MATEMATIK	Datum: 2012-05-30	Tid: 8:30
GU, Chalmers	Hjälpmedel: Beta	
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Tenta i matematisk modellering, MMG510, MVE160

1. **Liapunovs theory.**

   Formulate and give a proof for Liapunovs theorem on instability of a fixed point. **(4p)**

2. **Periodic solutions to ODE.**

   Show that the following system has at least one periodic solution.

   $$\begin{cases} x' = x - 2y - x\left(x^2 - xy + y^2\right) \\ y' = 2x + y - y\left(x^2 - xy + y^2\right) \end{cases}$$

   (4p)

3. **Bifurcations and stability of fixed points.**

   Consider the following system, find its fixed points, investigate their stability and identify bifurcations of the fixed points depending on the parameter μ for small absolute values of μ.

   $$\begin{cases} x' = x(\mu - x^2) \\ y' = -y\,(2 - y) \end{cases}$$

4. **Hopf bifurcation.**

   Explain the meaning of Hopf bifurcation and show that the system

   $$\begin{cases} x' = \mu x + y - xy^2 \\ y' = -x + \mu y - y^3 \end{cases}$$

   has a Hopf bifurcation at $\mu = 0$. **(4p)**

5. Consider the Lottka - Volterra system of equations

   $$\begin{cases} \frac{dF}{dt} = F(a - cS) \\ \frac{dS}{dt} = S(\lambda F - k) \end{cases}$$

   for populations of pray F (fish) and predators S (sharks). Explain the meaning of the terms in the equations.

   Show that the system has a fixed point in the first quadrant $(F, S > 0)$ of the phase. Find an equation representing closed orbits surrounding this fixed point. **(4p)**

   Hint: find a relation between S and F by solving the differential equation with separable variables for $\frac{dF}{dS}$ that follows from the system.

   Max. 20 points;

For GU: **VG:** 15 points; **G:** 10 points. For Chalmers: **5:** 17 points; **4:** 14 points; **3:** 10 points;
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1. **Liapunovs theory.**

   Formulate and give a proof for Liapunovs theorem on instability of a fixed point.

   See Arrowsmith Place Theorem 5.4.3 p. 203.

2. **Periodic solutions to ODE.**

   Show that the following system has at least one periodic solution.

   $$\begin{cases} x' = x - 2y - x\left(x^2 - xy + y^2\right) \\ y' = 2x + y - y\left(x^2 - xy + y^2\right) \end{cases}$$

   We multiply the equations by x and by y and add them:

   $$\begin{cases} xx' = x^2 - 2xy - x^2\left(x^2 - xy + y^2\right) \\ yy' = 2xy + y^2 - y^2\left(x^2 - xy + y^2\right) \end{cases}$$

   $$\frac{1}{2}\left(r^2\right)' = xx' + yy' = x^2 + y^2 - \left(x^2 + y^2\right)\left(x^2 - xy + y^2\right) = \left(x^2 + y^2\right)\left(1 - \left(x^2 - xy + y^2\right)\right)$$

   We observe that $\left(r^2\right)' = 0$ if and only if $(x,y) = (0,0)$ or $\left(x^2 - xy + y^2\right) = 1$.

   The last relation implies $\begin{cases} x' = x - 2y - x \\ y' = 2x + y - y \end{cases}$ $\begin{cases} x' = -2y \\ y' = 2x \end{cases}$ $\Longrightarrow (x,y) = (0,0)$ so the only fixed point is the origin.

   The sign of the $\left(r^2\right)'$ is the same as the sign of $1-\left(x^2 - xy + y^2\right)$. The expression $\left(x^2 - xy + y^2\right)$ is a positive definite quadratic form. It is easy to see by transforming it to the some of squares:

   $$\left(x^2 - xy + y^2\right) = \left(\left[x^2 - 2x\left(\tfrac{1}{2}y\right) + \tfrac{1}{4}y^2\right] - \tfrac{1}{4}y^2 + y^2\right) = \left(\left[x - \left(\tfrac{1}{2}y\right)\right]^2 + \tfrac{3}{4}y^2\right).$$

   Therefore the level sets of $\left(x^2 - xy + y^2\right)$ are ellipses. Choosing $D > 1$ and $d < 1$ we observe that the elliptic ring

   where $d \le \left(x^2 - xy + y^2\right) \le D$ is a positively invariant set for the given system because $\left(r^2\right)' > 0$ for $\left(x^2 - xy + y^2\right) = d < 1$ and $\left(r^2\right)' < 0$ for $\left(x^2 - xy + y^2\right) = D > 1$. This ring does not include the origin that is the only fixed point of the system.

