MATEMATIK | Datum: 2019-08-28 | Tid: 14-00 - 18-00
GU, Chalmers | Hjälpmedel: - Inga
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the theorem about the dimension of the solution space to a linear system of ODEs. **(4p)**

   Check "Lecture notes on general linear systems of ODEs with variable coefficients and Floquet theory"

   and "Introduction and autonomous linear systems".

2. Formulate and prove the LaSalle invariance principle. **(4p)**

   Check solution to Exercise 5.9 in the course book or in "Lecture notes on omega-limit sets and LaSalle's invariance principle with applications".

3. Consider the following system of ODEs: $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$, with the constant matrix

   $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

   i)Write down a Jordan canonical form for the matrix A.

   ii) Give general solution to the system.

   iii) Find all initial conditions giving bounded solutions. **(4p)**

   Solution. The characteristic polynomial for A is $p(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} =$

   $$\det \begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda - 1 & 1+\lambda \\ 1 & 1 & -\lambda \end{bmatrix} = \det \begin{bmatrix} 0 & 1+\lambda & 1-\lambda^2 \\ 0 & -\lambda-1 & 1+\lambda \\ 1 & 1 & -\lambda \end{bmatrix} = (1+\lambda) \det \begin{bmatrix} 0 & 1+\lambda & 1-\lambda^2 \\ 0 & -1 & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

   $$= (1+\lambda)\left[(1+\lambda) + (1-\lambda^2)\right] = (1+\lambda)^2 (2-\lambda)$$

   Eigenvalues are $\lambda_1 = \lambda_2 = -1$, (with multiplicity 2) $\lambda_3 = 2$.

   Eigenvectors are $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ corresponding to $\lambda_1 = \lambda_2 = -1$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_3 = 2$.

   Eigenvectors satisfy homogeneous systems with matrices $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (for $\lambda_1 = \lambda_2 = -1$) and $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

   The first one has two dimensional set of solutions (two free variable sin the system).

The second reduces after Gaussian elimination to the matrix $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ with one free variable.

Eigenvectors are $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ corresponding to $\lambda_1 = \lambda_2 = -1$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_3 = 2$.

There are three linearly independent eigenvectors, therefore the matrix A is diagonalisable. It means that the Jordan canonical form of the matrix is diagonal: $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

General solution to the system of ODEs is $\mathbf{x}(t) = C_1 v_1 e^{-t} + C_2 v_2 e^{-t} + C_3 e^{2t}$. Solutions are bounded if and only if the initial data $x(0)$ belongs to the $Span(v_1, v_2)$: $x(0) = C_1 v_1 + C_2 v_2$.

4. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction.

$$\begin{cases} x' = & 3y \\ y' = & -x - (4 - x^2) y \end{cases}$$ **(4p)**

Solution. We choose a Lyapunov function in the form $V(x, y) = x^2 + ay^2$. The choice $a = 3$ is optimal, it gives $V_f(x, y) = 6xy - 6xy - 2y^2(4 - x^2) = -2y^2(4 - x^2) \leq 0$ in the domain $|x| \leq 2$. Therefore the origin is a stable equilibrium point.

We apply the LaSalle invariance principle in the domain $|x| < 2$ to prove that the origin is also an asymptotically stable equilibrium.

We observe that $V_f^{-1}(0) = \{x - \text{axis}\}$ because $V_f(x, y) = 0$ if and only if $y = 0$. Checking the right hand side in the system for $y = 0$ implies $y' = -x$ that is zero only in the origin. It implies that the origin is the maximal invariant subset in $V_f^{-1}(0) = \{x - \text{axis}\}$. Therefore the origin is asymptotically stable. The domain of attraction is the set bounded by a level set of $V(x, y) = C = x^2 + 3y^2$ (an ellipse) that is inside the stripe $|x| \leq 2$. The largest such ellipse goes through points $x = \pm 2$, $y = 0$. Therefore $C = 4$ and the domain of attraction for the equilibrium in the origin consists of points (x, y) satisfying the inequality $x^2 + 3y^2 < 4$. All trajectories starting in this domain tend to the origin with $t \to \infty$.

