MATEMATIK Datum: 2019-01-07 Tid: 08-30 - 12-30
GU, Chalmers Hjälpmedel: - Inga
A.Heintz Telefonvakt: Linnea Hietala Tel.: 5325

Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

1. Formulate and give a proof to the theorem about the dimension of the solution space for the system of linear ODEs. **(4p)**

Theorem. (Proposition 2.7, p.30, L.R. in the case of non-autonomous systems).

Let $b_1, ..., b_N$ be a basis in $\mathbb{R}^N$(or $\mathbb{C}^N$). Then the functions $y_j : \mathbb{R} \to \mathbb{R}^N$(or $\mathbb{C}^N$) defined as solutions to the I.V.P.

$$x'(t) = A(t)x(t) \tag{1}$$

with $y_j(\tau) = b_j$, $j = 1, ...N$, by $y_j(t) = \exp(A(t-\tau))b_j$, form a basis for the space $\mathcal{S}_{\text{hom}}$ of solutions to (1). The dimension of the vector space $\mathcal{S}_{\text{hom}}$ of solutions to (1) is equal to N - the dimension of the system (1).

Hint to the proof. This property is a consequence of the linearity of the system and the uniqueness of solutions to the system and is independent of detailed properties of the matrix $A(t)$.

Proof. Consider a linear combination of $y_j(t)$ equal to zero for some time $\sigma \in \mathbb{R}$: $l(\sigma) = \sum_{j=1}^N \alpha_j y_j(\sigma) = 0$. The trivial constant zero solution coinsides with l at this time point.

But by the uniqueness of solutions to (1) it implies that $l(t)$ must coinside with the trivial zero solution for all times and in particular at time $t = \tau$. Therefore $\sum_{j=1}^N \alpha_j b_j = 0$. It implies that all coefficients $\alpha_j = 0$ because $b_1, ..., b_N$ are linearly independent vectors in $\mathbb{R}^N$(or $\mathbb{C}^N$). Therefore $y_1(t), ..., y_N(t)$ are linearly independent for all $t \in \mathbb{R}$ by definition. Arbitrary initial data $x(\tau) = \xi$ can be represented as a linear combination of basis vectors $b_1, ..., b_N$: $\xi = \sum_{j=1}^N C_j b_j$. The construction of $y_1(t), ..., y_N(t)$ shows that an arbitrary solution to (1) can be represented as linear combination of $y_1(t), ..., y_N(t)$.

$$x(t) = \exp(A(t-\tau))\xi = \exp(A(t-\tau)) \sum_{j=1}^N C_j b_j = \sum_{j=1}^N C_j y_j(t)$$

Therefore $\{y_1(t), ..., y_N(t)\}$ is the basis in the space of solutions $\mathcal{S}_{\text{hom}}$ and therefore $\mathcal{S}_{\text{hom}}$ has dimension N.$\blacksquare$

2. Formulate and give a proof to LaSalle's invariance principle. **(4p)**

Formulation.

Assume that f is locally Lipschitz as before and let $\varphi(t, \xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0.$ for all $z \in U$. If $\xi \in U$ is such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U, then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and $\varphi(t, \xi)$ approaches as $t \to \infty$ the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

Proof given in the solution of Exercise 5.9, on p. 312.

Exercise 5.9
Set $x(\cdot) := \varphi(\cdot, \xi)$. By continuity of V and compactness of $\mathrm{cl}(O^+(\xi))$, V is bounded on $O^+(\xi)$ and so the function $V \circ x$ is bounded. Since $(\mathrm{d}/\mathrm{d}t)(V \circ x))(t) = V_f(x(t)) \leq 0$ for all $t \in \mathbb{R}_+$, $V \circ x$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t)) =: \lambda$ exists and is finite. Let $z \in \Omega(\xi)$ be arbitrary. Then there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to z$ as $n \to \infty$. By continuity of V, it follows that $V(z) = \lambda$. Consequently,

$$V(z) = \lambda \quad \forall z \in \Omega(\xi). \tag{$*$}$$

By invariance of $\Omega(\xi)$, if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$ and so $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$. Therefore, $V_f(\varphi(t, z)) = 0$ for all $t \in \mathbb{R}$. Since $\varphi(0, z) = z$ and z is an arbitrary point of $\Omega(\xi)$, it follows that

$$V_f(z) = 0 \quad \forall\, z \in \Omega(\xi), \tag{$**$}$$

and so $\Omega(\xi) \subset V_f^{-1}(0)$. The claim now follows because, by Theorem 4.38, $\Omega(\xi)$ is invariant and $x(t)$ approaches $\Omega(\xi)$ as $t \to \infty$.

