MATEMATIK	Datum: 2017-08-21	Tid: 14.00-18:00
GU, Chalmers	Hjälpmedel: - Inga	
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Define what is a generalized eigenspace of a matrix. Formulate and proof the theorem on the properties of solutions $\overrightarrow{r}(t)$ to a linear system of ODEs $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$ with constant matrix A, when $t \to +\infty$ (when they all tend to zero and when they all are bounded) **(4p)**

2. Formulate and prove Bendixsons criterion for non-existence of periodic solutions. **(4p)**

3. Solve the initial value problem: $\dot{x} = t\,\exp(x)$, $x(0) = \ln(2)$, and find the maximal interval for the solution. **(2p)**

   Can one conclude which maximal interval have solutions to the equation $\dot{x} = t^3(1 + x\cos(x))$? **(2p)**

   Solution. The first equation can be solved by the separation of variables.

   $\exp(-x)dx = tdt$; $\int \exp(-x)dx = \int tdt + C$; $-\exp(-x) = \frac{t^2}{2} + C$. Using the intitial condition $x(0) = \ln(2)$ we arrive to $-\exp(-\ln(2)) = C$ and $C = -\frac{1}{2}$.

   Therefore $\exp(-x) = \frac{1}{2} - \frac{t^2}{2}$ and $x(t) = -\ln\left(\frac{1}{2} - \frac{t^2}{2}\right)$.

   The maximal interval for this solution is the open interval $(-1, 1)$. The solution tends to ∞ when t tends to -1 from above and to 1 from below and therefore cannot be extended to a larger interval.

   The equation $\dot{x} = t^3(1 + x\cos(x))$ has the right hand side that is evidently bounded by $C(1 + |x|)$ on any bounded time interval whith a constant C dependent on this time interval. It implies that all solution are defined for time t on the whole real line $\mathbb{R}$.

4. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are unbounded. **(4p)**

   Solution. The caracteristic polinomial fo the matrix $A = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$ characteristic polynomial: $p(\lambda) = \lambda^3 + 3\lambda^2$.

   Eigenvalues are $\lambda_1 = 0$ with multiplicity 2 and $\lambda_2 = -3$.

   Eigenvector v_1 corresponding to λ_1 satisfies the equation $Av_1 = 0$ and can be chosen as $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The generalized eigenvector v_1^1 satisfies the equation $Av_1^1 = v_1$ or and can be chosen as $\begin{bmatrix} 1/3 \\ 4/3 \\ 1 \end{bmatrix}$.The eigenvector v_2 corresponding to the simple eigenvalue $\lambda_2 = -3$ satisfies the equation $\begin{bmatrix} 2 & 1 & -1 \\ 2 & 4 & -1 \\ 0 & 3 & 0 \end{bmatrix}$ and can be chosen as $v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

The general solution $x(t) = e^{At}x_0$ to the equation can be written by choosing the initial data x_0 expressed in terms of the basis of eigenvectors and generalized eigenvectors $x_0 = C_1v_1 + C_2v_1^1 + C_3v_2$ and using for each term the representation for the exponent e^{At} acting on an element $x^{0,j}$ of a particular generalized eigenspace:

$e^{At}x^{0,j} = \left[\sum_{k=0}^{m_j-1} (A-\lambda_j I)^k \frac{t^k}{k!}\right] e^{\lambda_j t}x^{0,j}$

where m_j is the multiplicity of the eigenvalue λ_j and $x^{0,j}$ is an element of the corresponding generalized eigenspace. It implies that the general solution is the linear combination of expressions of this type and is expressed as

$x(t) = C_1v_1 + C_2v_1^1 + t\,C_2v_1 + C_3e^{-3t}v_2$

because $m_1 = 2$, $m_2 = 1$, and $(A - \lambda_1 I)v_1^1 = v_1$. The solutions will be bounded for all x_0 in the form $x_0 = C_1v_1 + C_3v_2$.

5. Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

   Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

   Hint: applying Lyapunovs theorem, use the elementary inequality $2xy \le \left(x^2 + y^2\right)$ to estimate indefinite terms with xy. **(4p)**

   Solution. Choose a test function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$

   $V_f = x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2$

   $= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + xy$

   We apply the inequality $2xy \le \left(x^2 + y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

   $V_f \le -x^2\left(0.5 - y^2\right) - y^2\left(2.5 - y^2\right)$

   It implies that $V_f < 0$ for $(x,y) \ne (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the origin is asymptotically stable.

   The attracting region is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely

   $\left(x^2 + y^2\right) < 1/2$.

   Another more clever choice of a test function is $V(x,y) = 3x^2 + 2y^2$.

   $V_f = 6x(-x - 2y + xy^2) + 4y(3x - 3y + y^3) = 4y^4 - 12y^2 - 6x^2 + 6x^2y^2 = -4y^2\left(3 - y^2\right) - 6x^2\left(1 - y^2\right) < 0$

   for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

   One can also observe the asymptotic stability of the origin by linearization with variational matrix

   $A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2 + 4\lambda + 9 = 0$, and calculating eigenvalues: $-i\sqrt{5} - 2, i\sqrt{5} - 2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the set of attraction.

6. Is it possible to find a non-linear autonomous system of ODE's in the plane with an unstable equilibrium in the origin, but having some trajectories tending to the origin when $t \to \infty$? If yes, give an example. **(4p)**

   Solution. It is possible. The simplest example is the linear autonomous system $x' = Ax$ in the plane with a saddle point. The matrix A has two eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$ with eigenvectors v_1 and v_2.General solution has the expression $x(t) = C_1e^{\lambda_1 t}v_1 + C_2e^{\lambda_2 t}v_2$.

The simplest example of such matrix is $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ with $v_1 = e_1$ and $v_2 = e_2$, $\lambda_1 = 1$, $\lambda_2 = -3$.

The origin is unstable because solutions with $C_1 \neq 0$ will not stay in any neighbourhood of the origin: for arbitrary small C_1 the term $C_1 e^{\lambda_1 t}$ will tend to infinity when $t \to \infty$.

On the other hand solutions with $C_1 = 0$ on the line through the origin parallel to v_2 will tend to the origin because $C_2 e^{\lambda_2 t} \to 0$ when $t \to 0$.

The Grobman Hartman theorem implies that the same property is valid even for nonlinear systems in the form $x' = Ax + h(x)$ with $\|h(x)\| / \|x\| \to 0$ when $\|x\| \to 0$, because in a neighbourhood of the origin the nonlinear system and its linearization $x' = Ax$ have "similar" phase portraits.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32 Assignments + 0.68 Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.