MATEMATIK Datum: 2013-06-01 Tid: 8:30
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Lösningar till Tenta i ODE och matematisk modellering, MMG511, MVE160
This exam is for students who learnd the course in year 2013.

1. Formulate and verify by direct calculation Duhamel's formula for solutions of a non-homogeneous linear system of ODE with constant matrix. **(4p)**

   if $\dot{x} = Ax + g(t)$ then Duhamel's formula gives general solution to the inhomogeneous syste:

   $$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-s))g(s)ds$$

   We verify the formula by direct calculation:

   $\dot{x}(t) = \frac{d}{dt}(\exp(At)x_0) + \exp(A(t-t))g(t) + \int_0^t \frac{d}{dt}(\exp(A(t-s))g(s))\,ds = A\exp(At)x_0 + \exp(A(0))g(t) + \int_0^t A\exp(A(t-s))g(s)ds =$

   $A\exp(At)x_0 + A\int_0^t \exp(A(t-s))g(s)ds + g(t) = A\left(\exp(At)x_0 + \int_0^t \exp(A(t-s))g(s)ds\right) + g(t) = Ax + g(t).$

2. Give definitions of a stable fixed point and of an asymptotically stable fixed point to an autonomous system of ODE. Formulate and give a proof of the theorem about asymptotic stability by linearization of a fixed point to an autonomous system of ODE.

   (4p)

   Consider an autonomous system of ODE $\dot{x} = f(x)$, $f \in C^1(M, R^n)$, $M \subset R^n$, open. Let $x_0 \in M$ be a fixed point: $f(x_0) = 0$.

   Definition. The fixed point x_0 is stable if for any neighborhood $U(x_0)$ of x_0 there is another neighborhood $V(x_0)$ such that for any $x \in V(x_0)$ the positive orbit $\gamma_+(x) \subset U(x_0)$. Or by the other words the trajectory $\phi_x(t)$ starting in $V(x_0)$ will stay in $U(x_0)$ for all $0 \le t$.

   Definition. The fixed point x_0 is asymptotically stable if it is stable and for there is a neighborhood $V_a(x_0)$ of x_0 such that such that for any $x \in V_a(x_0)$ it follows $\lim_{t\to+\infty} |\phi_x(t) - x_0| = 0$.

   Theorem on stability of a fixed point by linearization.

   Consider the ODE $\dot{x} = f(x)$, $f \in C^1(B_R(0), R^n)$,where $B_R(0)$ is a ball of radie R around the origin and $f(0) = 0$. Suppose that Jacobi matrix A of f at the origin has all eigenvalues with negative real part: $\operatorname{Re}\lambda_j < -\beta$, $\beta > 0$.

   Then there are $\varepsilon > 0$, $C > 0$, $\alpha > 0$ such that solutions to $\dot{x} = f(x)$, $x(0) = x_0$, satisfy the estimate

   $$|x(t)| \le Ce^{-\alpha t}|x_0| \text{ for } |x_0| \le \delta$$

   It implies that the fixed point in the origin is asymptotically stable.

   Proof: We express $f(x)$ as $f(x) = Ax + g(x)$ where A is the Jacobi matrix of f in the origin. According to Taylor expansion of $f \in C^1(B_R(0), R^n)$ the function g satisfies $g(x) = o(x)$, when $x \to 0$.It means that $g(x) = |x|\,\xi(x)$ with $\lim_{x\to 0}\xi(x) = 0$ Therefore for any $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ such that for any $|x| < \delta_\varepsilon$ we get $|\xi(x)| < \varepsilon$.

   If all eigenvalues $\operatorname{Re}\lambda_j$ to A have $\operatorname{Re}\lambda_j < -\beta$ then theory of linear systems with constant coefficients imply that

   $$\|\exp(At)\| \le Ce^{-t\beta}$$

for some $C > 0$. We can now choose $\varepsilon > 0$ such that $\varepsilon C < \beta$ and corresponding $\delta_\varepsilon > 0$ for the estimate with the Taylor expansion above.

The theorem about perturbed linear systems and the Gronwall inequality imply that

$$|x(t)| \leq C |x_0| e^{-(\beta - \varepsilon C)t}$$

Choosing $\alpha = \beta - \varepsilon C$ we finishes the proof.

3. Consider the following system of ODE:

$$\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t), \text{ with a constant matrix } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Give general solution to the system. Find all those initial vectors $\overrightarrow{r_0} = \overrightarrow{r}(0)$ that give bounded solutions to the system. **(4p)**

General solution is given by $\overrightarrow{r} = \exp(At)\overrightarrow{r_0}$ with arbitrary $\overrightarrow{r_0} \in R^3$.Matrix A is given in Jordan canonical form and its exponent has the following form: $\exp(At) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-3t) \end{bmatrix}$.

One of the eginevalues of A is $\lambda_1 = 0$ and has multiplicity 2 and only one egenvector $\overrightarrow{v_1} = [1, 0, 0]^T$. The second eigenvalue is $\lambda_2 = -3$ and is simple with the eigenvector $\overrightarrow{v_2} = [0, 0, 1]^T$.

