MATEMATIK | Datum: 2011-08-16 | Tid: 8:30
GU, Chalmers | Hjälpmedel: Beta
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Tenta i matematisk modellering, MMG510, MVE160

1. **Linear systems.**

   Consider the following ODE:
   $\frac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t),\ \overrightarrow{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$ with $A = \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$.

   Find the evolution operator for this system. **(2p)**

   Find which type has the stationary point at the origin and give a possibly exact sketch of the phase portrait. **(2p)**

   Eigenvectors and eigenvalues of the matrix $\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$, are: $\left\{\begin{bmatrix} -\sqrt{3}-1 \\ 1 \end{bmatrix}\right\} \leftrightarrow \lambda_1 = \sqrt{3}$, $\left\{\begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}\right\} \leftrightarrow \lambda_2 = -\sqrt{3}$

   The change of variables $\overrightarrow{r} = \begin{bmatrix} -\sqrt{3}-1 & \sqrt{3}-1 \\ 1 & 1 \end{bmatrix}\overrightarrow{y}$ reduces the system to two independent equations:$y_1' = \sqrt{3}y_1$ and $y_2' = -\sqrt{3}y_2$.

   Sylvesters method uses matrices $Q_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2}$ and $Q_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1}$ Such that $A = Q_1 + Q_2$ and the evolution operator $\exp(At) = \exp(\lambda_1 t)Q_1 + \exp(\lambda_2 t)Q_2$

   $Q_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \frac{1}{(\sqrt{3})-(-\sqrt{3})}\left(\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} - (-\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left(\frac{1}{2\sqrt{3}}\right)\begin{bmatrix} \sqrt{3}+1 & -2 \\ -1 & \sqrt{3}-1 \end{bmatrix}$

   $Q_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} = \frac{1}{(-\sqrt{3})-(\sqrt{3})}\left(\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} - (\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) == \left(\frac{1}{2\sqrt{3}}\right)\begin{bmatrix} \sqrt{3}-1 & 2 \\ 1 & \sqrt{3}+1 \end{bmatrix}$

   The stationary point is a saddel point. Trajectories ar hyperbolas that tend to lines through the origin parallel to eigenvectors.

2. **Ljapunovs functions and stability of fixed points.**

   Consider the system of equations: $\begin{cases} x' = -x + 2xy\sin(y) \\ y' = -\cos(x)y \end{cases}$

   Investigate stability of the fixed point in the origin. **(4p)**

   We try if function $V(x,y) = x^2 + y^2$ is a Lyapunovs funktion in some neghbourhood of origin.
   $V' = 2x(-x + 2xy\sin(y)) + 2y(-\cos(x)y) = -2x^2 - 2y^2 + 4x^2 y\sin(y) + 2y^2(1-\cos(y)) = -2(x^2+y^2) + 2\left(x^4+y^4\right) + x^2 O(y^3) + 2y^2 O(y^2)$

   where $O(z)/z < const$ when $z \to 0$. It shows that in a small neighbourhood of the origin all terms in V' are dominated by $-2(x^2+y^2)$ and that V is a strong Ljapunovs function and th eorigin is an asymptotically stable point for the system.

3. **Periodic solutions to ODE**

   Show that the system of equations

   $\begin{cases} x' = x - 2y - x\left(2x^2+y^2\right) has at least one periodic solution. **(4p)**

We consider the equation for the function $\varphi(x,y) = 2x^2+y^2$ by myltiplying the first equation by $2x$ and the second equaiton by y.

$\begin{cases} 2xx' = 2xx - 4xy - 2x^2\left(2x^2+y^2\right) \\ yy' = 4xy + y^2 - y^2\left(2x^2+y^2\right) \end{cases} \Longrightarrow 0.5\varphi' = 0.5\left(2x^2+y^2\right)' = \left(2x^2+y^2\right) - \left(2x^2+y^2\right)^2 =$
$\varphi(x,y)(1-\varphi(x,y))$

We observe that for (x,y) on the ellipse ring $2x^2+y^2 = 0.5$ we get $\varphi' > 0$ and on the ellipse $2x^2+y^2 = 2$ we get $\varphi' < 0$. It implies that the elliptic ring $0.5 < 2x^2+y^2 < 2$ is the invariant set for the system. On the other hand on the ellipse $2x^2+y^2 = 1$ where $\varphi' = 0$ velocities are not zero and system has no stationary points in the ring $0.5 < 2x^2+y^2 < 2$.Poincare'-Bendixson theorem implies that is must be a periodic solution inside this ring.

