Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE161

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the theorem about the limit and boundedness of solutions \( \overrightarrow{x}(t) \) to linear systems of ODE with constant coefficients when \( t \to +\infty \).

Check Theorem 3.3.5 in the course textbook. (4p)

2. Formulate and give a proof to Bendixsons criteria about non-existence of periodic solutions to autonomous ODE in plane.

Check Theorem 6.1.2 in the course textbook. (4p)

3. Consider the following system of ODE:

\[
\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}.
\]

Give a general real solution to the system. Find all those initial vectors \( \overrightarrow{r}_0 = \overrightarrow{r}(0) \) that give bounded solutions to the system. (4p)

Solution. The general solution has the form \( \overrightarrow{r}(t) = \exp(tA)\overrightarrow{r}_0 \) for \( \overrightarrow{r}(0) = \overrightarrow{r}_0 \). The matrix \( A \) is block-diagonal, so the matrix exponential has the form

\[
\exp(tA) = \begin{bmatrix} \exp(tJ) & \mathbb{0} \\ \mathbb{0} & \exp(tK) \end{bmatrix}
\]

with \( J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \), and \( \mathbb{0} \) - zero \( 2 \times 2 \) matrix.

We know that \( \exp(tJ) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \) and \( \exp(tK) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \).

The first relation follows from the power series for exponent \( \exp(tJ) \), that for the matrix \( J \) will consist just of two nonzero terms, because \( J^2 = 0 \). The second relation follows from the fact that the matrices in the form: \( \mathbb{K} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \) have the same algebraic properties as complex numbers \( a + ib \) and from the Euler formula for the exponent of complex numbers: \( \exp(a + ib) = \exp(a) (\cos(b) + i \sin(b)) \).

If one does not remember the formula for \( \exp \left( \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right) \), then one can easily derive it by solving the system \( \overrightarrow{w}' = \mathbb{K}\overrightarrow{w} \).

We observe that the arbitrary matrix in the form \( \mathbb{K} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), has complex conjugate eigenvectors and eigenvalues : \( v_2 = \left\{ \begin{bmatrix} -1 \\ i \end{bmatrix} \right\} \leftrightarrow \lambda_2 = a - ib \), and \( v_1 = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \leftrightarrow \lambda_1 = a + ib \). The general complex solution to the system \( \overrightarrow{w}' = \mathbb{K}\overrightarrow{w} \) has the form: \( \overrightarrow{w}(t) = C_1 \exp(\lambda_1 t)v_1 + C_2 \exp(\lambda_2 t)v_2 \).

We can instead choose real and imaginary parts of \( \exp(\lambda_1 t)v_1 \) as a basis for real solutions.

We use here the Euler formula for the exponent of complex numbers:
4. Formulate Banach’s contraction principle. Consider the following operator

\[ V_1(t) = \text{Re} \left( \exp((a + bi)t) \begin{bmatrix} 1 \\ i \end{bmatrix} \right) = \begin{bmatrix} e^{at} \cos(bt) \\ -e^{at} \sin(bt) \end{bmatrix}; \]

\[ V_2(t) = \text{Im} \left( \exp((a + bi)t) \begin{bmatrix} 1 \\ i \end{bmatrix} \right) = \begin{bmatrix} e^{at} \sin(bt) \\ e^{at} \cos(bt) \end{bmatrix} \]

We observe that \( V_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( V_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). It implies that the fundamental matrix solution \([V_1(t), V_2(t)]\) is the principal matrix solution: \([V_1(0), V_2(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I\) and \([V_1(t), V_2(t)] = \exp(tK) = \exp(at) \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}\). In particular we get the formula for the exponent of the matrix \(tK\) above.

Finally the general real solution to the given system is

\[
\vec{r}(t) = \exp(tA)\vec{r}_0 = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & e^{\lambda_2 t} \\ 0 & 0 & -e^{\lambda_1 t} & e^{\lambda_2 t} \end{bmatrix} \vec{r}_0 \text{ for an arbitrary initial vector } \vec{r}_0.
\]

It is easy to observe that the only initial data that give bounded solutions consist of vectors \(\vec{r}_0 \in \mathbb{R}^4\) with all components except the first one equal to zero, and of the zero initial data, because all columns in \(\exp(tA)\) except the first one include unbounded functions. The last question can be answered even using the simpler complex form of the general solution: \(\vec{r}(t) = \exp(tA)\vec{r}_0\) with \(\lambda_1 = 2 + i, \lambda_2 = 2 - i\) being complex eigenvalues to the matrix \(K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\).

4. Formulate Banach’s contraction principle. Consider the following operator

\[ K(x)(t) = A \int_0^\pi \sin(ts) x(s) ds + t^2, \]

for all \(t \in [0, \pi]\) acting in the Banach space \(C([0, \pi])\) of continuous functions with norm \(|x| = \sup_{t \in [0, \pi]} |x(t)|\).

Find using Banach’s contraction principle conditions on the constant \(A > 0\) such that the operator \(K(x)(t)\) has a fixed point. \(\textbf{(4p)}\)

\textbf{Solution.} Banach’s contraction principle states that if an operator \(K\) maps a closed subset \(U\) in a Banach space \(B\) into itself: \(K : U \rightarrow U\) and is a contraction on \(U\), that means that \(|Kx - Ky| \leq \theta \|x - y\|\) with \(\theta < 1\), then the operator \(K\) has a unique fixed point \(\vec{x} = K\vec{x}\) in \(U\) that can be found by iterations \(x_{n+1} = Kx_n\) with an arbitrary start approximation \(x_0 \in U\), so that \(x_{n+1} \rightarrow \vec{x}\).