In general situation we get bounded solutions with initial data $\overrightarrow{r_0}$ from the subspace spanned by generalised eigenvectors corresponding eigenvalues with $\operatorname{Re}\lambda < 0$ and by eigenvectors corresponding eigenvalues with $\operatorname{Re}\lambda = 0$.

For the given system bounded solutions correspond to initial data $\overrightarrow{r_0}$ from the subspace (a plane through the origin) in R^3 spanned by vectors v_1 and v_2: $\overrightarrow{r_0} = a\overrightarrow{v_1} + b\overrightarrow{v_2} = [a, 0, b]^T$ with arbitrary $a, b \in R$.

4. Formulate Banach's contraction principle.

Consider the following operator

$$K(x)(t) = \int_0^2 B(t, s)x(s)ds + g(t),$$

with $B(t, s)$ and $g(t)$ continuous functions and $|B(t, s)| < 0.25$ for all $t, s \in [0, 2]$ acting in the Banach space $C([0, 2])$ of continuous functions with norm $\|x\| = \sup_{t \in [0,2]} |x(t)|$.

Show using Banach's contraction principle that $K(x)(t)$ has a fixed point. **(4p)**

Banach's contraction principle. Let C be a nonempty closed subset of a Banach space X and let the non-linear operator $K : C \to C$ be a contraction.

$$\|K(x) - K(y)\| \leq \theta \|x - y\|, \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x) - \overline{x}\| \leq \frac{\theta^n}{1 - \theta}$$

for any $x \in C$.

We start with showing that the operator K is contraction in the space $C([0,2])$ of continuous function on the interval $[0,2]$with norm $\|f\| = \sup_{t\in[0,2]} |f(t)|$:

$\|K(x) - K(y)\| \leq \left\| \int_0^2 B(t,s)x(s)ds + g(t), - \int_0^2 B(t,s)y(s)ds + g(t) \right\| = \left\| \int_0^2 B(t,s)(x(s) - y)ds \right\| \leq \int_0^2 \|B(t,s)(x(s) - y)\| \, ds =$

$\int_0^2 \sup_{t,s\in[0,2]} |B(t,s)| \, \|(x-y)\| \, ds = \sup_{t,s\in[0,2]} |B(t,s)| \, \|(x-y)\| \int_0^2 ds \leq 0.25 \cdot 2 \, \|(x-y)\| = 0.5 \, \|(x-y)\|$

We need to find a closed set that K maps to itself.We take a ball with radius R in the space $C([0,2])$, a function f from this ball: $\|f\| = \sup_{t\in[0,2]} |f(t)| \leq R$ and estimate the norm of $K(f)$ similarly as above:

$\|K(f)\| = \left\| \int_0^2 B(t,s)f(s)ds + g(t) \right\| = \left\| \int_0^2 B(t,s)f(s)ds \right\| + \|g\| = \sup_{t\in[0,2]} \left| \int_0^2 B(t,s)f(s)ds \right| \leq$

$\sup_{t\in[0,2]} \left| \int_0^2 B(t,s)f(s)ds \right| + \|g(t)\| \leq 0.25 \cdot 2 \, \|f\| + \|g\| = 0.5 \, \|f\| + \|g\| \leq 0.5R + \|g\|$

So if we choose the radius R for the ball in the space $C([0,2])$ as $R = 2\,\|g\|$ we will see that $\|K(f)\| \leq R$ for $\|f\| \leq R$ and therefore the operator K maps the closed ball of radius R into itself. Therefore both requirements of the Banach's contraction principle are satisfied and the operator K must have a fixed point in the ball $\|f\| \leq R = 2\,\|g\|$.

5. Consider the following system of ODE and investigate stability of the fixed point in the origin.

$$\begin{cases} x' = 2y^3 - x^5 \\ y' = -x - y^3 + y^5 \end{cases} \qquad \textbf{(4p)}$$

We choose a test function $L(x,y) = x^2 + y^4$. $L(x,y) > 0$ for all $(x,y) \neq (0,0)$.

$\frac{d}{dt}(L(x(t),y(t))) = \nabla L \cdot [2y^3 - x^5, -x - y^3 + y^5]^T = [2x, 4y^3][2y^3 - x^5, -x - y^3 + y^5]^T = 4xy^3 - 2x^6 - 4xy^3 - 4y^6 + 4y^8 =$

$-2x^6 - 4y^6 + 4y^8 = -2x^6 - 4y^6(1 - y^2)$.

Observe that for $|y| < 1$ and $(x,y) \neq (0,0)$ we have $\frac{d}{dt}(L(x(t),y(t))) < 0$. Therefore the fixed point in the origin is asymptotically stable.

6. Show that the following system of ODE has no periodic solutions.

$$\begin{cases} x' = x^3 - y^2x + x \\ y' = -0.5y + y^3 + x^4y \end{cases} \qquad \textbf{(4p)}$$

We consider divergence of the right hand side of the system.

$div(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes). According to Bendixson's criterion the system cannot have periodic solutions anywhere in the plane.

Max. 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course will be the average of the points for the home assignments (30%) and for this exam (70%).

The same thresholding is valid for the exam, for the home assignments, and for the total points.