   Therefore by a corollary from the Poincare-Bendixson theorem this ring must include at least one periodic solution. ■

3. **Bifurcations and stability of fixed points.**

   Consider the following system, find its fixed points, investigate their stability and identify bifurcations of the fixed points depending on the parameter μ for small absolute values of μ.

   $$\begin{cases} x' = x(\mu - x^2) \\ y' = -y\,(2 - y) \end{cases}$$

   For $\mu < 0$ there are two fixed points: $r_1 = (0,0)$ and $r_2 = (0,2)$.

   For $\mu = 0$ there are also two fixed points $r_1 = (0,0)$ and $r_2 = (0,2)$.

   For $\mu > 0$ there are six fixed points: $r_1 = (0,0)$ and $r_2 = (0,2)$; $r_3 = (\sqrt{\mu}, 0)$ and $r_4 = (-\sqrt{\mu}, 0)$; $r_5 = (\sqrt{\mu}, 2)$. and $r_6 = (-\sqrt{\mu}, 2)$;

   The Jacobi matrix of the right hand side of the system is

   $$A(x,y,\mu) = \begin{bmatrix} \mu - 3x^2 & 0 \\ 0 & 2y - 2 \end{bmatrix}.$$

It implies that for $\mu < 0$ the fixed point $r_1 = (0,0)$ is a stable knot and the fixed point $r_2 = (0,2)$ is a saddle point.

For $\mu = 0$ fixed points $r_1 = (0,0)$ and $r_2 = (0,2)$ have a degenerate linearization:

$A(0,0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ and $A(0,0,2) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

The linearization

For $\mu > 0$:

the fixed point $r_1 = (0,0)$ is a saddle point and unstable: $A(0,0,\mu) = \begin{bmatrix} \mu & 0 \\ 0 & -2 \end{bmatrix}$

the fixed point $r_3 = (\sqrt{\mu}, 0)$ is a stable knot: $A(\sqrt{\mu}, 0, \mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & -2 \end{bmatrix}$

the fixed point $r_4 = (-\sqrt{\mu}, 0)$ is a stable knot: $A(-\sqrt{\mu}, 0, \mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & -2 \end{bmatrix}$

the fixed point $r_2 = (0,2)$ is an unstable knot: $A(x,y,\mu) = \begin{bmatrix} \mu & 0 \\ 0 & 2 \end{bmatrix}$

the fixed point $r_5 = (\sqrt{\mu}, 2)$ is a saddle point and unstable: $A(x,y,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & 2 \end{bmatrix}$

the fixed point $r_6 = (-\sqrt{\mu}, 2)$ is a saddle point and unstable: $A(x,y,\mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & 2 \end{bmatrix}$

It implies that we observe in this system two pitchfork bifurcations at $\mu = 0$:

the stable fixed point $r_1 = (0,0)$ splits into three fixed points r_1, r_3, r_4 : one unstable and two stable and

the unstable fixed point $r_2 = (0,2)$ splits into three fixed points r_2, r_5, r_6 : all three unstable.

4. **Hopf bifurcation.**

Explain the meaning of Hopf bifurcation and show that the system

$$\begin{cases} x' = \mu x + y - xy^2 \\ y' = -x + \mu y - y^3 \end{cases}$$

has a Hopf bifurcation at $\mu = 0$. **(4p)**

The system has a fixed point in the origin for all μ with small absolute value.

Linearization of the system is:

$$\begin{cases} x' = \mu x + y \\ y' = -x + \mu y \end{cases}$$

Eigenvalues of the corresponding matrix are $\lambda_{1,2}(\mu) = \mu \pm i$. It implies that $\frac{d(\operatorname{Re}\lambda_{1,2}(\mu))}{d\mu} = 1 > 0$.

For $\mu < 0$ we have stable focus(spiral), for $\mu > 0$ we have unstable focus (spiral) in a neighborhood of the origin.

When $\mu = 0$ the stability of the system cannot be investigated by lniearization because $\operatorname{Re}\lambda_{1,2}(0) = 0$.

