MATEMATIK Datum: 2016-05-30 Tid: 8-30 - 12-30
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Lösningar till **tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)**

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Give definitions to: monodromy matrix, characteristic multiplicators, characteristic exponentials. Formulate and give a proof to the theorem on stability of solutions to the linear system of ODE with periodic coefficients. **(4p)**

2. Formulate and give a proof to Picard - Lindelöf theorem on solvability of the initial value problem to a system of ordinary differential equation $x' = f(t,x)$, $x(t_0) = x_0$. **(4p)**

3. Calculate $\exp(At)$ for the constant matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$. **(4p)**

Solution.

$\exp(At)$ is a fundamental matrix to the system of differential equations $x' = Ax$. It means that columns in $\exp(At)$ are solutions to the system above with initial data $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The plan is to find first the general solution, then these two particular solution.

The characteristic polynom for A is $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $X^2 - 3X + 2 = (X-1)(X-2) = 0$, so eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$. Eigenvectors are $v_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \leftrightarrow \lambda_1$; $v_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2$

General solution is $x(t) = C_1 v_1 e^t + C_2 v_2 e^{2t}$. To satisfy the initial data $x(0) = C_1 v_1 e^t + C_2 v_2 e^{2t} = e_1$

we solve a system of two equations for C_1 and C_2:

$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or in matrix form $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow C_1 = -1$ and $C_2 = 2$. Therefore the first columnt in $\exp(At)$ is: $-v_1 e^t + 2 v_2 e^{2t} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^t + 2e^{2t} \\ -2e^t + 2e^{2t} \end{bmatrix}$

Similarly we find the second column:

$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\Longrightarrow C_1 = 1$ and $C_2 = -1$.

The second column in $\exp(At)$ is: $v_1 e^t - v_2 e^{2t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^t - e^{2t} \\ 2e^t - e^{2t} \end{bmatrix}$

and finally $\exp(At) = \begin{bmatrix} -e^t + 2e^{2t} & e^t - e^{2t} \\ -2e^t + 2e^{2t} & 2e^t - e^{2t} \end{bmatrix}$

An alternative but more complicated solution would be to represent $\exp(At)$ as $\exp(At) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}$, where the matrix P has columns of eigenvectors: $P = (v_1, v_2) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

and the inversion of P can be calculated by Kramer formulas: $P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

We derive the final expression by multiplication of the three matrices:

$\exp(At) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} e^t & e^{2t} \\ 2e^t & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} =$
$\begin{bmatrix} -e^t + 2e^{2t} & e^t - e^{2t} \\ -2e^t + 2e^{2t} & 2e^t - e^{2t} \end{bmatrix}$

4. Find all stationary points of the system of ODE $\begin{cases} x' = e^y - e^x \\ y' = \sqrt{3x + y^2} - 2 \end{cases}$

and investigate their stability by linearization. **(4p)**

Solution.

We find stationary points by pointing out that the first equation implies $y = x$ and then $\sqrt{3x + x^2} - 2 = 0$ implies $3x + x^2 - 4 = (x+4)(x-1) = 0$ and therefore two roots $x_1 = 1$ and $x_2 = -4$ follow.

We have two stationary points: $(1, 1)$ and $(-4, -4)$.

The Jacobi matrix is $J(x, y) = \begin{bmatrix} -e^x & e^y \\ \frac{3}{2\sqrt{3x+y^2}} & \frac{y}{\sqrt{3x+y^2}} \end{bmatrix}$

$J(1, 1) = \begin{bmatrix} -e & e \\ \frac{3}{2\sqrt{3+1}} & \frac{1}{\sqrt{3+1}} \end{bmatrix} = \begin{bmatrix} -e & e \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$ The trace of $J(1, 1)$ is $tr\,(J(1, 1)) = 1/2 - e < 0$

$\det(J(1, 1)) = e(-1/2 - 3/4) = -\frac{5}{4}e < 0$ it implies that the stationary point $(1, 1)$ is has one negative and one postive eigenvalue and therefore is a saddle point and is unstable by the Grobman Hartman theorem.

The characteristic equation for a 2x2 matrix A is $\lambda^2 - tr(A)\lambda - \det(A) = 0$.

In this particular situation it is $\lambda^2 + \left(e - \frac{1}{2}\right)\lambda - \frac{5}{4}e = 0$.

Eigenvalues are: $\lambda_1 = -\frac{1}{2}e + \frac{1}{4} - \frac{1}{4}\sqrt{16e + 4e^2 + 1}$, $\lambda_2 = -\frac{1}{2}e + \frac{1}{4} + \frac{1}{4}\sqrt{16e + 4e^2 + 1}$.

$J(-4, -4) = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & \frac{-4}{2} \end{bmatrix} = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & -2 \end{bmatrix}$.

The trace of $J(-4, -4)$ is $tr\,(J(-4, -4)) = -2 - e^{-4} < 0$.

$\det(J(-4, -4)) = e^{-4}\left(2 - \frac{3}{4}\right) = \frac{5}{4}e^{-4} > 0$. Therefore the the real parts of eigenvalues are negative and the stationary point $(-4, -4)$ is an asymptotically stable node by the Grobman Hartman theorem.

The characteristic equation is $\lambda^2 + \left(e^{-4} + 2\right)\lambda + \frac{5}{4}e^{-4} = 0$.

Eigenvalues are : $\lambda_1 = -\frac{1}{2}e^{-4} - 1 - \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$, $\lambda_2 = -\frac{1}{2}e^{-4} - 1 + \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$

5. Investigate stability of the origin and find a domain of stability for the following system of ODE by using an appropriate Lyapunov function.