5. Show that the following system of ODE-s has a periodic solution.

$$\begin{cases} x' = x + y - x\left(x^2 - xy + y^2\right) \\ y' = -x + y - y\left(x^2 - xy + y^2\right) \end{cases}$$ **(4p)**

Hint. The Cauchy inequality $|xy| \leq 0.5\left(x^2 + y^2\right)$ can be useful for analysis here.

Solution. We like to apply the Poincare-Bendixson theorem to prove that the system has a periodic solution by showing that some of it's trajectories must have a periodic orbit as an ω-limit set. To show it we find a positively - invariant set that does not include equilibrium points. By the Poincare-Bendixson theorem all trajectories starting in this positively invariant set will have an ω limit set that is a periodic orbit.

We try the test function $V(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$ and try to find two such circles (level sets to $V(x, y)$) that they bound a positively invariant set .

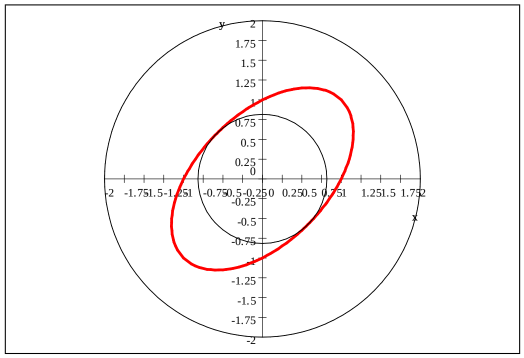
$$V_f(x, y) = \begin{bmatrix} x + y - x\left(x^2 - xy + y^2\right) \\ -x + y - y\left(x^2 - xy + y^2\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$x\left(x + y - x\left(-xy + x^2 + y^2\right)\right) + y\left(-x + y - y\left(-xy + x^2 + y^2\right)\right) =$$

$\left[1-\left(-xy+x^2+y^2\right)\right]\left(x^2+y^2\right).$

We see that $V_f(x,y)<0$ for $1-\left(-xy+x^2+y^2\right)<0$ and $V_f(x,y)>0$ for $1-\left(-xy+x^2+y^2\right)>0$.

The curve $1=\left(-xy+x^2+y^2\right)$ is an ellipse (red curve on the picture) because the expression $-xy+x^2+y^2$ is positive definite by the Cauchy inequality $|xy|\leq 0.5\left(x^2+y^2\right)$. One can also observe it by introducing new variables in the plane: $u=x+y$, $w=x-y$.



This ellipse separates the area where $V_f(x,y)<0$ and trajectories of the system go inside circles, that are level sets of $V(x,y)$ from the area where $V_f(x,y)>0$ and trajectories of the system go outside circles that are level sets of $V(x,y)$.

Finding two circles $x^2+y^2=R^2$ and $x^2+y^2=r^2$, $R>r>0$ such that the first one is completely outside the ellipse $1=\left(-xy+x^2+y^2\right)$ and the second one is completely inside the ellipse, will give us the desired ring shaped positively invariant set. It is intuitively evident that such R - large enough and r - small enough exist. Then we must check that it does not contain any equilibrium points. We find first some explicit estimates for R and r.

We consider the expression $\left[1-\left(-xy+x^2+y^2\right)\right]$ and try to find a circle $x^2+y^2=R^2$ such that $\left[1-\left(-xy+x^2+y^2\right)\right]<0$.

$\left[1-\left(-xy+x^2+y^2\right)\right]\leq\left[1-x^2-y^2+|xy|\right]\leq$

$\left[1-x^2-y^2+0.5\left(x^2+y^2\right)\right]=1-0.5\left(x^2+y^2\right)\leq 0.$

Therefore for $x^2+y^2=R^2\geq 2$ the desired inequality $V_f(x,y)\leq 0$ is valid. We found the outer boundary of the ring shaped positively invariant set.