Comment. It might be tempting to conclude from $(*)$ that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$, which then immediately would yield $(**)$. However, this conclusion is not correct: the set $\Omega(\xi)$ is not open and therefore $(*)$ does not imply that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$. (The invalidity of the conclusion is illustrated by the following simple example: if $V(z) = \|z\|^2$ and $\Omega(\xi) = \{z \in \mathbb{R}^N : \|z\| = 1\}$, then $V(z) = 1$ for all $z \in \Omega(\xi)$, but $(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.)

3. Consider the following matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

a) Write down a canonical Jordan form J for the matrix A and find the matrix P in the relation $J = P^{-1}AP$ using eigenvectors and generalised eigenvectors to A (do not calculate P^{-1}).

b) Write down general solution to the system $x' = Ax$ with this matrix A. **(4p)**

Solution.

a)

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{bmatrix} =$

$(2-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = (2-\lambda)\left(\lambda^2 - 4\lambda + 4\right) = (2-\lambda)^3.$

The only eigenvalue $\lambda = 2$ has multiplicity 3.

Find eigenvectors. $(A - 2I) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. There are two linearly independent eigenvectors, for example $w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We need one more generalized eigenvector to construct the representation $J = P^{-1}AP$. We consider equations $(A - 2I)x = w_1$ and $(A - 2I)x = v_2$ for possible generalized eigenvectors. They both are not solvable because the range of the matrix $(A - 2I)$ is one dimensional and consists of vectors parallel to the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_1 + v_2$. It motivates to choose eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and a generalized eigenvector $v_1^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ that is a solution to the equation $(A - 2I)x = v_1$.

We can finally make conclution that the Jordan form corresponding matrix A is $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and corresponding matrix P in the relation $J = P^{-1}AP$ has columns v_1, $v_1^{(1)}$ and v_2: $P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. $\blacksquare$

Solution.

b) The general solution $x(t) = e^{At}x_0$ to the equation can be written by choosing the initial data x_0 expressed in terms of the basis of eigenvectors and generalized eigenvectors $x_0 = C_1 v_1 + C_2 v_1^{(1)} + C_3 v_2$ and using for each term the representation for the exponent e^{At} acting on an element $x^{0,j}$ of a particular generalized eigenspace:

$$e^{At}x^{0,j} = \left[\sum_{k=0}^{m_j - 1} (A - \lambda_j I)^k \frac{t^k}{k!} \right] e^{\lambda_j t} x^{0,j}$$

where m_j is the multiplicity of the eigenvalue λ_j and $x^{0,j}$ is an element of the corresponding generalized eigenspace.

General solution to the particular system of interest with initial data $\xi = C_1 v_1 + C_2 v_1^{(1)} + C_3 v_2$is

$$x(t) = e^{2t} \left(C_1 v_1 + tC_2 v_1 + C_2 v_1^{(1)} + C_3 v_2 \right)$$

follows from the general expression

4. Find for which values of the parameter a the origin is an asymptotically stable equilibrium, stable equilibrium, unstable equilibrium of the following system:

$$\begin{cases} x' = y \\ y' = -ay - x^3 - a^2 x \end{cases} \qquad \textbf{(4p)}$$

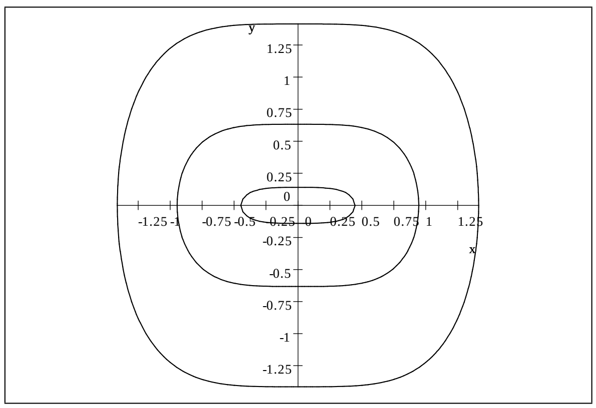
Solution. Consider the Jacoby matrix of the right hand side in the equatiuon.

