MATEMATIK	Datum: 2012-08-21	Tid: 8:30-12:30
GU, Chalmers	Hjälpmedel: Beta	
A.Heintz	Telefonvakt: Peter Helgesson	Tel.: 0703-088304

Tenta i matematisk modellering, MMG510, MVE160

1. **Liapunovs theory.**

   Formulate and give a proof for Liapunovs theorem on stability of a fixed point together with definitions of the notions used in the formulation. (4p)

2. **Liapunovs functions and stability**

   Consider the following system of ODE and investigate stability of the fixed point in the origin. (4p)

$$\begin{cases} x' = 2y^3 - x^5 \\ y' = -x - y^3 + y^5 \end{cases}$$

3. **Periodic solutions to ODE.**

   Show that the following system of ODE has no periodic solutions in the circle $x^2 + y^2 \leq 4$.

$$\begin{cases} x' = x^2 + y^4 + 1 \\ y' = \left(x^2 - xy + y^2\right) + 2 \end{cases}$$

   (4p)

4. **Bifurcations and stability of fixed points.**

   Consider the following system, find its fixed points, investigate their stability for arbitrary values of the parameter μ and identify bifurcations of the fixed points depending on the parameter μ for small absolute values of μ.

$$\begin{cases} x' = \mu + x^2 \\ y' = x - y \end{cases}$$

   (4p)

5. Consider traffic on a road with a given relation between the traffic flow $q = q(\rho)$ and the density of cars ρ. Traffic flow is defined as $q = \rho v$ where v is the velocity of cars.

   Write down a formula for the velocity v_s of a shock wave (jump in cars density) in the discontinuous traffic in terms of the densities ρ^+ and ρ^- and velocities v^+ and v^- at the left and at the right of the shock wave. Give a proof of the formula. (4p)

   Hint: write an expression for the number $N(t)$ of cars on an arbitrary interval around the shock wave with coordinate $x_s(t)$, calculate time derivative of N, and its limit for infinitely short interval. $\frac{dx_s}{dt} = v_s$

   Max. 20 points;

For GU: **VG:** 15 points; **G:** 10 points. For Chalmers: **5:** 17 points; **4:** 14 points; **3:** 10 points; Total points for the course will be the average of points for the project (60%) and for this exam together with bonus points for home assingments (40%).

MATEMATIK
GU, Chalmers
A.Heintz

Datum: 2012-08-21
Hjälpmedel: Beta
Telefonvakt: Peter Helgesson

Tid: 8:30

Tel.: 0762-721860.

Lösningar till tenta i matematisk modellering, MMG510, MVE160

1. **Liapunovs theory.**

   Formulate and give a proof for Liapunovs theorem on stability of a fixed point together with definitions of the notions used in the formulation. **(4p)**

2. **Liapunovs functions and stability**

   Consider the following system of ODE and investigate stability of the fixed point in the origin. **(4p)**

   $$\begin{cases} x' = 2y^3 - x^5 \\ y' = -x - y^3 + y^5 \end{cases}$$

   Consider a positive definite function $V(x,y) = x^2 + y^4$ and check if it is a Liapunovs function for the system.

   $$V'(x,y) = 2x\left(2y^3 - x^5\right) + 4y^3\left(-x - y^3 + y^5\right) = -2x^6 - 4y^6 + 4y^8$$

   For small enough $|y|$ we get that $V'(x,y) \leq -2x^6 - 3y^6$. It implies that $V'$is negative definite and

   the origin is asymptotically stable.

3. **Periodic solutions to ODE.**

   Show that the following system of ODE has no periodic solutions in the circle $x^2 + y^2 \leq 4$.

   $$\begin{cases} x' = x^2 + y^4 + 1 \\ y' = \left(x^2 - xy + y^2\right) + 2 \end{cases}$$

   Suppose that there is a periodic solution $(x(t), y(t))$. Choose a rectangle $[a,b] \times [c,d]$ such that the closed trajectory is completely inside the rectangle. **(4p)**

   Observe that $x' \geq 1$ and $y' \geq 2$ It implies that any trajectory of the system leaves the rectangle $[a,b] \times [c,d]$ in finite time. It contradicts to our hypothesis on the existence of the periodic solution.

