MATEMATIK | Datum: 2018-08-29 | Tid: 14-00 - 18-00
GU, Chalmers | Hjälpmedel: - Inga
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Lösningsförslag till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Formulate and give a proof to the Picard - Lindelöf existence and uniqueness theorem. **(4p)**

   Check the formulation and the proof in lecture notes on the home page of the course or in the course book .

2. Formulate and give a proof to Bendixsons kriterium for non-existence of periodic solutions to systems of ODEs in the plane. **(4p)**

   Check the formulation and the proof in lecture notes on the home page of the course.

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$,

   $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$. Find a general solution to the system by using generalised eigenvectors to A. Write down a canonical Jordan form of the matrix A. **(4p)**

   Solution.

   $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$, The characteristic polynomial of the matrix A is $p(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$. It is easy to see that

   $p(\lambda) = (\lambda - 1)^3$ and therefore $\lambda = 1$ is an eigenvalue with multiplicity 3.

   We check eigenvectors of the matrix $\dot{A}$.

   $(A - I)\, v = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The last equation implies that $x = 0$. The second equation implies that $z = 0$. Therefore y is arbitrtary and the eigenspace is one dimensional and is spanned by the eigenvector $v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

   We find generalized eigenvector $v^{(1)}$ satisfying the equation $(A - I)\, v^{(1)} = v$. The system reads as:

   $$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

   The third equation implies $x = 0$. The second equation implies that y is arbitrary, for example $y = 1$ and $z = 1$.

$$v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We find one more generalized eigenvector $v^{(2)}$ satisfying the equation $(A - I)\, v^{(2)} = v^{(1)}$. This chain connection guarantees that the found generalized eigenvectors will be linearly independent.

The system reads as:

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The third equation implies $x = 1/3$; The second equation reads: $-1/3 + 2z = 1$; $2z = 4/3$; $z = 2/3$; we choose y that is arbitrary as $y = 0$;

$$v^{(2)} = 1/3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

By choosing initial data $x_0 = x(0)$ expressed as a linear combination of the eigenvector v and generalised eigenvectors $v^{(1)}$and $v^{(2)}$

we can express general solution to the system with help of the following formula

$$\begin{aligned} x(t) &= e^{At} x_0 = \left(\left[\sum_{k=0}^{2} (A - I)^k \frac{t^k}{k!} \right] x_0 e^t \right) \\ x_0 &= C_1 v_j + C_2 v^{(1)} + C_3 v^{(2)} \end{aligned}$$

for solutions with initial data $x(0) = x_0$. The matrix A has only one eigenvalue $\lambda = 1$. It makes the formula simpler comparing with the general case. We rewrite this formula separately for each of the therm in x_0.

$e^{At} C_1 v = C_1 v e^t$

$e^{At} C_2 v^{(1)} = C_2 \left(\left[\sum_{k=0}^{1} (A - I)^k \frac{t^k}{k!} \right] v^{(1)} e^t \right) = C_2 e^t \left[v^{(1)} + t (A - I)\, v^{(1)} \right] = C_2 e^t \left[v^{(1)} + t v \right]$

$e^{At} C_3 v^{(2)} = C_3 \left(\left[\sum_{k=0}^{2} (A - I)^k \frac{t^k}{k!} \right] v^{(2)} e^t \right) = C_3 e^t \left(v^{(2)} + t (A - I)\, v^{(2)} + \frac{t^2}{2} (A - I)^2 v^{(2)} \right) = C_3 e^t \left(v^{(2)} + t v^{(1)} + \frac{t^2}{2} v \right)$

Therefore the general solution to the system of ODEs with initial data $x_0 = C_1 v_j + C_2 v^{(1)} + C_3 v^{(2)}$ is equal to

$$x(t) = C_1 e^t v + C_2 e^t \left(v^{(1)} + t v \right) + C_3 e^t \left(v^{(2)} + t v^{(1)} + \frac{t^2}{2} v \right)$$

The Jordan canonical form of the matrix A is $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

4. Find the monodromy matrix (scalar here) for the following linear equation with periodic coefficients

$$x' = \left(a + \sin^2 t \right) x$$

Find for which real values of parameter a all solutions are bounded. **(4p)**

Solution. We calculate the primitiv function of the coefficient in the equation to write down explicit solution.

