1. Liapunov's theory.
   Formulate and give a proof for Liapunov's theorem on stability of a fixed point together with definitions of the notions used in the formulation.  

2. Liapunov's functions and stability
   Consider the following system of ODE and investigate stability of the fixed point in the origin. 
   
   \[
   \begin{cases}
   x' = 2y^3 - x^5 \\
   y' = -x - y^3 + y^5
   \end{cases}
   \]

3. Periodic solutions to ODE.
   Show that the following system of ODE has no periodic solutions in the circle \( x^2 + y^2 \leq 4 \).
   
   \[
   \begin{cases}
   x' = x^2 + y^4 + 1 \\
   y' = (x^2 - xy + y^2) + 2
   \end{cases}
   \]

4. Bifurcations and stability of fixed points.
   Consider the following system, find its fixed points, investigate their stability for arbitrary values of the parameter \( \mu \) and identify bifurcations of the fixed points depending on the parameter \( \mu \) for small absolute values of \( \mu \).
   
   \[
   \begin{cases}
   x' = \mu + x^2 \\
   y' = x - y
   \end{cases}
   \]

5. Consider traffic on a road with a given relation between the traffic flow \( q = q(\rho) \) and the density of cars \( \rho \). Traffic flow is defined as \( q = \rho v \) where \( v \) is the velocity of cars.

   Write down a formula for the velocity \( v_\alpha \) of a shock wave (jump in cars density) in the discontinuous traffic in terms of the densities \( \rho^+ \) and \( \rho^- \) and velocities \( v^+ \) and \( v^- \) at the left and at the right of the shock wave. Give a proof of the formula.  

   **Hint:** write an expression for the number \( N(t) \) of cars on an arbitrary interval around the shock wave with coordinate \( x_\alpha(t) \), calculate time derivative of \( N \), and its limit for infinitely short interval. \( \frac{dN}{dt} = v_\alpha \)

Max. 20 points;

For GU: VG: 15 points; G: 10 points. For Chalmers: 5: 17 points; 4: 14 points; 3: 10 points; Total points for the course will be the average of points for the project (60%) and for this exam together with bonus points for home assignments (40%).
1. Liapunov’s theory.

Formulate and give a proof for Liapunov’s theorem on stability of a fixed point together with definitions of the notions used in the formulation.

2. Liapunov functions and stability

Consider the following system of ODE and investigate stability of the fixed point in the origin.

\[
\begin{align*}
    x' &= 2y^3 - x^5 \\
    y' &= -x - y^3 + y^5
\end{align*}
\]

Consider a positive definite function \( V(x, y) = x^2 + y^4 \) and check if it is a Liapunov function for the system.

\[
V'(x, y) = 2x (2y^3 - x^5) + 4y^3 (-x - y^3 + y^5) = -2x^6 - 4y^6 + 4y^8
\]

For small enough \(|y|\) we get that \( V'(x, y) \leq -2x^6 - 3y^6 \). It implies that \( V \) is negative definite and the origin is asymptotically stable.

3. Periodic solutions to ODE.

Show that the following system of ODE has no periodic solutions in the circle \( x^2 + y^2 \leq 4 \).

\[
\begin{align*}
    x' &= x^2 + y^4 + 1 \\
    y' &= (x^2 - xy + y^2) + 2
\end{align*}
\]

Suppose that there is a periodic solution \((x(t), y(t))\). Choose a rectangle \([a, b] \times [c, d]\) such that the closed trajectory is completely inside the rectangle.

Observe that \( x' \geq 1 \) and \( y' \geq 2 \). It implies that any trajectory of the system leaves the rectangle \([a, b] \times [c, d]\) in finite time. It contradicts to our hypothesis on the existence of the periodic solution.

4. Bifurcations and stability of fixed points.

Consider the following system, find its fixed points, investigate their stability for arbitrary values of the parameter \( \mu \) and identify bifurcations of the fixed points depending on the parameter \( \mu \) for small absolute values of \( \mu \).

\[
\begin{align*}
    x' &= -\mu + x^2 \\
    y' &= x - y
\end{align*}
\]

There are no fixed points for \( \mu < 0 \). If \( \mu > 0 \) there are two fixed points: \((\sqrt{\mu}, \sqrt{\mu})\) and \((-\sqrt{\mu}, -\sqrt{\mu})\).

