MATEMATIK	Datum: 2017-05-29	Tid: 08-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
A.Heintz	Telefonvakt: Mattias Lennartsson	Tel.: 5325

Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to Gronwall inequality. **(4p)**

   Check the proof to Lemma 2.4 , p. 24 in the course book.

2. Formulate and prove the theorem about the stability of equilibrium points to autonomous ODEs by Lyapunov's method. **(4p)**

   Check the constructive proof on the home page, or alternatively the proof by contradiction to the Theorem 5.2 , p. 170 in the course book.

3. Solve the initial value problem

$$\dot{x} = t\,x^3, \qquad x\,(1) = 1$$

   and find maximal interval for the solution. **(2p)**

   Can one conclude which maximal interval have solutions to the similar equation $\dot{x} = t^3 x$ without solving it? **(2p)**

   Solution. It is the equation with separable variables.

$$\begin{aligned}
\frac{dx}{dt} &= tx^3 \\
\int \frac{dx}{x^3} &= \int tdt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2} = 1 \\
\frac{-1}{2x^2} &= \frac{t^2 - 2}{2}; \quad \frac{1}{x^2} = 2 - t^2 \\
x^2 &= \frac{1}{(2 - t^2)} \\
x &= \frac{1}{\sqrt{2 - t^2}}; \quad x(1) = 1; \quad t \in (-\sqrt{2}, \sqrt{2})
\end{aligned}$$

   Checking the solution: $\frac{d}{dt}\left(\frac{1}{\sqrt{2-t^2}}\right) = \frac{t}{2\sqrt{2-t^2} - t^2\sqrt{2-t^2}} = \left(2 - t^2\right)^{-1}\left(\sqrt{2 - t^2}\right)^{-1} t = \left(\sqrt{2 - t^2}\right)^{-3} t = tx^3.$

   The maximal interval for this solution is $I = (-\sqrt{2}, \sqrt{2})$ and is open in accordance with the general theory.

   The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $\left|t^3 x\right| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

4. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 3 & 0 & 4 \\ -3 & 1 & -7 \\ -2 & 0 & -3 \end{bmatrix}$. Give general solution to the system. Find all initial data such that corresponding solutions are unbounded. **(4p)**

Solution.

Characteristic polynomial is:

$\det\begin{bmatrix} 3-\lambda & 0 & 4 \\ -3 & 1-\lambda & -7 \\ -2 & 0 & -3-\lambda \end{bmatrix} = (1-\lambda)\left((3-\lambda)(3-\lambda)+8\right) = (1-\lambda)(-9+\lambda^2+8) = (1-\lambda)(\lambda^2-1) =$
$(1-\lambda)\left(\lambda^2-1\right) : \lambda+\lambda^2-\lambda^3-1$

$\lambda_1 = -1$. Corresponding eigenvector satisfies $[A-\lambda_1 I]\,v_1 = 0$;

$$\begin{bmatrix} 4 & 0 & 4 \\ -3 & 2 & -7 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 4 \\ -3 & 2 & -7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 12 & 0 & 12 \\ -12 & 8 & -28 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 12 & 0 & 12 \\ 0 & 8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 12 & 0 & 12 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvector: $v_1 = \left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\} \leftrightarrow \lambda_1 = -1$

$\lambda_2 = 1$. Corresponding eigenvector satisfies $[A-\lambda_2 I]\,v_2 = 0$;

$$\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 8 \\ -6 & 0 & -14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

, $v_2 = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow \lambda_2 = 1$ has multiplicity 2 and geometric multiplicity 1.

A generalized eigenvector $v_2^{(1)}$ satisfies the equation $[A-\lambda_2 I]\,v_2^{(1)} = v_2; \quad \lambda_2 = 1$

$$\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow v_3 = v_2^{(1)} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ -3 & 0 & -7 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -8 \\ 8 & 0 & 16 \\ 4 & 0 & 8 \end{bmatrix}$$

The initial data is represented in terms of two linearly independent eigenvectors and one generalized eigenvector: $\xi = C_1 v_1 + C_2 v_2 + C_3 v_3$

Using the formula for general solution, and pointing out that $(A-\lambda_2)\,v_3 = v_2$

$$x(t) = \exp(At)\xi = \sum_i C_i e^{\lambda_i t} \sum_{k=1}^{m(\lambda_i)} \frac{t^k}{k!}\,(A-\lambda_i)^k = C_1 e^{-t} v_1 + C_2 e^t v_2 + C_3 e^t v_3 + C_3 t\, e^t v_2$$

For all initial points $\xi \in \mathbb{R}^3$ outside the line $\xi = v_1\mu$ where $\mu \in \mathbb{R}$ the solutions will be unbounded.

5. Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x-(1-x^2)y \end{cases}$.