We calculate supremum norm of the value of the operator \(K(x)(t)\):

\[ \|Kx\| = \sup_{t \in [0, \pi]} |K(x)(t)| = \sup_{t \in [0, \pi]} |A \int_0^\pi \sin(ts) x(s) ds + t^2| \leq \pi^2 + A\pi \sup_{s \in [0, \pi]} |x(s)| \]

Therefore \(|Kx| \leq \pi^2 + A\pi \|x\|\). It implies that the operator \(K\) maps a ball with radius \(r\) in the Banach space \(C([0, \pi])\) into the ball of radius \(\pi^2 + A\pi r\).
We like to find such radius $R$ of the ball $B(0, R)$ in $C([0, \pi])$ and such constant $A > 0$ that $K$ would map the ball $B(0, R)$ into itself. Namely that $\pi^2 + A\pi R \leq R$. It implies $\pi^2 \leq R(1 - A\pi)$.

We see that $A$ must be chosen smaller than $1/\pi$: $A < \frac{1}{\pi}$ and $R$ must be chosen large enough: $\frac{\pi}{(1 - A\pi)} \leq R$. Then $K : B(0, R) \to B(0, R)$. The next step is to find conditions that imply that $K$ is a contraction on $B(0, R)$. We estimate the norm $\|Kx - Ky\|:

\[\|Kx - Ky\| \leq \sup_{t \in [0, \pi]} |A \int_0^\pi \sin(ts) (x(s) - y(s)) \, ds| \leq A\pi \sup_{s \in [0, \pi]} |x(s) - y(s)| = A\pi \|x - y\|\]

We have chosen already $A < \frac{1}{\pi}$. It implies that $K$ is a contraction on $B(0, R)$ with $\frac{\pi}{(1 - A\pi)} \leq R$ chosen above so that $K : B(0, R) \to B(0, R)$. It implies by the Banach contraction principle that $K$ has a unique fixed point in $B(0, R)$.

5. Consider the following system of ODE and investigate the stability of the stationary point in the origin depending on the real constant $a \in \mathbb{R}$.

\[
\begin{align*}
x' &= y \\
y' &= -x + (a - x^2)y
\end{align*}
\] (4p)

**Solution.** We try to use the linearization of the system. The variational matrix in the origin is $A = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$.

Eigenvalues of $A$ are $\lambda_1 = \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4}, \lambda_2 = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4}$. For $a > 0$ we see that $\text{Re}(\lambda_1) > 0$ and the Grobman-Hartman theorem implies that the origin is unstable (even repeller). Similarly for $a < 0$, $\text{Re}(\lambda_1) < 0$ and the origin is asymptotically stable. For $0 < a < 2$ it will be an unstable spiral, for $2 < a$ it will be an unstable node. For $a = 2$ it will be an unstable improper node. For $-2 < a < 0$ it will be stable spiral, for $a < -2$ it will be a stable node. For $a = -2$ it will be a stable improper node.

For $a = 0$ we cannot use Grobman Hartman theorem because the origin is not hyperbolic: both eigenvalues have real part zero.

We try instead to use a simple test function $V(x, y) = \frac{1}{2} (x^2 + y^2)$ in this case. Introducing the vector notation $f(x, y)$ for the right hand side of the equation we get

$$V'(x, y) = \nabla V \cdot f(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ -x - x^2y \end{bmatrix} = xy - yx - y(x)^2 = -x^2y^2 \leq 0$$

for $(x, y) \neq (0, 0)$. Therefore the origin is a stable stationary point. We use Lasalle’s invariance principle to check if the origin is asymptotically stable or not. $V'(x, y) = 0$ on the set $S$ where $x=0$ or $y = 0$: the union of coordinate axises. We check if this set includes invariant sets other than the origin. For $x = 0$, $x' = y \neq 0$ for $y \neq 0$. For $y = 0$, $y' = -x \neq 0$ for $x \neq 0$. Therefore the set $S$ includes only one invariant set - the origin $(0, 0)$, that by a corollary to Lasalle’s invariance principle must be asymptotically stable.

6. Show that all solutions to the following system of ODE exist for arbitrary large time $t > 0$

\[
\begin{align*}
x' &= -4x^3 + 2xy \\
y' &= -2y + x^2
\end{align*}
\] (4p)

**Solution.** We try to show that all solutions stay within a finite domain. It would imply that they all are extendable for any time $t > 0$. We use a simple test function $V(x, y) = \frac{1}{2}x^2 + y^2$ in this case. Introducing vector notations for the right hand side $f(x, y)$ of the equation we get

$$V'(x, y) = \nabla V \cdot f(x, y) = \begin{bmatrix} x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -4x^3 + 2xy \\ -2y + x^2 \end{bmatrix} = 2x^2y - 4y^2 - 4x^4 + 2x^2y = -4 \left( x^4 - x^2y + y^2 \right) < 0$$

for $(x, y) \neq (0, 0)$.
because the quadratic form $a^2 - ab + b^2$ is positive definite.

It means that solutions starting inside an ellipse $\frac{1}{2}x^2 + y^2 < C$ of radius $C > 0$ will never leave it and therefore can be extended for any time $t > 0$ because the right hand side of the equation is a smooth function in the whole plane $\mathbb{R}^2$ (and therefore is Lipschitz in any bounded domain).

Max. 24 points;

Thresholding for marks: for GU: **VG**: 19 points; **G**: 12 points. For Chalmers: **5**: 21 points; **4**: 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.3 \cdot Assignments + 0.7 \cdot Exam$ - the average of the points for the home assignments (30%) and for this exam (70%). The same thresholding is valid for the exam, for the home assignments, and for the total points for the course.