1. Liapunovs theory.

Formulate and give a proof for Liapunovs theorem on instability of a fixed point. (4p)

2. Periodic solutions to ODE.

Show that the following system has at least one periodic solution.
\[
\begin{align*}
x' &= x - 2y - x\left(x^2 - xy + y^2\right) \\
y' &= 2x + y - y\left(x^2 - xy + y^2\right)
\end{align*}
\]

(4p)

3. Bifurcations and stability of fixed points.

Consider the following system, find its fixed points, investigate their stability and identify bifurcations of the fixed points depending on the parameter \( \mu \) for small absolute values of \( \mu \).
\[
\begin{align*}
x' &= x(\mu - x^2) \\
y' &= -y(2 - y)
\end{align*}
\]


Explain the meaning of Hopf bifurcation and show that the system
\[
\begin{align*}
x' &= \mu x + y - xy^2 \\
y' &= -x + \mu y - y^3
\end{align*}
\]

has a Hopf bifurcation at \( \mu = 0 \). (4p)

5. Consider the Lotka - Volterra system of equations
\[
\begin{align*}
\frac{dF}{dt} &= F(a - cS) \\
\frac{dS}{dt} &= S(AF - k)
\end{align*}
\]

for populations of pray \( F \) (fish) and predators \( S \) (sharks). Explain the meaning of the terms in the equations.

Show that the system has a fixed point in the first quadrant \((F, S > 0)\) of the phase. Find an equation representing closed orbits surrounding this fixed point. (4p)

Hint: find a relation between \( S \) and \( F \) by solving the differential equation with separable variables for \( \frac{dF}{dS} \) that follows from the system.

Max. 20 points;

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1. Liapunov's theory.

Formulate and give a proof for Liapunov's theorem on instability of a fixed point.

See Arrowmith Place Theorem 5.4.3 p. 203.

2. Periodic solutions to ODE.

Show that the following system has at least one periodic solution.

\[
\begin{cases}
x' = x - 2y - x\left(x^2 - xy + y^2\right) \\
y' = 2x + y - y\left(x^2 - xy + y^2\right)
\end{cases}
\]

We multiply the equations by \(x\) and \(y\) and add them:

\[
\begin{cases}
x = x^2 - 2xy - x^3 + y^3 \\
y = 2xy + y^2 - x^3 + y^3
\end{cases}
\]

\[
\frac{1}{2}\left(r^2\right)' = xx' + yy' = x^2 + y^2 - \left(x^2 + y^2\right)\left(x^2 - xy + y^2\right) = \left(x^2 + y^2\right)\left(1 - \left(x^2 - xy + y^2\right)\right)
\]

We observe that \(\left(r^2\right)' = 0\) if and only if \((x, y) = (0, 0)\) or \((x^2 - xy + y^2) = 1\).

The last relation implies \(\begin{cases} x' = x - 2y - x \\ y' = 2x + y - y\end{cases} \Rightarrow (x, y) = (0, 0)\) so the only fixed point is the origin.

The sign of the \(\left(r^2\right)\)' is the same as the sign of \(1 - \left(x^2 - xy + y^2\right)\). The expression \(x^2 - xy + y^2\) is a positive definite quadratic form. It is easy to see by transforming it to the some of squares:

\[
(x^2 - xy + y^2) = \left(x^2 - 2x\left(\frac{3}{2}y\right) + \frac{3}{2}y^2\right) - \frac{3}{4}y^2 = \left(x - \left(\frac{3}{2}y\right)\right)^2 + \frac{3}{4}y^2.
\]

Therefore the level sets of \(x^2 - xy + y^2\) are ellipses. Choosing \(D > 1\) and \(d < 1\) we observe that the elliptic ring where \(d \leq (x^2 - xy + y^2) \leq D\) is a positively invariant set for the given system because \(\left(r^2\right)' > 0\) for \(\left(x^2 - xy + y^2\right) = d < 1\) and \(\left(r^2\right)' < 0\) for \(\left(x^2 - xy + y^2\right) = D > 1\). This ring does not include the origin that is the only fixed point of the system.