Linearization around the first fixed point has matrix \[
\begin{bmatrix}
2\sqrt{\mu} & 0 \\
1 & -1
\end{bmatrix}
\]. This fixed point is a saddle point.

Linearization around the second fixed point has matrix \[
\begin{bmatrix}
-2\sqrt{\mu} & 0 \\
1 & -1
\end{bmatrix}
\]. This fixed point is a stable node.

In the point \( \mu = 0 \) the linearization is degenerate and the system has a saddle-node bifurcation.
5. Consider traffic on a road with a given relation between the traffic flow $q = q(\rho)$ and the density of cars $\rho$. Traffic flow is defined as $q = \rho v$ where $v$ is the velocity of cars.

Write down a formula for the velocity $v_s$ of a shock wave (jump in cars density) in the discontinuous traffic in terms of the densities $\rho^+$ and $\rho^-$ and velocities $v^+$ and $v^-$ at the left and at the right of the shock wave. Give a proof of the formula.

\textbf{Hint:} write an expression for the number $N(t)$ of cars on an arbitrary interval around the shock wave with coordinate $x_s(t)$, calculate time derivative of $N$, and its limit for infinitely short interval. $\frac{dx_s}{dt} = v_s$

Consider the number $N(t)$ of cars in the interval $[a, b]$ such that $[a, b] = [a, x_s(t)] \cup [x_s(t), b]$:

\[ N(t) = \int_a^b \rho(x, t) dx \]

Calculate derivative of $N$:

\[ \frac{dN}{dt} = \frac{d}{dt} \int_a^b \rho(x, t) dx = q(a, t) - q(b, t) \]

We use elementary formula for derivative of an integral depending on a parameter:

\[ \frac{d}{dt} \int_a^{\beta(t)} f(x, t) dx = \frac{d\beta}{dt} f(\beta, t) - \frac{d}{dt} f(\alpha, t) + \int_{\alpha(t)}^{\beta(t)} \frac{d}{dt} f(x, t) dx \]

that is valid for a smooth function $f$.

Density $\rho$ has a jump in point $a < x_s < b$. The last formula cannot be directly applied to the expression of $N$. We split the integral over $(a, b)$ into two integrals:

\[ N(t) = \int_a^{x_s(t)} \rho(x, t) dx + \int_{x_s(t)}^b \rho(x, t) dx \]

compute derivative separately for these two integrals:

\[ \frac{d}{dt} \int_a^{x_s(t)} \rho(x, t) dx = \frac{dx_s}{dt} \rho(x_s^-, t) + \int_a^{x_s(t)} \frac{d}{dt} \rho(x, t) dx \]

\[ \frac{d}{dt} \int_{x_s(t)}^b \rho(x, t) dx = -\frac{dx_s}{dt} \rho(x_s^+, t) + \int_{x_s(t)}^b \frac{d}{dt} \rho(x, t) dx \]

Here $\rho(x_s^-, t)$ and $\rho(x_s^+, t)$ are left and right limits of $\rho(x, t)$ at the jump point $x_s(t)$.

We put these formulas into the expression for $\frac{dN}{dt}$.

\[ \frac{dx_s}{dt} \rho(x_s^-, t) - \frac{dx_s}{dt} \rho(x_s^+, t) + \int_a^{x_s(t)} \frac{d}{dt} \rho(x, t) dx + \int_{x_s(t)}^b \frac{d}{dt} \rho(x, t) dx = q(a, t) - q(b, t) \]

We let $a \to x_s$ and $b \to x_s$. Integrals in the last equation go to zero with the length of the integration intervals and we get the desired formula:

\[ \frac{dx_s}{dt} \left( \rho(x_s^-, t) - \rho(x_s^+, t) \right) = q(x_s^-, t) - q(x_s^+, t) \]

or with jump notations $[\rho] = \rho(x_s^-, t) - \rho(x_s^+, t)$

\[ v_s = \frac{dx_s}{dt} = \frac{[q]}{[\rho]} \]

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