$A(x, y) = \begin{bmatrix} 0 & 1 \\ -a^2 - 3x^2 & -a \end{bmatrix}$. It's value in the origin is $A(0, 0) = \begin{bmatrix} 0 & 1 \\ -a^2 & -a \end{bmatrix}$, with characteristic polynomial: $p(\lambda) = \lambda^2 + a\lambda + a^2$.

Eigenvalues are $\lambda_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - a^2} = -\frac{a}{2} \pm i\sqrt{\frac{3a^2}{4}}$

The Grobman - Hartman theorem about stability by linearization imples that the origin is asymptotically stable when $a > 0$ and is unstable when $a < 0$. For $a = 0$ linearization does not give any information about stability because in this case $\operatorname{Re}\lambda = 0$. In this case the system is reduced to $\begin{cases} x' = y \\ y' = -x^3 \end{cases}$ and we can find an equation for trajectories of the system from an ODE with separable variables:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-x^3}{y} \\
ydy &= -x^3dx \\
\int ydy &= -\int x^3dx \\
\frac{y^2}{2} &= -\frac{x^4}{4} = +C \\
\frac{x^4}{4} + \frac{y^2}{2} &= 1
\end{aligned}$$



We observe that trajectories for the system are closed curves like "flattened circles" parametrized by $C \geq 0$. It implies that the origin is a stable equilibrium when $a = 0$.■

◊ One could also just guess that the positive definite test function $V(x,y) = \frac{x^4}{2} + y^2$ is a weak Lyapunov function for the system with the right hand side $f(x,y)$:

$$\begin{aligned}
f(x,y) &= \begin{bmatrix} y \\ -x^3 \end{bmatrix} \\
f \cdot \nabla V &= 0
\end{aligned}$$

and make the same conclusion that the origin is a stable equilibrium in the case of $a = 0$.■

5. Consider the following system of ODEs. Prove the instability of the equilibrium point in the origin, of the following system

$$\begin{cases} x' = & x^5 + y^3 \\ y' = & x^3 - y^5 \end{cases}$$ **(4p)**

using the test function $V(x,y) = x^4 - y^4$ and Lyapunov's instability theorem.

Solution.

Denoting $f(x,y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix}$, consider how $V(x,y) = x^4 - y^4$changes along trajectories of the system. $f(x,y) \cdot \nabla V(x,y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix} \cdot \begin{bmatrix} 4x^3 \\ -4y^3 \end{bmatrix} =$

$x^5 4x^3 + y^3 4x^3 - x^3 4y^3 + y^5 4y^3 = x^5 4x^3 + y^5 4y^3 = 4(x^8 + y^8) > 0.$

Point out that the function $V(x,y) = x^4 - y^4$ is positive along the line $y = x/2$, $x > 0$ arbitrarily close to the origin. It implies according to the instability theorem, that the origin is an unstable equilibrium.■

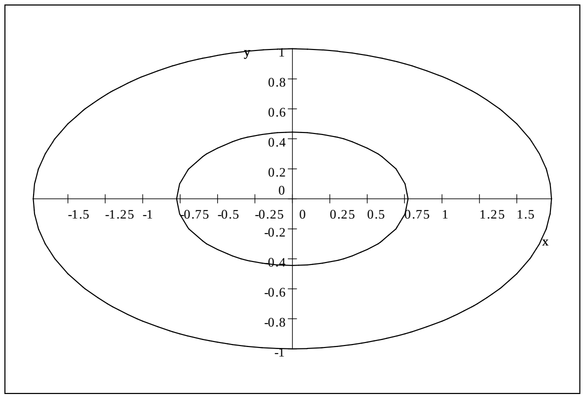
6. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = y \\ y' = -\frac{1}{3}x + (1 - 4x^2 - y^2)y \end{cases}$$ **(4p)**

Solution.

We intend to proof that conditions of the Poincare - Bendixsons theorem can be satisfied. Nemaly, that there is a ring shaped positively invariant set that does not include equilibrium points of the system. We try to find such an invariant set bounded by level sets of a test function $V(x,y)$.