$\begin{cases} x' = y \\ y' = -y + y^3 - x^5 \end{cases}$ **(4p)**

Solution.

We choose a test function $V(x, y) = x^6 + ay^2$ with unknown positive coefficient a because there are terms x^5 in the second equation and y both in the first and in the second equation. We calculate

$\nabla V \cdot f = \begin{bmatrix} 6x^5 \\ 2ay \end{bmatrix} \cdot \begin{bmatrix} y \\ -y + y^3 - x^5 \end{bmatrix} = 6x^5 y - 2ay^2 + 2ay^4 - 2ayx^5$

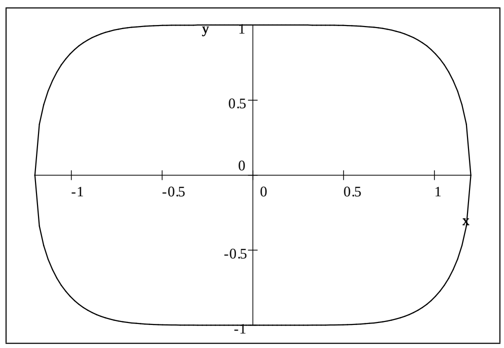
and observe that with the choice $a = 3$ and $V(x, y) = x^6 + 3y^2$ we get:

$\nabla V \cdot f = 6x^5y - 6y^2 + 6y^4 - 6yx^5 = -6y^2 \left(1 - y^2\right) \leq 0$

for $|y| \leq 1$. Therefore the stationary point in the origin is stable by Lyapunov's theorem.

To decide if it asymptotically stable or not we check the set of points (x, y) where $\nabla V \cdot f = 0$ These are points on the x - axis $y = 0$.

We observe that trajectories starting on the x - axis have velocities in y direction $y' = -x^5$ that are zero only in the origin $(0, 0)$. Therefore all trajectories starting on the x - axis leave it except the trajectory starting in the origin that is a stationary point. Therefore there are no complete orbits on the x axis except the origin and the origin is asymptotically stable by a corollary to the Krasovsky - la'Salle principle. Level sets of of the Lyapunovs function $V(x, y) = x^6+3y^2$ are ellipse like closed curves symmetric with respect to coordinate axes. The "largest" such level set inside the stripe $|y| \leq 1$ must, because of the symmetry, go through the point $(0, 1)$ and is $V(0, 1) = 3$. Therefore a domain of asymptotic stability that we can identify using this Lyapunov function is the domain inside this level set: $S = \left\{(x, y) : x^6 + 3y^2 < 3\right\}$:



6. Consider the following system of ODE $\begin{cases} x' = -ay + x\left(1 - x^2 - y^2\right) \\ y' = ax + y\left(1 - x^2 - y^2\right) - B \end{cases}$ where a and B are arbitrary constants.

   i) Show that there exists a region $K = \left\{(x, y) : \quad x^2 + y^2 \leq r^2\right\}$ such that all trajectories eventually enter K.

   ii) Do all solutions to this system exist on infinite interval of time and why?

   iii) Show that the system has a periodic solution when $B = 0$. **(4p)**

   Solution.

   i) We derive the equation for $r = \sqrt{x^2 + y^2}$ by muliplying the firs equation by x, the second equation by y

   and adding the equations and using that $x'x + y'y = \frac{1}{2}\left(x^2 + y^2\right)' = \frac{1}{2}\left(r^2\right)' = rr'$. It implies that

$$rr' = r^2\left(1 - r^2\right) - Br\sin(\theta)$$

   where θ is the polar angle. Finally

$$r' = r\left(1 - r^2\right) - B\sin(\theta)$$

   The last equation implies that for $r(r^2 - 1) > |B| + 1$ the derivative $r' < -1$ and therefore choosing an r_* satisfying this inequality, we get that any trajectory starting outside the set $K = \left\{(x, y) : x^2 + y^2 \leq r_*^2\right\}$ will enter this set after a finite time, because $r(t) < r(0)\exp(-t)$ when points of the trajectory are outside the set K.

ii) Solutions having bounded maximal interval of existence must leave any compact set in finite time.The fact that all trajectories of this system enter the compact set K and stay there implies that all solutions can be extended to infinite interval of time, because they all stay within this compact set forever.

iii) If $B=0$ we observe that for example the annulus $0.5<r<2$ is a positively invariant set that in case $a\neq 0$ will include no stationary points. Therefore, according to the Poincare Bendixson theorem this annulus must include at least one periodic orbit.

One can also observe that in this case $r'=0$ for $r=1$ and therefore the circle $r=1$ must be periodic orbit because there are no stationary points on this circle if $a\neq 0$.

One can also derive a differential equation for the polar angle:

$$\begin{aligned}
(\tan(\theta))' &= \frac{1}{\cos^2(\theta)}\theta' = \left(\frac{y}{x}\right)' = \frac{y'x-x'y}{x^2} \\
&= \frac{\left[ax+y\left(1-x^2-y^2\right)\right]x-\left[-ay+x\left(1-x^2-y^2\right)\right]y}{x^2} \\
&= \frac{a\left(x^2+y^2\right)}{x^2} = \frac{a}{\cos^2(\theta)}
\end{aligned}$$

Therefore $\theta'=a$. It implies that the periodic solution with $r=1$ will evolve uniformly around the circle $r=1$ with angle speed a. If $a=0$ there will be movement only along the straight lines through the origin towards the whole circle $r=1$ of stationary points and no periodic solutions.

Max. 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32Assignments + 0.68Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same thresholding is valid for the exam, for the home assignments, and for the total points for the course.