MATEMATIK	Datum: 2019-06-03	Tid: 08-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
A.Heintz	Telefonvakt: Felix Held	Tel.: 5325

Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the Floquet theorem about the structure of transfer matrices to periodic linear systems of ODEs. **(4p)**

   Check the proof in lecture notes or in the course book.

2. Formulate and prove the theorem about stability of equilibrium points to autonomous ODEs by Lyapunovs method. **(4p)**

   Check the proof in lecture notes.

3. Consider the following system of ODEs: $\frac{dx(t)}{dt} = Ax(t)$, with a constant matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ having two eigenvalues 2 and 0.

i) Give general solution to the system.
ii) Find all initial conditions giving bounded solutions.
iii) Write down a Jordan canonical form for the matrix A. **(4p)**

Solution.

The general solution is constructed by applying the general formula

$$x(t) = e^{At}x_0 = \sum_{j=1}^{s}\left(\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k \frac{t^k}{k!}\right]x^{0,j}e^{\lambda_j t}\right)$$

where initial data is represented as $x(0) = x_0 = \sum_{j=1}^{s} x^{0,j}$ with $x^{0,j} \in E(\lambda_j, A)$ - components of x_0 in the generalized eigenspaces $E(\lambda_j, A) = \ker(A-\lambda_j)^{m_j}$ of the matrix A. Here s is the number of distinct eigenvalues λ_j to A and m_j is the algebraic multiplicity of the eigenvalue λ_j. We point out that $\mathbb{C}^n = E(\lambda_1, A) \oplus E(\lambda_2, A) \oplus ... \oplus E(\lambda_s, A)$.

General solution can be expressed more explicitely by finding a basis of $\mathbb{C}^n$ in terms of eigenvectors v_j and generalized eigenvectors $v_j^{(k)}$ $k = 1, ...l \leq m_j - 1$ corresponding to all distinct eigenvalues to A: λ_j, $j = 1, ...s$, so that components $x^{0,j}$ of x_0 on to the generalized eigenspaces are expressed in the form

$$x^{0,j} = ...C_p v_j + C_{p+1}v_j^{(1)} + C_{p+2}v_j^{(2)}...$$

including all linearly independent eigenvectors corresponding to each distinct λ_j (it might be several eigenvectors v_j corresponding to one λ_j) and corresponding linearly independent generalized eigenvectors for example calculated as it is suggested below.

Eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ are given and we will find corresponding eigenvectirs first. Eigenvectors corresponding to $\lambda_1 = 0$ satisfy the homogenious system $Av = 0$. We carry out Gauss elimination

$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There is only one linearly independent eigenvector corresponding to $\lambda_1 = 0$

we can choose it as $v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Eigenvectors corresponding to $\lambda_2 = 2$ satisfy the homogenious system $(A - 2I)\,v = 0$. We carry

$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.There is only one linearly independent eigenvector corresponding to $\lambda_2 = 2$

we can choose it as $v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. To find the general solution we need a generalized eigenvector, but we do not know to which eigenvalue, because we do not know their multiplicities. We can proceed by calculating the characteristic polynomial ot just can try to find a generalized eigenvector as a solution to one of the equations $(A - 2I)\, v_2^{(1)} = v_2$ or $Av_1^{(1)} = v_1$. We show both approachs.

$p(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & 2 \\ 1 & 0 & -\lambda \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 2-\lambda & 2 \end{bmatrix} + (-\lambda) \det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} =$

$= \lambda - \lambda \left((2-\lambda)^2 + 1 \right) = \lambda(1 - 1 - (2-\lambda)^2) = -\lambda(2-\lambda)^2.$

We see that the eigenvalue $\lambda_2 = 2$ has multiplicity 2 and the eigenvalue $\lambda_1 = 0$ has multiplicity 1. It means that the generalized eigenspace $E(2, A)$ has multiplicity 2 and there are generalized eigenvectors linearly independent of v_2 in this eigenspace. The generalized eigenspace $E(0, A)$ has multiplicity 1 and we have already found an eigenvectore there. We solve the equation $(A - 2I)\, v_2^{(1)} = v_2$:

$\begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We choose a generalized eigenvector (not unique!) as $v_2^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Together with two found eigenvectors it builds a basis in $\mathbb{R}^3$

We try just for fun, to find a generalised eigenvector corresponding to the eigenvalue $\lambda_1 = 0$ by considering solutions to the system $Av_1^{(1)} = v_1$.(we know we will not find any, but just like to check what will happend!)

$\begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 2 & 2 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ -1 & 2 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ *the system is not compatible.*

If we try to find a solution to the system $A^2 v = 0$ instead, we arrive after Gauss elimination to the same system as when solving the system $Av = 0$ that gave us the only linearly independent eigenvector earlier. We see that no generalised eigenvector of rank higher then one exist.