We try the set function $V(x,y) = x^2 + y^2$ and see that when $\mu\mu = 0$ $V' = -y^2(x^2 + y^2) \leq 0$. It implies that the origin is a stable fixed point. On the other hand the line $y^2 = 0$ where $V' = 0$ does not include any whole trajectory except the origin because $y' = -x$ when $y = 0$. It makes that any non-trivial trajectory can only cross the line $y = 0$ where $V' = 0$ and cannot belong to it.■

5. Consider the Lottka - Volterra system of equations

$$\begin{cases} \frac{dF}{dt} = F(a - cS) \\ \frac{dS}{dt} = S(\lambda F - k) \end{cases}$$

for populations of pray F (fish) and predators S (sharks). Explain the meaning of the terms in the equations.

Show that the system has a fixed point in the first quadrant ($F, S > 0$) of the phase. Find the equation representing closed orbits surrounding this fixed point. (4p)

Hint: find the relation between S and F by solving the equation with separable variables for $\frac{dF}{dS}$ that follows from the system.

The system has only one fixed point with positive F and S that is: $(a/c, k/\lambda)$.

We consider the differential relation between F and S:

$$\left\{ \frac{dF}{dS} = \frac{F(a-cS)}{S(\lambda F-k)} \right.$$

It implies:

$$\left\{ \frac{(\lambda F-k)dF}{F} = \frac{(a-cS)dS}{S} = \right. \qquad \left(\lambda - \frac{k}{F}\right) dF = \left(\frac{a}{S} - c\right) dS$$

$$\int \left(\lambda - \frac{k}{F}\right) dF = \int \left(\frac{a}{S} - c\right) dS$$

The integration gives the following functional relation between F and S:

$E + a \ln S - Sc = F\lambda - k \ln F$.

Computing exponent of the left and right hand sides gives

$Z = \exp(E + a \ln S - Sc) = E_0 S^a e^{-Sc}$; with $E_0 = e^E$ an arbitrary constant.

$Z = \exp(F\lambda - k \ln F) = F^{-k} e^{\lambda F}$;

The desired functional relation between F and S is:

$$E_0 S^a e^{-Sc} = F^{-k} e^{\lambda F}$$

■

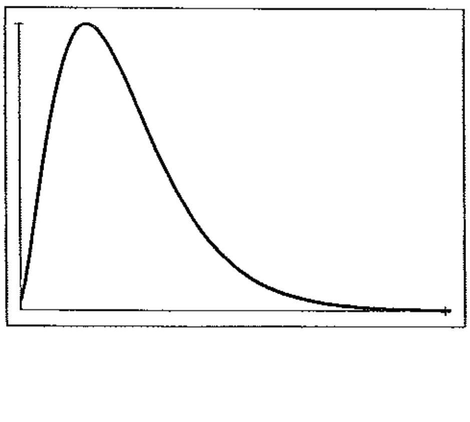
An extra(!) investigation of the left and the right hand sides of this relation implies that points (S, F) satisfying it constitute for different constants E_0 closed curves around the fixed point $(a/c, k/\lambda)$.

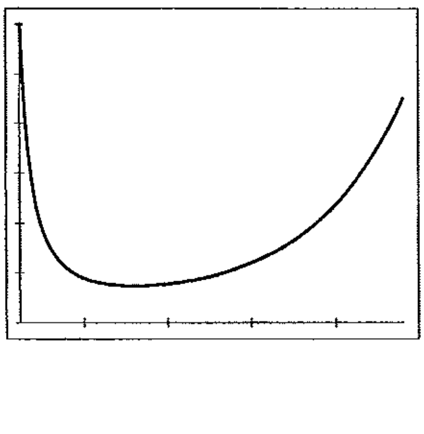
$\frac{d}{dS}\left(E_0 S^a e^{-Sc}\right) = E_0 S^a e^{-Sc}\left(a/S - c\right)$. This derivative is zero for $S = c/a$ that is S - coordinate of the fixed point $(a/c, k/\lambda)$ of the system

$\frac{d}{dS}\left(F^{-k} e^{\lambda F}\right) = F^{-k} e^{\lambda F}\left(-k/F + \lambda\right)$.This derivative is zero for $F = \lambda/k$ that is F - coordinate of the fixed point $(a/c, k/\lambda)$ of the system.

It is easy to see that these two functions $Z(S)$ and $Z(F)$ have a local maximum and a local minimum in these points.

Graphics of these two functions look as the following:





The analysis of F and S - values where $Z(S) = Z(F)$ on these graphs implies that the points (S, F) constitute closed curves in the $S - F$ plane. There are no other fixed points except $(c/a, \lambda/k)$ in the $S - F$ plane. It implies that these closed curves must be closed periodic orbits.

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