MATEMATIK | Datum: 2015-04-13 | Tid: efterrmiddag
GU, Chalmers | Hjälpmedel: inga
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Tenta i ODE och matematisk modellering, MMG511, MVE161

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Give definition of a Banach space. Formulate and give a proof to the Banach's contraction principle. **(4p)**

2. Formulate and prove the theorem on stability and asymptotical stability of equilibrium points to autonomous ODEs by Lyapunovs functions. **(4p)**

   Check proofs of theorems and definitions in the course book. Point out that the poof of uniquness in contraction principle is missed in the book.

3. Consider the following system of ODE:

   $$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix},$$

   Give general solution to the system. Find the solution to initial value problem with initial conditions: $x(0) = 1$, $y(0) = 1/2$, $z(0) = 1/2$. **(4p)**

   Solution:

   Characteristic polynomial of A: $\det \begin{bmatrix} -\lambda & -1 & 1 \\ 0 & -\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - \lambda^3 + 1 = (1-\lambda)\left(\lambda^2+1\right)$

   $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, eigenvectors and eigenvalues: $v_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_1 = 1; v_2 = \left\{ \begin{bmatrix} 1+i \\ i \\ 1 \end{bmatrix} \right\} \leftrightarrow$ $\lambda_2 = -i; v_3 = \left\{ \begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2 = i$

   All eigenvalues are simple and a complex general solution is expressed as $\overrightarrow{r}(t) = \sum_{k=1}^{3} C_k v_k \exp(\lambda_k t)$.

   We choose $\operatorname{Re}[v_3 \exp(\lambda_3 t)]$ and $\operatorname{Im}[v_3 \exp(\lambda_3 t)]$ as two real basis vectors of the solution space in addition to $[v_1 \exp(\lambda_1 t)]$:

   $$\operatorname{Re}[v_3 \exp(\lambda_3 t)] = \operatorname{Re}\left[\begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix} (\exp(it))\right] = \begin{bmatrix} \cos t + \sin t \\ \sin t \\ \cos t \end{bmatrix}; \quad \operatorname{Im}\left[\begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix} (\exp(it))\right] = \begin{bmatrix} -\cos t + \sin t \\ -\cos t \\ \sin t \end{bmatrix}.$$

   to build a real general solution as $\overrightarrow{r}(t) = C_1[v_1 \exp(\lambda_1 t)] + C_2 \operatorname{Re}[v_3 \exp(\lambda_3 t)] + C_3 \operatorname{Im}[v_3 \exp(\lambda_3 t)]$. Expressing this in coordinates we get the real general solution as:

   $$\begin{aligned} x &= C_2(\cos t + \sin t) + C_3(-\cos t + \sin t); \\ y &= C_1 e^t + C_2 \sin(t) + C_3(-\cos(t)); \\ z &= C_1 e^t + C_2 \cos(t) + C_3 \sin(t). \end{aligned}$$

To solve a particular I.V.P. with $x(0) = 1$, $y(0) = 1/2$, $z(0) = 1/2$ we substitute these values into the general solution for $t = 0$ and find coefficients

$1 = C_2 - C_3$; $1/2 = C_1 - C_3$; $1/2 = C_1 + C_2$. Substracting two last equations gives $0 = C_2 + C_3$. Therefore $C_2 = 1/2$; $C_3 = -1/2$; $C_1 = 0$ and

solution to I.V.P. is:

$$\begin{aligned} x &= 1/2\,(\cos t + \sin t) - 1/2\,(-\cos t + \sin t) = \cos t; \\ y &= 1/2 \sin(t) + 1/2 \cos(t); \\ z &= 1/2\,(\cos(t) - \sin(t)). \end{aligned}$$

4. Consider the following system of equations:

$$\begin{cases} x' = & 2y - x \\ y' = & 3x - 2y \end{cases}$$

a) can the system have a trajectory going from the point $(-a^2 - 1, -1)$ to the point $(1, a^2 + 1)$?

b) which type of fixed point is the origin?

c) draw a sketch of the phase portrait. **(4p)**

Solution

Matrix of the system is $A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$. Characteristic polynomila is $\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} = 3\lambda + \lambda^2 - 4$. Eigenvalues and eigenvectors are $\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$, eigenvalues: $\lambda_1 = -4, \lambda_2 = 1$. Eigenvectors $v_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \leftrightarrow \lambda_1 = -4$; $\quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 1$,

Origin is a saddle point and is unstable. Trajectories are hyperbolas asymptotically approaching with $t \to \infty$ or $t \to -\infty$ trajectories L_1, L_2, L_3, L_4, that are straight lines through the origin and are parallell to the eigenvectors.

