MATEMATIK	Datum: 2013-08-20	Tid: eftermiddag
GU, Chalmers	Hjälpmedel: inga	
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Tenta i ODE och matematisk modellering, MMG511, MVE161

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

1. Formulate and give a proof to the Picard-Lindelöf theorem on existence and uniquness of solutions to initial value problem for systems of ODE. **(4p)**

   See Theorem 2.2.6 on pages pagest 22-23 in the course book.

2. Formulate and give a proof to the theorem about stability of linear systems of ODE with periodic coefficients. **(4p)**

   See Theorem 3.5.5 on pages 65-66 in the course book.

3. Consider the following system of ODE:

$$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 2 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

Give general solution to the system. Find all those initial vectors $\overrightarrow{r}_0 = \overrightarrow{r}(0)$ that give bounded solutions to the system.

Solution. General solution is can be given in two ways: as $\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r}_0$ for arbitrary vector $\overrightarrow{r}_0$ of initial data at $t = 0$, alternativly as a linear combination of columns in an arbitrary fundamental matrix solution $\Phi(t)$.

Practical calculation of $\exp(tA)\overrightarrow{r_0}$ is based on the idea of considering $\exp(tA)\overrightarrow{r}_0 = e^{\lambda_j t}e^{t(A-\lambda_j)}\overrightarrow{r}_0$ for an eigenvalue λ_j.

It is easy to observe that the expression $\exp(tA)\overrightarrow{r_0} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(A-\lambda_j)^k\overrightarrow{r}_0$ will be finite sum for $\overrightarrow{r}_0$ being a linear combination of the eigenvectors and generalized eigenvectors to A corresponding to the eigenvalue λ_j or $\overrightarrow{r}_0 \in M(\lambda_j, A)$:

$$\exp(tA)\overrightarrow{r}_0 = \sum_{k=0}^{n_j-1} \frac{t^k}{k!}(A-\lambda_j)^k\overrightarrow{r}_0$$

The set of all eigenvectors and generalized eigenvectors to A build a basis to the space $\mathbb{C}^n$ and therefore general solution $\exp(tA)\overrightarrow{r}_0$ can be explicitely expressed as a sum of the from as above for all eigenvalues of A. We remind that generalised eigenvalues $v^{j,(k)}$ satisfy the following equations $Av^{j,(k)} = \lambda_j v^{j,(k-1)}$ for $k > 1$, and $Av^{j,(1)} = \lambda_j v^j$ where v^j is a eigenvector corresponding to λ_j and $v^{j,(k)}$ are associated generalized eigenvectors. For the given matrix A, the characteristic polynomial is $\lambda^3+2\lambda^2 = 0$ the characteristic values are $\lambda_1 = 0$ (with multiplicity 2) and $\lambda_1 = -2$ (simple). Eigenvectors satisfy equations $Av^1 = \begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ 0, and $(A+2I)v^2 = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. Or equivalently

$$\left\{\begin{array}{c} -3x+y+2z=0 \\ -x-y+2z=0 \\ -2x+2z=0 \end{array}\right. , \left\{\begin{array}{c} -x+y+2z=0 \\ -x+y+2z=0 \\ -2x+4z=0 \end{array}\right. .$$

It is easy to see that $v^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v^2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

There is one generalized eigenvector $v^{1,(1)}$ corresponding to the eigenvalue $\lambda_1 = 0$ because it has multiplicity 2. The dimension of the space is 3 and we have already got two linearly independent eigenvectors corresponding to two different eigenvalues λ_1 and λ_2.

$(A - 0I)\, v^{1,(1)} = v^1$ or $\left\{\begin{array}{c} -3x+y+2z=1 \\ -x-y+2z=1 \\ -2x+2z=1 \end{array}\right.$ $2z = 1+2x$; $\left\{\begin{array}{c} -3x+y+1+2x=1 \\ -x-y+1+2x=1 \end{array}\right.$, $x = y$

and the generalized eigenvector can be chosen (not uniquely!) as $v^{1,(1)} = \begin{bmatrix} 1 \\ 1 \\ 3/2 \end{bmatrix}$.

