MATEMATIK GU, Chalmers A.Heintz

Datum: 2013-08-20
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE160
This exam is for students who learned the new version of the course in the year 2013.

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and proof the theorem about the principal matrix solution and the dimension of the space of solutions to a non-autonomous linear system of ODE. **(4p)**

   Check the proof in lecture notes.

2. Formulate and give a proof to the theorem on stability of fixed points to non-linear autonomous systems of ODE by Liapunovs functions.

   Check the proof in lecture notes. **(4p)**

3. Consider the following system of ODE:

$$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t),\ \overrightarrow{r}(0) = \overrightarrow{r_0} \text{ with a constant matrix } A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   Find general solution to the system. Find all $\overrightarrow{r_0}$ such that solution $\overrightarrow{r}(t)$ is bounded for $t \to +\infty$. **(4p)**

   General solution is given by $\overrightarrow{r} = \exp(tA)\overrightarrow{r_0}$ with arbitrary $\overrightarrow{r_0} \in R^3$.

   Matrix A has the block diagonal form $A = \begin{bmatrix} J & \\ & C \end{bmatrix}$ and $\exp(At) = \begin{bmatrix} \exp(tJ) & \\ & \exp(tC) \end{bmatrix}$

   with Jordan block J and matrix $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   $\exp(tJ) = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ and $\exp(tC) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} = \cos(t)I + \sin(t)C$, that gives general solution to the system.

   The last relation follows from observation that $C^2 = -I$ and from Taylor series for the exponential function and for $\sin(t)$ and $\cos(t)$.

   The solution $\overrightarrow{r} = \exp(tA)\overrightarrow{r_0}$ is bounded if vetor r_0 has zero components at the second and the third places:

   $r_0 = [a, 0, 0, b, c]^T$ with arbitrary real numbers a, b, c as other components.

   One can also make similar conclution from the general theorem about bounded solutions and generalized eigenspaces.

4. Formulate Banach's contraction principle.

   Consider the following (nonlinear!) operator

$$K(x)(t) = \int_0^2 B(t,s)\,[x(s)]^2\,ds + g(t),$$

acting on the Banach space $C([0,2])$ of continuous functions with norm $\|x\| = \sup_{t\in[0,2]} |x(t)|$. Here $B(t,s)$ and $g(t)$ are continuous functions and $|B(t,s)| < 0.5$ for all $t,s \in [0,2]$.

Estimate the norm $\|K(x)\|$ of the operator $K(x)(t)$.Find requirements on the function $g(t)$ and on a set B in $C([0,2])$such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point in B. **(4p)**

Banach's contraction principle. Let $\mathbb{B}$ be a nonempty closed subset of a Banach space X and let the non-linear operator $K : \mathbb{B} \to \mathbb{B}$ be a contraction.

$$\|K(x) - K(y)\| \le \theta \|x - y\|, \; \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x) - \overline{x}\| \le \frac{\theta^n}{1-\theta}$$

for any $x \in \mathbb{B}$.

We like to have the estimate $\|K(x) - K(y)\| \le \theta \|x - y\|$ for x, y in some closed subset $\mathbb{B}$ of $C([0,2])$.

$\|K(x) - K(y)\| \le \sup_{t\in[0,2]} \left| \int_0^2 B(t,s) \left([x(s)]^2 - [y(s)]^2 \right) ds \right| = \sup_{t\in[0,2]} \left| \int_0^2 B(t,s) \left(x(s) - y(s) \right) \left(x(s) + y(s) \right) ds \right| \le$

$\left| \int_0^2 \sup_{t,s\in[0,2]} B(t,s) ds \right| \|x - y\| \|x + y\| \le \|x - y\| \|x + y\| \le \|x - y\| \left(\|x\| + \|y\| \right)$

We can choose a ball $\mathbb{B} \subset C([0,2])$ such that for any $x,\, y \in \mathbb{B}$ it follows $\|x\| + \|y\| \le \theta < 1$, for example $\mathbb{B}$ can be taken as a set of functions with $\|x\| \le 1/4$. On this set K will be a contraction because $\|K(x) - K(y)\| \le \|x - y\| (0.5)$.