4. **Hopf bifurcation.**

Explain the notion Hopf bifurcation.

Show that the system $\begin{cases} x' = y \\ y' = -x + \mu y - y^3 \end{cases}$

has a Hopf bifurcation at $\mu = 0$. **(4p)**

The linearizaton of the equation around the origin is $\begin{cases} x' = y \\ y' = -x + \mu y \end{cases}$

The matrix $\begin{matrix} 0 & 1 \\ -1 & \mu \end{matrix}$, has eigenvalues: $\lambda_1 = \frac{1}{2}\mu - \frac{1}{2}i\sqrt{4-\mu^2}$; $\lambda_2 = \frac{1}{2}\mu + \frac{1}{2}i\sqrt{4-\mu^2}$.

The sustem stable focus for $\mu < 0$ and unstable focus for $\mu > 0$. $\frac{d\,\mathrm{Re}\,\lambda_i(\mu)}{d\mu} > 0$.

We check stability of the stationary point in the origin for $\mu = 0$, by computing $(x^2+y^2)'$:

$0.5(x^2+y^2)' = -y^4 \leq 0$. It implies that (x^2+y^2) is a weak Lyapunovs function.

On the other hand system has no trajectories imbedded in the line $y = 0$, because $y' = -x$ on this line and trajectories cross it everywhere exept the origin. It implies that the origin is even asymptotically stable stationary point. It implies that the system has a Hopf bifurcatiuon for $\mu = 0$. For $\mu < 0$ system has stable focus in the origin. When μ changes from negative to positive valjue it appears a limit cycle around the origin and the stationary point becomes unstable.

5. **Chemical reactions by Gillespies method**

Consider the following reactions: $X + P \underset{c_2}{\overset{c_1}{\rightleftarrows}} W, \qquad Z + Z \underset{c_4}{\overset{c_3}{\rightleftarrows}} P$ where $c_i\,dt$ is the probability that during time dt the reaction with index i will take place $i = 1,2,3,4$.

a) Write down differential equations for the number of particles for these reactions. **(2p)**

$X' = -c_1PX + c_2W$

$P' = -c_1PX + c_3\frac{1}{2}Z(Z-1) - c_4P$

$W' = c_1PX - c_2W$

$Z' = -c_3Z(Z-1) + c_42P$

b) Give formulas for the algorithm that models these reactions stochastically by Gillespies method. **(2p)**

$P(\tau,\mu)d\tau$ is the probability that the reaction of type μ will take place during the time interval $d\tau$ after the time τ when no reactions were observed.

$P(\tau,\mu) = P_0(\tau)h_\mu c_\mu d\tau$.

$h_\mu c_\mu d\tau$ is the probability that only the reaction μ will be observed during the time $d\tau$.

h_μ is the number of combinations of particles necessary for the reaction μ. For reaction 1 in the example: $h_1 = XP$, for reaction 2: $h_2 = W$, for reaction 3: $h_3 = \frac{1}{2}Z(Z-1)$, for reaction 4: $h_4 = P$.

For $P_0(\tau) = exp(-a\tau)$ with $a = \sum_{\mu=1}^{4} h_\mu c_\mu$.

Algorith to model reactions:

0) initialize variables X, Z, W, P for time $t = 0$.

1) Compute h_i, a for actual values of variables.

2) Generate two random numbers r and p uniformly distributed over the interval $(0, 1)$.

Random time τ before the next reaction is $\tau = 1/a \ln(1/r)$.

Choose the next reaction μ so that $\sum_{i=1}^{\mu-1} h_i c_i \leq p\, a \leq \sum_{i=1}^{\mu} h_i c_i$.

3) Add τ to the time variable t.Change the numbers of particles after the chosen reaction:

$\mu = 1 \quad \rightarrow X = X - 1,\ P = P - 1,\ W = W + 1.$

$\mu = 2 \quad \rightarrow X = X + 1,\ P = P + 1,\ W = W - 1.$

$\mu = 3 \quad \rightarrow Z = Z - 2,\ P = P + 1.$

$\mu = 4 \quad \rightarrow Z = Z + 2,\ P = P - 1.$

3) If time is larger then the maximal time we are interested in - finish computation, otherwise go to the step 1.

Max. 20 points;

For GU: **VG:** 15 points; **G:** 10 points. For Chalmers: **5:** 17 points; **4:** 14 points; **3**: 10 points; Total points for the course will be an average of points for the project (60%) and for this exam together with bonus points for home assignments(40%).