We consider the expression $\left[1-\left(-xy+x^2+y^2\right)\right]$ and try to find a circle $x^2+y^2=r^2$ such that $\left[1-\left(-xy+x^2+y^2\right)\right]\geq 0$.

$\left[1-\left(-xy+x^2+y^2\right)\right]\geq\left[1-x^2-y^2-|xy|\right]\geq$

$\left[1-x^2-y^2-0.5\left(x^2+y^2\right)\right]=1-1.5\left(x^2+y^2\right)\leq 0.$

Therefore for $x^2+y^2=r^2<1/1.5$ the desired inequality $V_f(x,y)\geq 0$ is valid. We have found the internal boundary for the ring shaped positively invariant set that finally is defined by $\left\{1/1.5<x^2+y^2<2\right\}$.

In any equilibrium point we must have $V_f(x,y)=0$. It implies that $\left[1-\left(-xy+x^2+y^2\right)\right]\left(x^2+y^2\right)=0$ that gives us that equilibrium point must be in the origin, that is outside our positively invariant set, or on the ellipse $1=-xy+x^2+y^2$. We check that on this ellipse $x'=y$ and $y'=-x$. Therefore equilibrium points can be only the origin $(x,y)=(0,0)$. It is outside the ellipse and outside the positively invariant set defined by the inequality $1/1.5<x^2+y^2<2$.

Therefore all trajectories starting in our positively invariant set $\left\{1/1.5<x^2+y^2<2\right\}$ must have a limit set inside it that must be a periodic orbit by the Poincare-Bendixson theorem. Therefore the system must have at least one periodic orbit inside the positively invariant set.

6. Give definition of the monodromy matrix for the linear system $\mathbf{x}' = A(t)\mathbf{x}(t)$ with periodic matrix $A(t+p) = A(t)$.

   Find the monodromy matrix (scalar here) for the following linear equation with periodic coefficients

   $$x' = \left(a + \sin^3 t\right) x$$

   Find for which real values of parameter a all solutions are bounded. **(4p)**

   Solution.

   The transition matrix function $\Phi(t,\tau)$ is a solution to the matrix equation $\frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau)$, $\Phi(\tau,\tau) = I$.

   The monodromy matrix is the value $\Phi(p,0)$ where p is the period of the matrix $A(t)$. In the scalar case we can find $\Phi(t,\tau)$ explicitely.

   We calculate the primitive function of the coefficient in the equation to write down an explicit expression for $\Phi(t,\tau)$.

   $P(t) = \int \left(a + \sin^3 t\right) dt = at - \frac{3}{4}\cos t + \frac{1}{12}\cos 3t = at + \int \left(1 - \cos^2(t)\right)\sin t dt = at - \int \left(1 - \cos^2(t)\right) d\left(\cos(t)\right) = at - \cos(t) + \frac{1}{3}\cos^3(t)$.

   The transition matrix function (scalar for the scalar equation) is in our case $\Phi(t,\tau) = \exp(P(t) - P(\tau))$.

   $x(t) = \exp(P(t) - P(\tau))x_0$ is a solution of the given equation with initial data $x(\tau) = x_0$.

   The period of the coefficient in the ODE is $p = 2\pi$.

   The monodromy matrix (scalar here) is the value of the transition matrix function with starting time $\tau = 0$ and at time t equal to the period p:

   $\Phi(p,0) = \exp(P(2\pi) - P(0)) = \exp\left(2\pi a - 1 + \frac{1}{3} + 1 - 1/3\right) = \exp\left(2\pi a\right)$

   All solutions will be bounded if the Floquet multiplicator $\exp(2\pi a) \leq 1$. It is valid if and only if $a \leq 0$.

   Max. 24 points;

   Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.