General solution for $\overrightarrow{r_0} = C_1 v^1 + C_2 v^{1,(1)} + C_3 v^2$ with arbitrary C_1, C_2, C_3 is expressed as $\overrightarrow{r}(t) = \left(C_1 v^1 + (I + At)\, C_2 v^{1,(1)}\right) e^{0t} + C_3 v^2 e^{-2t} = \left(C_1 v^1 + C_2 \left(v^{1,(1)} + t v^1\right)\right) + C_3 v^2 e^{-2t}$ or in terms of coordinates:

$x(t) = C_1 + C_2(1+t) + 2C_3 e^{-2t}$, $y(t) = C_1 + C_2(1+t)$, $z(t) = C_1 + C_2(3/2+t) + C_3 e^{-2t}$.

We point out that one can get different equivalent expressions of general solution depending on how one chooses eigenvectors and corresponding generalized eigenvectors.

The only initial data giving bounded solutions are vectors are linear combinatoins eigenvectors: $x(0) = C_1 v^1 + C_3 v^2$.

4. The following system of equations describes the evolution of numbers x and y of two competeng species.

$$\left\{\begin{array}{ll} x' = & x\,(2-x-y) \\ y' = & y\,(3-2x-y) \end{array}\right.$$

Explain by analysing their equilibrium points, how these equations make it mathematically possible but extremely unlikely for both species to survive. **(4p)**

Solution. There is only one possible nonzero equilibrium point: $x = 1$, $y = 1$. It is the only possible point for both species to survive.

We try to analyse stability of this point using linearization. The variational matrix is $A(x,y) = \left.\begin{bmatrix} 2-2x-y & -x \\ -2y & 3-2x-2y \end{bmatrix}\right|_{(x,y)=(1,1)} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$,

Characteristic equation: $\lambda^2 + 2\lambda - 1 = 0$.

It's eigenvectors and eigenvalues in the point $(x,y) = (1,1)$ are: $v^1 = \left\{\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow \lambda^1 = -\sqrt{2} - 1 < 0$; $v^2 = \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = \sqrt{2} - 1 > 0$;

The linearized system has a saddle point in the origin, that is hyperbolic because both eigenvalues have nonzero real part. General solution to the linearized system is $r = C_2 e^{-(\sqrt{2}+1)t} v^1 + C_2 e^{(\sqrt{2}-1)t} v^2 +$. The only initial data giving solutions tending to the origin with $t \to \infty$ are those on the line $r = C_1 v^1$.

The Grobman-Hartmann theorem states that in the neighbourhood of the point $(1,1)$ the phase portrait of the original nonlinear system is homeomorfic to one of the linear system $x' = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} x$. Therefore there is a curve through the equilibrium point $(1,1)$ such that evolution of the original system starting on this curve leads to the equilibrium point. All other trajectories escape this equilibrium point as they do for the linearized equation.

Another possibility for both species to survive could be rising infinitely. One can see that it is impossible from the following analysis. It is easy to observe that coordinate axises consist of trajectories to the system: $x' = \ x(2-x)$ and $y' = \ y(3-y)$ that have for $x > 0$ and $y > 0$ the only attracting equilibrium points $(2,0)$ and $(0,3)$. The right hand of the ODE side is a smooth function, therefore solutions are unique and trajectories starting in the first quadrant cannot cross coordinate axises and will stay in the first quadrant, $x(t), y(t) > 0$. It implies that $x' \leq \ x(2-x)$ and $y' \leq \ y(3-y)$ everywhere in the first quadrant and implies that $x(t)$, $y(t)$ are bounded for $t \to \infty$.