$P(t) = \int \left(a + \sin^2 t\right) dt = \frac{1}{2}t + at - \frac{1}{4}\sin 2t$. The transition matrix function (scalar for the scalar equation) is $\Phi(t,\tau) = \exp(P(t) - P(\tau))$.

$x(t) = \exp(P(t) - P(\tau))x_0$ is a solution of the given equation with initial data $x(\tau) = x_0$.

The period of the coefficient in the ODE is $p = \pi$, because $\sin^2 t = \frac{1-\cos(2t)}{2}$.

The monodromy matrix (scalar here) is the value of the transition matrix function with starting time $\tau = 0$ and at time t equal to the period p:

$\Phi(p, 0) = \exp(P(\pi)) = \exp(\frac{1}{2}\pi + a\pi - \frac{1}{4}\sin 2\pi) = \exp(\pi\left(\frac{1}{2} + a\right))$

All solutions will be bounded if the Floquet multiplica$\exp(\pi\left(\frac{1}{2} + a\right)) \leq 1$. It is valid if $a \leq -1/2$.

5. Consider the following system of ODEs. Investigate stability of the equilibrium point in the origin, and find a possible domain of attraction.

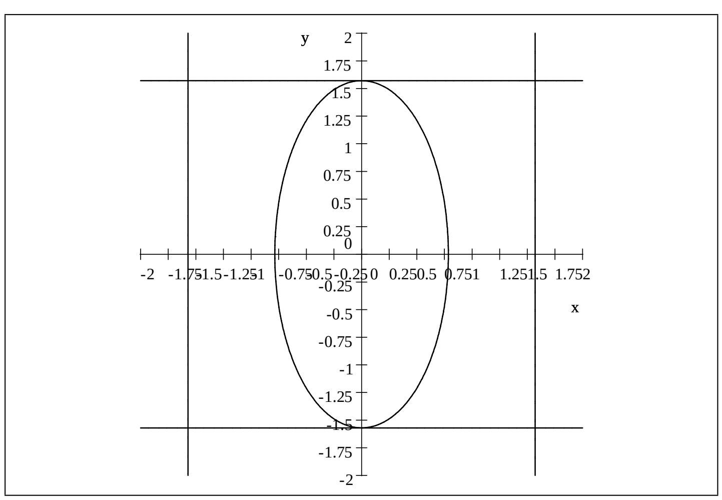
$$\begin{cases} x' = & -\sin(x) + y \\ y' = & -4x - 3\tan(y) \end{cases}$$ **(4p)**

Solution.

We can choose a test function $V(x,y) = 2x^2 + \frac{1}{2}y^2$ to cancel terms with indefinite sign in $\nabla V \cdot f(x,y)$.

$\nabla V \cdot f(x,y) = V_f(x,y) = 4xy - 4xy - 4x\sin(x) - 3y\tan(y) = -4x\sin(x) - 3y\tan(y)$

We observe that $-4x\sin(x) \leq 0$ and $-3y\tan(y) \leq 0$ for $(x,y) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ because sin and tan are odd functions. Therefore $\nabla V \cdot f(x,y) \leq 0$ and is zero only in the point $(0,0)$. Therefore the origin is asymptotically stable. The maximal ellipse that fits to the rectangle $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is a boundary of a domain of attraction (not necessary the largest one but one we get with help of the chosen Lyapunov function). It is the ellipse $V(x,y) = const$ with center in the origin and half axes $\pi/2$ in y direction and $\pi/4$ in x direction: $\dfrac{x^2}{(\pi/4)^2} + \dfrac{y^2}{(\pi/2)^2} = 1$ or $V(x,y) = 2x^2 + \frac{1}{2}y^2 = \frac{\pi^2}{8}$.