4. **Bifurcations and stability of fixed points.**

   Consider the following system, find its fixed points, investigate their stability for arbitrary values of the parameter μ and identify bifurcations of the fixed points depending on the parameter μ for small absolute values of μ.

   $$\begin{cases} x' = -\mu + x^2 \\ y' = x - y \end{cases}$$

   There are no fixed points for $\mu < 0$. If $\mu > 0$ there are two fixed points: $(\sqrt{\mu}, \sqrt{\mu})$ and $(-\sqrt{\mu}, -\sqrt{\mu})$. **(4p)**

   Linearization around the first fixed point has matrix $\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 1 & -1 \end{bmatrix}$. This fixed point is a saddle point.

   Linearization around the second fixed point has matrix $\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 1 & -1 \end{bmatrix}$. This fixed point is a stable node.

   In the point $\mu = 0$ the linearization is degenerate and the sistem has a saddle-node bifurcation

5. Consider traffic on a road with a given relation between the traffic flow $q = q(\rho)$ and the density of cars ρ. Traffic flow is defined as $q = \rho v$ where v is the velocity of cars.

Write down a formula for the velocity v_s of a shock wave (jump in cars density) in the discontinuous traffic in terms of the densities ρ^+ and ρ^- and velocities v^+ and v^- at the left and at the right of the shock wave. Give a proof of the formula. (4p)

Hint: write an expression for the number $N(t)$ of cars on an arbitrary interval around the shock wave with coordinate $x_s(t)$, calculate time derivative of N, and its limit for infinitely short interval. $\frac{dx_s}{dt} = v_s$

Consider the number $N(t)$ of cars in the interval $[a, b]$ such that $[a, b] = [a, x_s(t)] \cup [x_s(t), b]$:
$N(t) = \int_a^b \rho(x, t)dx$; Calculate derivative of N : $\frac{dN}{dt} = \frac{d}{dt}\int_a^b \rho(x, t)dx = q(a, t) - q(b, t)$

We use elementary formula for derivative of an integral depending on a parameter:
$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x, t)dx = \frac{d\beta}{dt} f(\beta, t) - \frac{d\alpha}{dt} f(\alpha, t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x, t)dx$ that is valid for a smooth function f.

Density ρ has a jump in point $a < x_s < b$. The last formula cannot be directly applied to the expression of N. We split the integral over (a, b) into two integrals: $N(t) = \int_a^b \rho(x, t)dx = \int_a^{x_s(t)} \rho(x, t)dx + \int_{x_s(t)}^b \rho(x, t)dx$ and compute derivative separately for these two integrals:
$\frac{d}{dt}\int_a^{x_s(t)} \rho(x, t)dx = \frac{dx_s}{dt}\rho(x_s^-, t) + \int_a^{x_s(t)} \frac{d}{dt}\rho(x, t)dx$;
$\frac{d}{dt}\int_{x_s(t)}^b \rho(x, t)dx = -\frac{dx_s}{dt}\rho(x_s^+, t) + \int_{x_s(t)}^b \frac{d}{dt}\rho(x, t)dx$. Here $\rho(x_s^-, t)$ and $\rho(x_s^+, t)$ are left and right limits of $\rho(x, t)$ at the jump point $x_s(t)$.

We put these formulas into the expression for $\frac{dN}{dt}$.

$\frac{dx_s}{dt}\rho(x_s^-, t) - \frac{dx_s}{dt}\rho(x_s^+, t) + \int_a^{x_s(t)} \frac{d}{dt}\rho(x, t)dx + \int_{x_s(t)}^b \frac{d}{dt}\rho(x, t)dx = q(a, t) - q(b, t)$

We let $a \to x_s$ and $b \to x_s$. Integrals in the last equation go to zero with the length of the integration intervals and we get the desired formula:

$\frac{dx_s}{dt}\left(\rho(x_s^-, t) - \rho(x_s^+, t)\right) = q(x_s^-, t) - q(x_s^+, t)$ or with jump notations $[\rho] = \rho(x_s^-, t) - \rho(x_s^+, t)$

$$v_s = \frac{dx_s}{dt} = \frac{[q]}{[\rho]}$$

Max. 20 points;

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