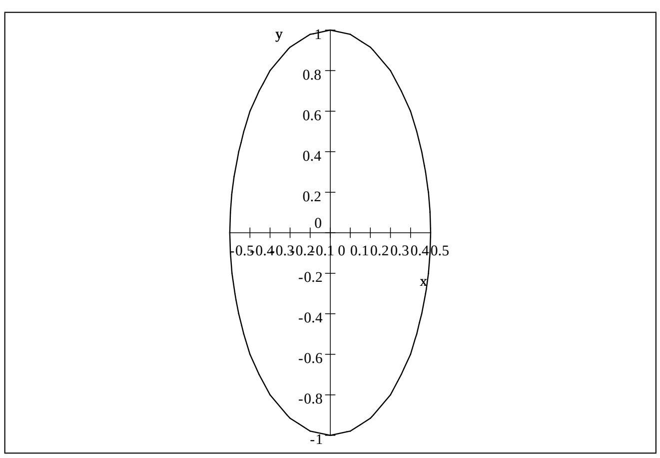
A convenient choice of the test function is $V(x,y) = \frac{1}{3}x^2 + y^2$. Its level sets are ellipses with center in the origin: $\frac{1}{3}x^2 + y^2 = C$:



For $f(x,y) = \begin{bmatrix} y \\ -\frac{1}{3}x + (1 - 4x^2 - y^2)y \end{bmatrix}$ we have

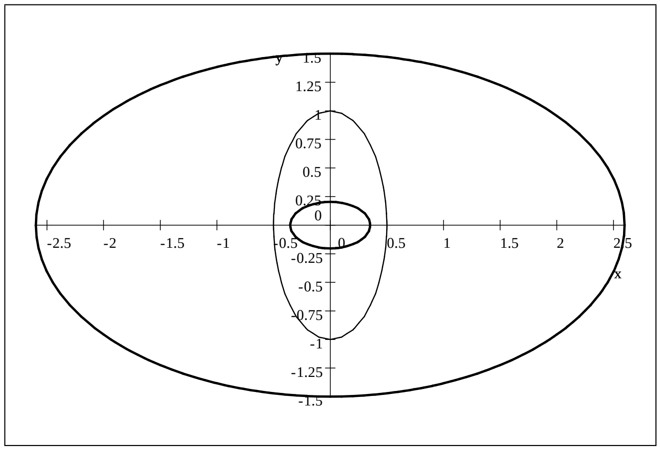
$$\begin{aligned} f(x,y) \cdot \nabla V(x,y) &= \begin{bmatrix} y \\ -\frac{1}{3}x + (1 - 4x^2 - y^2)y \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3}x \\ 2y \end{bmatrix} = \\ &= y\frac{2}{3}x + 2y\left(-\frac{1}{3}x + (1 - 4x^2 - y^2)y\right) \\ &= 2y^2(1 - 4x^2 - y^2) \end{aligned}$$

we observe that $f(x,y) \cdot \nabla V(x,y) < 0$ outside the ellipse $\mathcal{E}$: $4x^2 + y^2 = 1$



and $f(x,y) \cdot \nabla V(x,y) > 0$ inside this ellipse. It means that inside the ellipse $\mathcal{E}$ all trajectories cross level sets of the test function V in the direction out of the origin. Similarly outside the ellipse $\mathcal{E}$ all trajectories cross level sets of the test function V in the direction towards the origin.

To find the desired ring shaped invariant set we must find a level set of V that lays completely outside $\mathcal{E}$ for the outer boundary of the invariant set, and a level set of V that lays completely inside $\mathcal{E}$ for the inner boundary of the invariant set. We choose the ellipse $V(x,y) = \frac{1}{3}x^2 + y^2 = (1.5)^2$ as the outer boundary so that it upper point is above the upper point of the elipse $\mathcal{E}$ and the elllipse $V(x,y) = \frac{1}{3}x^2 + y^2 = \frac{1}{6}(0.5)^2$as the inner boundary, so that its left and right points are inside the ellipse $\mathcal{E}$.



The ring shaped set can be specified by inequalities $\frac{1}{6}(0.5)^2 \leq V(x,y) \leq (1.5)^2$.

The system has only one equilibrium point in the origin. It follows from the observation that equilibrium points must be on the x - axis and from the fact that $y' = -\frac{1}{3}x$ on the x - axeln and is zero only in the origin.

We can conclude that the system must have at least one periodic orbit inside the set defined by $\frac{1}{6}(0.5)^2 \leq V(x,y) \leq (1.5)^2$.■