6. Show that the following system of ODEs has a periodic solution. **Hint:** use polar coordinates r, θ.

$$\begin{cases} x' = -y - x(x^2+y^2-3x-1) \\ y' = x - y(x^2+y^2-3x-1) \end{cases} \qquad \textbf{(4p)}$$

Solution.

We can write an equation for $r = \sqrt{x^2+y^2}$:

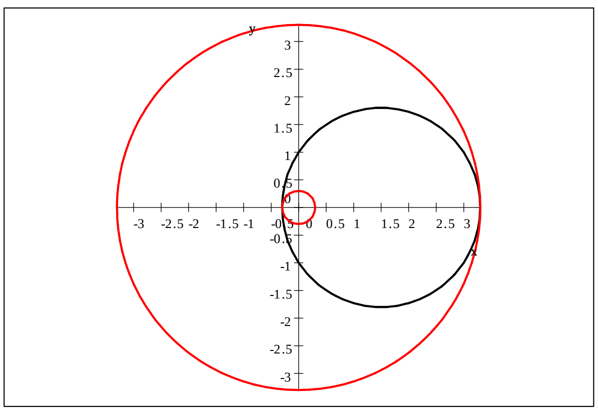
$xx' + yy' = \frac{1}{2}\left(x^2+y^2\right)' = rr' = -\left(x^2+y^2\right)\left(x^2+y^2-3x-1\right) = -r^2(r^2 - 3r\cos\theta - 1)$

$r' = -r(r^2 - 3r\cos\theta - 1)$

It easy to observe that $(r^2 - 3r\cos\theta - 1) < 0$ for $r < \varepsilon$ with some $\varepsilon > 0$ small enough, and that $(r^2 - 3r\cos\theta - 1) > 0$ for $r > R$ for some $R > 0$ large enough. Therefore trajectories of solutions starting in the ring shaped domain $\varepsilon < r < R$ do not leave this domain.

A more precise analysis is also possible. The sign of r' depends only on the sign of the quadratic form x^2+y^2-3x-1 having level sets circles with center in the point $(3/2, 0)$ on the x - axis.

$x^2+y^2-3x-1 = (x-3/2)^2 + y^2 - 13/4$. It means that the circle C with the equation $(x-3/2)^2 + y^2 = \left(\sqrt{13}/2\right)^2$ with radius $\sqrt{13}/2$ separates domains where r' is positive and negative. Choosing a circle with center on the origing that is contained inside the circle C and another one that contains the circle C gives an annulus domain that is positively invariant set for our equation. See the picture:



Therefore ε above can be chosen as $\varepsilon = \sqrt{13}/2 - 3/2$ and R can be chosen as $R = \sqrt{13}/2 + 3/2$.

To conclude about the number of equilibrium points inside this positively invariant set we derive also an expression for the derivative of the polar angle: $\theta' = 1$ by differentiating $(\tan\theta)'$ and using expressions for x' and y' from the system.

$(\tan\theta)' = \frac{1}{\cos^2(\theta)}\theta' = \left(\frac{y}{x}\right)' = \frac{xy'-yx'}{x^2} = \frac{x\left(x-y(x^2+y^2-3x-1)\right)-y\left(-y-x(x^2+y^2-3x-1)\right)}{x^2} =$

$\frac{x^2-xy(x^2+y^2-3x-1)+y^2+xy(x^2+y^2-3x-1)}{x^2} = \frac{x^2+y^2}{x^2} = \frac{r^2}{r^2\cos^2\theta}$

It implies that $\theta' = 1$ and is never zero. Therefore the system has no equilibrium points except the origin. The Poincare Bendixson theorem implies that the domain $\varepsilon < r < R$ must contain at least one periodic orbit.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3:** 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course are calculated as:

$Total = 0.16\,Assignment1 + 0.16\,Assignment2 + 0.68\,Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will collect all necessary points.