To apply Banachs principle we need also that K maps $\mathbb{B}$ into itself, namely that $\|K(x)\| \le 1/4$ for $\|x\| < 1/4$.

It gives a requirement on function $g(t)$. Estimate the operator K:

$\|K(x)\| \le \sup_{t\in[0,2]} \left| \int_0^2 B(t,s) \left([x(s)]^2 \right) ds \right| + \sup_{t\in[0,2]} |g(t)| \le \|x\|^2 + \|g\|$

If $\|x\| < 1/4$ then we like to have that $\|K(x)\| \le 1/4$ that follows if $\|K(x)\| \le 1/16 + \|g\| \le 1/4$

It is satisfied if $\|g\| \le 3/16$. Therefore for $\|g\| \le 3/16$ the operator K has a unique fixed point in the ball $\mathbb{B}$ $\|x\| \le 1/4$.

5. Consider the system of ODE:

$$\begin{cases} x_1' = & x_1 \sin(x_2) + x_2 \cos(x_1) \\ x_2' = & x_1 x_2 / (1 + |x_2|) + t \\ & \overrightarrow{x}(0) = \overrightarrow{x}_0 \end{cases}$$

Prove that all solutions to the initial value problem can be extended to an arbitrary time interval. **(4p)**

Solution. Observe that the right hand side of the system $x' = F(t,x)$ above has linear growth, namely that for any $T > 0$ there are constants $M(T)$ and $L(T)$ such that $|F(t,x)| \le M(T) + L(T)|x|$, for $(t,x) \in [-T,T] \times R^2$

A theorem about extensibility of solutions states that all solutions to such systems are defined for all $t \in R$.

To show that the estimate $|F(t,x)| \le M(T) + L(T)|x|$ is valid, we observe that $|x_1 \sin(x_2) + x_2 \cos(x_1)| \le |x_1| + |x_2| \le \sqrt{2}|x| \le \sqrt{2}|x| + |t|$

$|x_1 x_2/(1+|x_2|)+t| \le |x_1|+|t| \le \sqrt{2}\,|x|+|t|$

It implies the desired estimate: $|F(t,x)| \le 2\,|x| + \sqrt{2}\,|t|$ and the conclusion on extensibility to all times t.

6. Formulate Poincare-Bendixson theorem. Find a positively invariant set for the following system of ODE. Show that the system has at least one periodic solution.

$$\begin{cases} x' = & -y/3 + x(1-3x^2-y^2) \\ y' = & x + y(1-3x^2-y^2) \end{cases} \qquad \textbf{(4p)}$$

Poincare-Bendixson theorem. Consider a system $r' = f(r)$ in the plane R^2. If a limit set $\omega_\sigma(r)$ of a point r is not empty, compact and contains no fixed points, it is a regular periodic orbit.□

A corollary of the theorem is that if C is a compact positively invariant set to a system of ODE in the plane and C does not contain any fixed points, it must contain at least one regular periodic orbit.□

Multiply the first equation by $3x$ and the second equation by y and add:

$\frac{1}{2}\left(3x^2+y^2\right)' = \left(3x^2+y^2\right)\left(1-\left(3x^2+y^2\right)\right)$

The function $V(x,y) = 3x^2+y^2$ satisfies the equation: $V'(t) = 2V(1-V)$.

We observe that $V(t)$ increases along trajectories of the system for $V<1$ and V decreases for $V>1$. It implies that the set $G = \left\{(x,y) : 0.5 \le 3x^2+y^2 \le 2\right\}$ (an elliptic ring round the origin) is a positively invariant set.

The same calculation shows that the origin is the only stationary point, because stationary point

must satisfy the equation $V(1-V)=0$. By inserting $1-3x^2-y^2=0$ into the equations one can see that points on the ellips $3x^2-y^2=1$ are not stationary, because thay must at the same time be in the origin. It leaves the only fixed point in the origin.

The corollary to Poincare Bendixson theorem states that in a compact positively invariant set without fixed points there must be at least one periodic solution.

Max: 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course will be the average of the points for the home assignments (30%) and for this exam (70%).

The same thresholding is valid for the exam, for the home assignments, and for the total points.