2+9\right)^{-2}\left(A^2-27\right)A - (-12)\left(A^2+9\right)^{-2}A^3 = 9\left(A^2+9\right)^{-1}A > 0$ for $A > 0$.

Therefore $\det(J(x_*,y_*)) > 0$ for all $A > 0$.

Therefore for $A > \frac{3}{2}\sqrt{13} + \frac{3}{2}$ the equilibrium point (x_*,y_*) is a repeller and therefore no one trajectory can enter an arbitrarily small neighbourhood D of (x_*,y_*) from outside of it.

Excluding a small neighbourhood D with the center in the equilibrium point (x_*,y_*), from the positively invariant set $[0,A] \times \left[0, 1+A^2\right]$, we get a positively invariant compact set $[0,A] \times \left[0, 1+A^2\right] \backslash D$ without equilibrium points that according to the Poincare - Bendixson theorem must contain at least one periodic orbit.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.