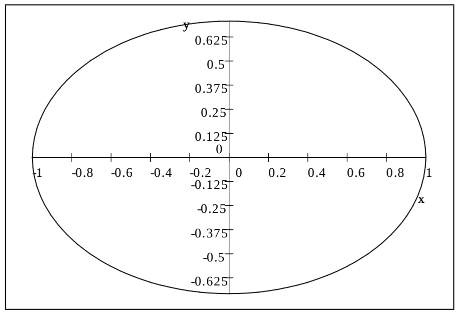
   Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction. **(4p)**

   Solution.

   Jacobi matrix in the origin is $\begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}$, eigenvalues: $-\frac{1}{2}i\sqrt{7}-\frac{1}{2}, \frac{1}{2}i\sqrt{7}-\frac{1}{2}$, characteristic polynomial: $\lambda^2+\lambda+2=0$

   We try the test function $V(x,y) = x^2+2y^2$ that leads to cancellation of mixed terms in the directional derivative along trajectories:

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2)$ that is not positive for $|x| \le 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x,y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x,y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalles invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x,y) = x^2 + 2y^2$ inside the stripe $|x| \le 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$:



6. Consider the following system of ODEs: $\begin{cases} x' = & 10 - x - \frac{4xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$

a) show that the point (x_*, y_*) with coordinates $x_* = 2$ and $y_* = 5$ is the only equilibrium point and is a repeller; **(2p)**

b) find a rectangle $[0,a] \times [0,b]$ in the first quadrant $x > 0$, $y > 0$ bounded by coordinate axes and by two lines parallel to them, that is a positively invariant set. Explain why the system must have at least one periodic orbit in this rectangle. **(2p)**

Solution.

a) $x_* = 2$ and $y_* = 5$ is an equilibrium point: $\left(1 - \frac{5}{1+2^2}\right) = 0$; and $10 - 2 - \frac{4 \cdot 2 \cdot 5}{5} = 10 - 2 - 8 = 0$.

The Jacobi matrix is $A = \begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2 \frac{y}{(x^2+1)^2} - 1 & -4\frac{x}{x^2+1} \\ -\frac{y}{x^2+1} + 2x^2 \frac{y}{(x^2+1)^2} + 1 & -\frac{x}{x^2+1} \end{bmatrix}$. It is calculated as:

$\nabla\left(10 - x - \frac{4xy}{1+x^2}\right) = \left.\begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2 \frac{y}{(x^2+1)^2} - 1 \\ -4\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5} = \begin{bmatrix} -4\frac{5}{5} + 8(4)\frac{5}{25} - 1 \\ -4 * \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -4 + \frac{32}{5} - 1 \\ -\frac{8}{5} \end{bmatrix} =$
$\begin{bmatrix} 1.4 \\ -1.6 \end{bmatrix}$

$\nabla\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = \left.\begin{bmatrix} -\frac{y}{x^2+1} + 2x^2 \frac{y}{(x^2+1)^2} + 1 \\ -\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5} = \begin{bmatrix} -\frac{5}{5} + 2(4)\frac{5}{25} + 1 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} -1 + \frac{8}{5} + 1 \\ -\frac{2}{5} \end{bmatrix} =$
$\begin{bmatrix} 1.6 \\ -0.4 \end{bmatrix}$

The Jacobi matrix in x_*, y_* is $A = \begin{bmatrix} 1.4 & -1.6 \\ 1.6 & -0.4 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - \lambda + 2 = 0$,

$trace(A) = 1 > 0$, $\det(A) = 2 > \frac{[trace(A)]^2}{4} = \frac{1}{4}$ that corresponds to unstable spiral and it is a repeller, eigenvalues are: $\lambda_1 = 0.5 + \sqrt{0.25 - 2} = 0.5 + i\sqrt{1.75}$, $\lambda_2 = 0.5 - \sqrt{0.25 - 2} = 0.5 - i\sqrt{1.75}$.

It implies that trajectories cannot enter a small open ball $B((x_*, y_*), \varepsilon)$ with the center in the equilibrium point $(2,5)$ and some small radius ε.

b) Observe that the first quadrant is a positively invariant set. For $y = 0$ we have $\dot{x} = 10 > 0$ and for $y = 0$ and $x > 0$ we have $y' = x > 0$.

Observe also that $\dot{y} < 0$ for $y > 1 + x^2$ and $x > 0$; $\dot{x} < 0$ for $x > 10$ and $y > 0$.

It implies that the rectangle $[0, 10] \times [0, 101]$ is a positively invariant compact set. Excluding a small open ball $B((x_*, y_*), \varepsilon)$ with the center in the equilibrium point $(2, 5)$ and small radius ε we get a positively invariant compact set $[0, 10] \times [0, 101] \setminus B((x_*, y_*), \varepsilon)$ without equilibrium points that according to the Poincare Bendixson theorem must contain a periodic orbit.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32 Assignments + 0.68 Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.