Therefore by a corollary from the Poincare-Bendixson theorem this ring must include at least one periodic solution.

3. Bifurcations and stability of fixed points.

Consider the following system, find its fixed points, investigate their stability and identify bifurcations of the fixed points depending on the parameter \(\mu\) for small absolute values of \(\mu\).

\[
\begin{cases}
x' = x(\mu - x^2) \\
y' = -y(2 - y)
\end{cases}
\]

For \(\mu < 0\) there are two fixed points: \(r_1 = (0, 0)\) and \(r_2 = (0, 2)\).

For \(\mu = 0\) there are also two fixed points \(r_1 = (0, 0)\) and \(r_2 = (0, 2)\).

For \(\mu > 0\) there are six fixed points: \(r_1 = (0, 0)\) and \(r_2 = (0, 2)\); \(r_3 = (\sqrt{\mu}, 0)\) and \(r_4 = (-\sqrt{\mu}, 0)\); \(r_5 = (\sqrt{2}, 2)\) and \(r_6 = (-\sqrt{2}, 2)\).

The Jacobian matrix of the right hand side of the system is

\[
A(x, y, \mu) = \begin{bmatrix}
\mu - 3x^2 & 0 \\
0 & 2y - 2
\end{bmatrix}.
\]
It implies that for $\mu < 0$ the fixed point $r_1 = (0,0)$ is a stable knot and the fixed point $r_2 = (0,2)$ is a saddle point.

For $\mu = 0$ fixed points $r_1 = (0,0)$ and $r_2 = (0,2)$ have a degenerate linearization:

\[ A(0,0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } A(0,0,2) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. \]

The linearization

For $\mu > 0$:

- the fixed point $r_1 = (0,0)$ is a saddle point and unstable: $A(0,0,\mu) = \begin{bmatrix} \mu & 0 \\ 0 & -2 \end{bmatrix}$
- the fixed point $r_3 = (\sqrt{\mu},0)$ is a stable knot: $A(\sqrt{\mu},0,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & -2 \end{bmatrix}$
- the fixed point $r_4 = (-\sqrt{\mu},0)$ is a stable knot: $A(-\sqrt{\mu},0,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & -2 \end{bmatrix}$
- the fixed point $r_2 = (0,2)$ is an unstable knot: $A(x,y,\mu) = \begin{bmatrix} \mu & 0 \\ 0 & 2 \end{bmatrix}$
- the fixed point $r_5 = (\sqrt{\mu},2)$ is a saddle point and unstable: $A(x,y,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & 2 \end{bmatrix}$
- the fixed point $r_6 = (-\sqrt{\mu},2)$ is a saddle point and unstable: $A(x,y,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & 2 \end{bmatrix}$

It implies that we observe in this system two pitchfork bifurcations at $\mu = 0$:

- the stable fixed point $r_1 = (0,0)$ splits into three fixed points $r_1, r_3, r_4$: one unstable and two stable and
- the unstable fixed point $r_2 = (0,2)$ splits into three fixed points $r_2, r_5, r_6$: all three unstable.

4. **Hopf bifurcation.**

Explain the meaning of Hopf bifurcation and show that the system

\[
\begin{cases}
  x' = \mu x + y - xy^2 \\
  y' = -x + \mu y - y^3
\end{cases}
\]

has a Hopf bifurcation at $\mu = 0$. \hfill (4p)

The system has a fixed point in the origin for all $\mu$ with small absolute value.

Linearization of the system is:

\[
\begin{cases}
  x' = \mu x + y \\
  y' = -x + \mu y
\end{cases}
\]

Eigenvalues of the corresponding matrix are $\lambda_{1,2}(\mu) = \mu \pm i$. It implies that $\frac{d(\text{Re}\lambda_{1,2}(\mu))}{d\mu} = 1 > 0$.