5. Consider the system of ODE:

$$\begin{cases} x' = & y \\ y' = & -y - 6x - 3x^2 \end{cases} \qquad \textbf{(4p)}$$

i) Using the test function $V(x,y) = \frac{1}{2}y^2 + 3x^2 + x^3$ show that the origin is asymptotically stable and

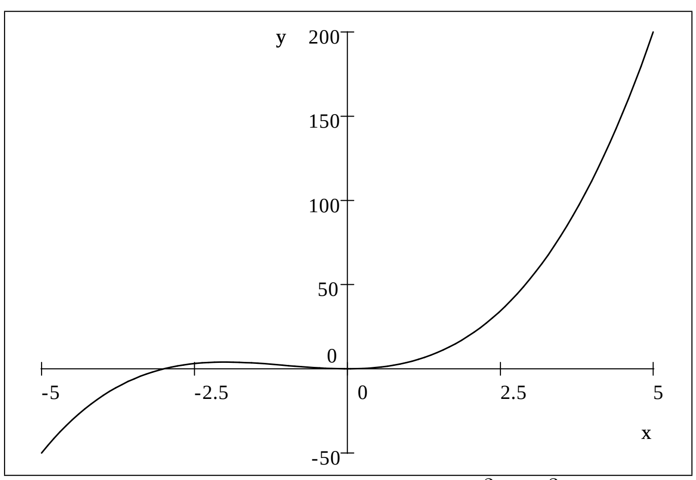
ii) Estimate the domain of attraction for the equilibrium point.

Solution. The system has two equilibrium points: $(-2,0)$ and $(0,0)$.

$V_f(x,y) = V'(x(t),y(t)) = \left(6x + 3x^2\right) y + y\left(-y - 6x - 3x^2\right) = -y^2 \leq 0.$

The expression $3x^2 + x^3 \geq 0$ for $x \in (-3, \infty$ because $3x^2 + x^3 = x^2(x+3)$, see the picture.

Therefore V is a Lyapunov function in half plane $x > -3$ and the origin is a stable equilibrium point.

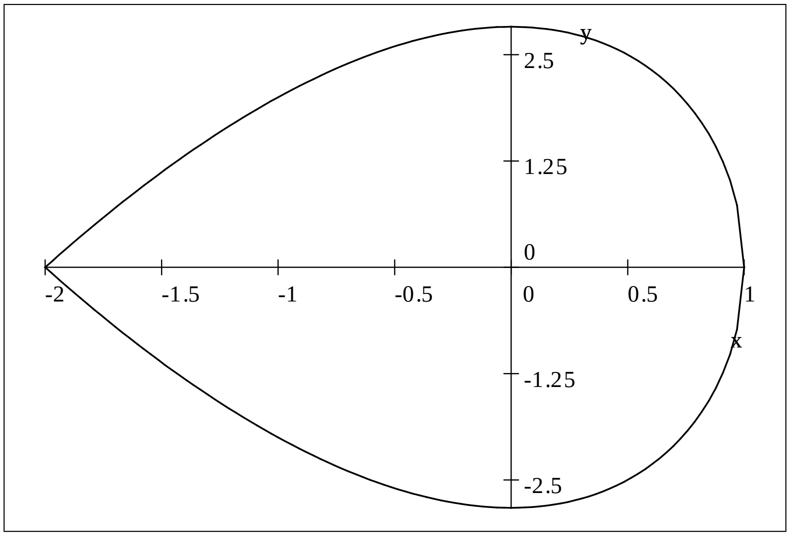


graph of the polynomial $3x^2 + x^3$

To show that the origin is also an asymptotically stable equilibrium we apply the La Salle theorem. On the set $V_f^{-1}(0)$ where $V_f(x,y) = 0$ is the x-axis, we have $x' = 0$, $y' = -6x - 3x^2 = -3x(2+x)$. All trajectories starting at points of the x axis with $x > -2$ exept the equilibrium point $(0,0)$ cross it and cannot belong to it. Therefore this halfline $x > -2$ includes no ω - invariant sets exept the equilibrium point $(0,0)$. Therefore the origin is asymptotically stable.