Now representing arbitrary initial data as $x_0 = C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}$ we can write down general solution

$$\begin{aligned} x(t) &= e^{At} x_0 = C_1 e^{0t} v_1 + C_2 e^{2t} v_2 + C_3 e^{2t} \left(v_2^{(1)} + t\,(A - 2I)\, v_2^{(1)} \right) = \\ & C_1 v_1 + C_2 e^{2t} v_2 + C_3 e^{2t} \left(v_2^{(1)} + t v_2 \right) \end{aligned}$$

Initial conditions giving bounded solution are on the line through the origin and parallel to v_1.

Jordan canonical form for the matrix A is $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ if we choose transformation matrix $V = \left[v_1, v_2, v_2^{(1)} \right]$.■

4. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction. **Hint.** Try test functions in the form $V(x,y) = x^{2n} + ay^{2m}$.

$$\begin{cases} x' = & y \\ y' = & -y + y^3 - 2x^3 \end{cases}$$

Solution.

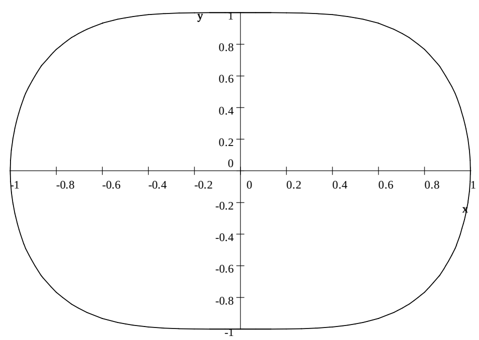
We take the test function $V(x,y) = x^4 + y^2$ that leads to cancelling indefinite terms, and consider

$$\begin{aligned} V_f(x,y) &= \nabla V \cdot f = \begin{bmatrix} 4x^3 \\ 2y \end{bmatrix} \cdot \begin{bmatrix} y \\ -y + y^3 - 2x^3 \end{bmatrix} = \\ &= 4x^3 y - 4yx^3 - 2y^2 + 2y^4 = -2y^2 + 2y^4 \\ &= -2y^2(1 - y^2) \leq 0, \text{ if } \quad |y| < 1 \end{aligned}$$

It implies that the equilibrium point in the origin is stable according to Lyapunovs stability theorem.

The set $V_f^{-1}(0)$ consists of the x - axis and contains the only invariant set consisting of the origin. It implies according to a Corollary from the LaSalles invariance principle that the origin is asymptotically stable.

Level curves of the Lyapunov function $V(x,y) = x^4 + y^2$ are closed curves looking as ellipses flattended in x - direction. They bound positively invariant sets if they lay inside the stripe $|y| < 1$, where the estimate $V_f(x,y) \leq 0$ is valid. The largest such curve must cross the point $(0,1)$ laying on the boundary of the stripe $|y| < 1$. It is the curve $V(x,y) = x^4 + y^2 = 1$. It bounds the region of attraction for the equilibrium in the origin. All trajectories starting in this domain tend to the origin when $t \to \infty$.



.

5. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x + 3y - x\left(x^2 + y^2\right) \\ y' = -x + y - y\left(x^2 + y^2\right) \end{cases}$$

(4p)

Solution.

We like to find a ring shaped posiively invariant set Ω, excluding the equilibrium point in the origin and such that it does not contain another equilibrium points if such exist. Then, according to Poincare Bendixson theorem, we can state that the system has at least one periodic solution in the set Ω.

We try to check directions of the gradient of the test function $V(x,y) = x^2 + 3y^2$.

$$\begin{aligned} V_f(x,y) &= \begin{bmatrix} 2x \\ 6y \end{bmatrix} \cdot \begin{bmatrix} x + 3y - x\left(x^2 + y^2\right) \\ -x + y - y\left(x^2 + y^2\right) \end{bmatrix} = \\ &= 2x^2 + 6xy - 2x^2\left(x^2 + y^2\right) - 6xy + 6y^2 - 6y^2\left(x^2 + y^2\right) \\ &= \left(2x^2 + 6y^2\right)\left(1 - \left(x^2 + y^2\right)\right) \end{aligned}$$

We observe that $V_f(x,y) \geq 0$ for $\left(x^2 + y^2\right) \leq 1$ and $V_f(x,y) \leq 0$ for $\left(x^2 + y^2\right) \geq 1$. The circle $\left(x^2 + y^2\right) = 1$ separates areas where trajectories go outside and inside level sets of the test function

The compact positively invariant set Ω that wee look for must be a "ring" bounded by level sets of the test function $V(x,y) = x^2 + 3y^2$. These level sets are ellipses $\mathfrak{e}$ and $\mathfrak{E}$, oriented along coordinate axes. We need to choose their sizes in such a way that trajectories go out from the smaller ellipse $\mathfrak{e}$ and go into the larger ellipse $\mathfrak{E}$. It means that we need to choose $\mathfrak{e}$ inside the circle $\left(x^2+y^2\right) = 1$ and $\mathfrak{E}$ - outside the circle $\left(x^2+y^2\right) = 1$.