For $\mu < 0$ we have stable focus(spiral), for $\mu > 0$ we have unstable focus (spiral) in a neighborhood of the origin.

When $\mu = 0$ the stability of the system cannot be investigated by linearization because $\text{Re} \lambda_{1,2}(0) = 0$.

We try the set function $V(x,y) = x^2 + y^2$ and see that when $\mu = 0$ $V' = -y^2(x^2 + y^2) \leq 0$. It implies that the origin is a stable fixed point. On the other hand the line $y^2 = 0$ where $V' = 0$ does not include any whole trajectory except the origin because $y' = -x$ when $y = 0$. It makes that any non-trivial trajectory can only cross the line $y = 0$ where $V' = 0$ and cannot belong to it.
5. Consider the Lotka-Volterra system of equations
\[
\begin{align*}
\frac{dF}{dt} &= F(a - cS) \\
\frac{dS}{dt} &= S(\lambda F - k)
\end{align*}
\]
for populations of pray \(F\) (fish) and predators \(S\) (sharks). Explain the meaning of the terms in the equations.

Show that the system has a fixed point in the first quadrant \((F, S > 0)\) of the phase. Find the equation representing closed orbits surrounding this fixed point. \((4p)\)

**Hint:** find the relation between \(S\) and \(F\) by solving the equation with separable variables for \(\frac{dF}{dS}\) that follows from the system.

The system has only one fixed point with positive \(F\) and \(S\) that is: \((a/c, k/\lambda)\).

We consider the differential relation between \(F\) and \(S\):
\[
\begin{align*}
\frac{dF}{dS} &= \frac{F(a-cS)}{S(\lambda F-k)}
\end{align*}
\]

It implies:
\[
\begin{align*}
\frac{(\lambda F-k) dF}{F} &= \frac{(a-cS) dS}{S} = \left(\lambda - \frac{k}{F}\right) dF = (\frac{a}{S} - c) dS \\
\int \left(\lambda - \frac{k}{F}\right) dF &= \int \left(\frac{a}{S} - c\right) dS
\end{align*}
\]

The integration gives the following functional relation between \(F\) and \(S\):
\[
E + a \ln S - Sc = F\lambda - k \ln F.
\]

Computing exponent of the left and right hand sides gives
\[
Z = \exp(E + a \ln S - Sc) = E_0 S^a e^{-Sc}, \quad \text{with } E_0 = e^E \text{ an arbitrary constant.}
\]

\[
Z = \exp(F\lambda - k \ln F) = F^{-k} e^{\lambda F}.
\]

The desired functional relation between \(F\) and \(S\) is:
\[
E_0 S^a e^{-Sc} = F^{-k} e^{\lambda F}
\]

An extra(!) investigation of the left and the right hand sides of this relation implies that points \((S, F)\) satisfying it constitute for different constants \(E_0\) closed curves around the fixed point \((a/c, k/\lambda)\).

\[
\frac{d}{dS} (E_0 S^a e^{-Sc}) = E_0 S^a e^{-Sc} (a/S - c). \quad \text{This derivative is zero for } S = c/a \text{ that is } S \text{- coordinate of the fixed point } (a/c, k/\lambda) \text{ of the system.}
\]

\[
\frac{d}{dF} (F^{-k} e^{\lambda F}) = F^{-k} e^{\lambda F} (-k/F + \lambda). \quad \text{This derivative is zero for } F = \lambda/k \text{ that is } F \text{- coordinate of the fixed point } (a/c, k/\lambda) \text{ of the system.}
\]

It is easy to see that these two functions \(Z(S)\) and \(Z(F)\) have a local maximum and a local minimum in these points.

Graphics of these two functions look as the following:
The analysis of $F$ and $S$ - values where $Z(S) = Z(F)$ on these graphs implies that the points $(S,F)$ constitute closed curves in the $S - F$ plane. There are no other fixed points except $(c/a, \lambda/h)$ in the $S - F$ plane. It implies that these closed curves must be closed periodic orbits.

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