The largest of the level set of the Lyapunov function V including only one equilibrium is one going through the second equilibrium point $(-2,0)$ namely $V(-2,0) = 4$. This domain is bounded by the closed curve $\frac{1}{2}y^2 + 3x^2 + x^3 = 4$. This curve is symmetric with respect to the x - axis and can be represented by two graphs: $y(x) = \pm\sqrt{4 - 3x^2 - x^3}\sqrt{2}$ with $x \in [-2, 1]$, $y(-2) = y(1) = 0$, $y(0) = \pm 2\sqrt{2}$, see picture.

The domain inside this curve is positively invariant set because of $V_f(x, y) = -y^2 \leq 0$ and evidently V decreases inside the level set with equation $\frac{1}{2}y^2 + 3x^2 + x^3 = 4$. Therefore La Salle invariance principle implies that the origin is asymptotically stable. Trajectories starting inside the identified ω - invariant set are attracted to the origin with $t \to \infty$



Domain of attraction

6. i) Formulate definition of an ω -limit set $\omega(p)$ of a point p for a continuous dynamical system.

   ii) What general properties have ω -limit sets of continuous dynamical systems?

   iii) Formulate the Poincare- Bendixon theorem and use it to show that the following system of ODE has an ω - limit set that is a periodic orbit.

$$\begin{cases} x' = & -x\left(x^2 + y^2 - 3x - 1\right) + y \\ y' = & -y\left(x^2 + y^2 - 3x - 1\right) - x \end{cases}$$

(4p)

Solution.

i) $\omega(p)$ is an ω -limit set of a point p for a continuous dynamical system $\pi(t, x)$ if for any point $z \in \omega(p)$ there is a sequence of times $\{t_n\}_{n=1}^{\infty}$ depending on z, such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and $\pi(t_n, p) \underset{n\to\infty}{\longrightarrow} z$.

ii) If the trajectory $\pi([0, \infty], x)$ of the dynamical system is bounded, then the ω -limit set $\omega(p)$ is not empty, closed, connected and positively (or ω) - invariant set.

iii) Poincare Bendixson theorem states that if an ODE in plane has a positively invariant set U without equilibrium points then it must include a periodic orbit that is an ω - limit set for all points in thes set U.

We try to use a simple test function $V(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$ to localize a positively invariant set without equilibrium points.

$V'(x(t), y(t)) = -x^2\left(x^2 + y^2 - 3x - 1\right) + xy + -y^2\left(x^2 + y^2 - 3x - 1\right) - xy = -\left(x^2 + y^2\right)\left(x^2 + y^2 - 3x - 1\right)$

It implies that for large enough $x^2 + y^2$ we have $V'(x(t), y(t)) \leq 0$ and for small enough $x^2 + y^2 \neq 0$ we have $V'(x(t), y(t)) > 0$.

Therefore there are δ and R, $0 < \delta < R$ such that the ring $\delta \leq \sqrt{x^2 + y^2} \leq R$ is a positively invariant set and therefore must include at least one periodic orbit that is an ω - limit set for all points in this ring.

$$\begin{bmatrix} 2x - 3 \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x\left(x^2 + y^2 - 3x - 1\right) + y \\ -y\left(x^2 + y^2 - 3x - 1\right) - x \end{bmatrix} =$$

$\left(-x\left(x^2 + y^2 - 3x - 1\right) + y\right)(2x - 3) + \left(-y\left(x^2 + y^2 - 3x - 1\right) - x\right)(2y) = 9x^3 - 3y - 7x^2 - 3x + 2y^2 - 2x^4 - 2y^4 + 9xy^2 - 4x^2y^2$

Max: 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course will be the average of the points for the home assignments (30%) and for this exam (70%).

The same thresholding is valid for the exam, for the home assignments, and for the total points.