Cheching points $(-a^2 - 1, -1)$ and $(1, a^2 + 1)$ we observe that they are separated by the above mentioned straight trajectories L_1, L_2, L_3, L_4. Therefore no one trajectory can go between these two points because such a trajectory should cross one of L_1, L_2, L_3, L_4 that is impossible because of the uniquness of solutions to linear systems.

5. Consider the system of ODE:

$$\begin{cases} x' = & y \\ y' = & a\,y - \tan(x) \end{cases}$$ **(4p)**

Find for which real values a the zero solution is a) asymptotically stable, b) stable, but not asymptotically c) unstable.

Calculate Jacoby matrix for the right hand side of the equation. $J = \left.\begin{bmatrix} 0 & 1 \\ -1/\cos^2(x) & a \end{bmatrix}\right|_{x=0,y=0} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$, characteristic polynomial: $\lambda^2 - a\lambda + 1$, eigenvalues are $\lambda_1 = \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4}$; $\quad \lambda_1 = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4}$. It is easy to see that $\operatorname{Re}\lambda_k < 0$ for $a < 0$, in this case the origin is an asymptotically stable stationary point.

$\operatorname{Re}\lambda_k > 0$ for $a > 0$, in this case the origin is an unstable stationary point.

For $a = 0$ $\operatorname{Re}\lambda_k = 0$ and we cannot make a conclusion about stability using only the linearized equation.

Try a test function $V(x,y) = \int \tan x dx + y^2 = -2\ln(\cos(x)) + y^2$. V is positive definite for small (x,y). $\nabla V(x,y) = 2\begin{bmatrix} \tan x \\ y \end{bmatrix}$.

$V'(x,y) = 2\begin{bmatrix} \tan x \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ a\,y - \tan(x) \end{bmatrix} = 2y(\tan x) - 2y\tan(x) + 2ay^2 = 2ay^2$. Therefore the origin is stable for $a = 0$ because $V'(x,y) = 0$.

Origin is not asymptotically stable for $a = 0$ because $V(x,y)$ is constant on trajectories starting close to the origin and V is positive definite. In fact close to the origin we observe that level sets of V will be almost circles. There are no other stationary points close to the origin. It makes that these level sets will be closed orbits corresponding to be periodic solutions going around the origin and the origin is a center.

6. Find the characteristic multiplicator for the scalar linear equation with periodic coefficient: **(4p)**

$$x' = (a + \sin^2 t)x$$

The characteristic multiplicator is eigenvalue of the monodromy matrix denoted by $M = e^{TR}$ in the course book, where T is the period of the right hand side in the equation. One builds a monodromy matrix (it will be a number in our case with one scalar equation) of solutions to initial value problems with initial data $x(0)$ that are standard basis vectors in R^n calculated in the time point T - equal to the period of the right hand side.In our case we have just one scalar equation, so the monodromy matrix will be a number. We find the value of the solution to I.V.P. to the given equation with initial data $x(0) = 1$ at the time $t = 2\pi$ that is a period of the right hand side in our case. The equation is linear, so the solution is found with help of a primitive function of the coefficient:

$P(t) = \int_0^t (a + \sin^2 s)ds = \frac{1}{2}t + at - \frac{1}{4}\sin 2t$.

$x(t) = \exp(P(t))x(0) = \exp\left(\frac{1}{2}t + at - \frac{1}{4}\sin 2t\right)x(0)$.

The monodromy "matrix" in our case is the value of the solution $x(t)$ in $t = 2\pi$ such that $x(0) = 1$.

$x(2\pi) = \exp\left(\frac{1}{2}2\pi + a2\pi\right) = \exp\left(\pi(1 + 2a)\right)$.

The characteristic multiplicator is the same number: $\exp(\pi(1 + 2a))$.

Checking when this number is larger or smaller than one in absolute value we can make conclusions about asymptotic properties of solutions when $t \to \infty$.

Max: 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course will be the average of the points for the home assignments (30%) and for this exam (70%).

The same thresholding is valid for the exam, for the home assignments, and for the total points.