We can just choose $\mathfrak{e}$ small enough and $\mathfrak{E}$ - large enough. For example the largest possible Ω satisfying our requirements is bounded by $\mathfrak{e}=\left\{(x,y) : x^2+3y^2 = 1 \quad\right\}$ and $\mathfrak{E} = \left\{(x,y) : x^2+3y^2 = 3\right\}$.

Now we check if there are any equilibrium points in Ω. We observe that $V_f(x,y) = 0$ outside the origin only if $\left(x^2+y^2\right) = 1$. By checking ODEs for $\left(x^2+y^2\right) = 1$ namely $x' = 3y = 0$, and $y' = -x = 0$, we observe that there are no equilibriums on this circle, because ODEs imply that $x = y = 0$. Therefore Ω is positively invariant compact set containing no equilibrium points and must contain at least one periodic orbit.■

6. Consider the following operator

$$K(x)(t) = \int_0^2 \sin(ts)\ [x(s)]^2\, ds + Ct,$$

for all $t \in [0,2]$ acting in the Banach space $C([0,2])$ of continuous functions with the norm $\|x\|_C = \sup_{t\in[0,2]} |x(t)|$.

i) Formulate Banach's contraction principle.

ii) Find a subset A in $C([0,2])$ and conditions on the constant C such that the operator $K(x)(t)$ has a fixed point in A by the Banach's contraction principle. **(4p)**

Banach's contraction principle. Let A be a nonempty closed subset of a Banach space X and let the non-linear operator $K : A \to A$ be a contraction:

$$\|K(x) - K(y)\|_X \le \theta \|x-y\|_X, \quad \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x_0) - \overline{x}\|_X \le \frac{\theta^n}{1-\theta}$$

for any $x_0 \in A$. Here $K^n(x_0) = (K(K(...K(x_0)...))$ is the n -fold superposition of the operator K with itself.

Solution.

We like to prove the estimate $\|K(x)-K(y)\|_C \le \theta\|x-y\|_C$ for x,y in some closed subset A of $C([0,2])$. Consider the absolute value of the difference $K(x) - K(y)$:

$$\begin{aligned}
|K(x)-K(y)| &\le \left|\int_0^2 |\sin(ts)| \left|[x(s)]^2 - [y(s)]^2\right| ds\right| \\
&= \left|\int_0^2 |\sin(ts)|\,|x(s)-y(s)|\,|x(s)+y(s)|\, ds\right| \overset{\text{taking } \sup_{t,s\in[0,2]}}{\le} \\
&\le \int_0^2 ds \sup_{t,s\in[0,2]} |\sin(ts)| \sup_{s\in[0,2]} |x(s)-y(s|) \left(\sup_{s\in[0,2]}|x(s)| + \sup_{s\in[0,2]}|y(s)|\right) \\
&= 2\|x-y\|_C (\|x\|_C + \|y\|_C) = \|x-y\|_C (2(\|x\|_C + \|y\|_C))
\end{aligned}$$

Then we take $\sup_{t,s\in[0,2]}$ of the left hand side of the inequality. We now can choose a ball $A \subset C([0,2])$ such that for any $x,\ y \in A$ it follows $(2(\|x\|_C + \|y\|_C)) \le \theta < 1$, for example A can be taken as a set of continuous functions on $[0,2]$ with $\|x\|_C \le \theta/4$. On this set K will be a contraction because $\|K(x)-K(y)\|_C \le \theta\|x-y\|_C$, $\theta < 1$.

To apply Banach's contraction principle we need also that K maps the set A where the contraction property is valid into itself. Namely we need that $\|K(x)\|_C \le \theta/4$ for $\|x\|_C \le \theta/4$

It will give a requirement on the constant C in the definition of the operator K. Estimate the operator K :

$\|K(x)\|_C \le \sup\limits_{t\in[0,2]} \left| \int_0^2 \sin(ts) \left([x(s)]^2\right) ds \right| + \sup\limits_{t\in[0,2]} |Ct| \le \|x\|_C^2 + 2\,|C|$

If $\|x\|_C < \theta/4$ then we like to have that $\|K(x)\|_C \le \theta/4$ that follows if $\|K(x)\|_C \le \theta^2/16 + 2\,|C| \le \theta/4$. Rewrite the inequality: $2\,|C| \le \theta/4 - \theta^2/16$

It is satisfied if $|C| \le \theta/8 - \theta^2/32 = \theta/8\,(1-\theta/4)$.

Therefore for $|C| \le \theta/8\,(1-\theta/4)$ the operator K has a unique fixed point in the ball A in $C([0,2]) : \|x\|_C \le \theta/4$.$\blacksquare$

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as

$Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for home assignments, and for the total amount of points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will pass all obligatory parts of the course.