1. Linear systems.

Consider the following ODE:
\[
\frac{d\vec{r}(t)}{dt} = A\vec{r}(t), \quad \vec{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}
\text{ with } A = \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix},
\]
Find the evolution operator for this system. \hfill (2p)

Find which type has the stationary point at the origin and give a possibly exact sketch of the phase portrait. \hfill (2p)

**Solution**

Eigenvectors and eigenvalues of the matrix \[
\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}
\] are: \[
\left\{ \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_1 = 1 - 2\sqrt{2} < 0,
\left\{ \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 2\sqrt{2} + 1 > 0
\]

The change of variables \[
\vec{r} \rightarrow \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1 \\ 1 \end{bmatrix} \vec{y}
\]
reduces the system to two independent equations: \[y'_1 = (1 - 2\sqrt{2}) y_1 \text{ and } y'_2 = (1 + 2\sqrt{2}) y_2.\]

The solution can be expressed in the form:
\[
\left( \begin{array}{c} r_1(t) \\ r_2(t) \end{array} \right) = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{t \left( 1 + 2\sqrt{2} \right)}
\]
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{t \left( 1 - 2\sqrt{2} \right)}
\]

It is in fact the evolution operator:
\[
\varphi_t(\vec{r}(0)) = M^{-1} \exp(t D) M = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \exp(t (1-2\sqrt{2})) & 0 \\ 0 & \exp(t (1+2\sqrt{2})) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \vec{r}(0)
\]
\[
\varphi_t(\vec{r}(0)) = e^{t \left( 2\sqrt{2} \right)} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \vec{r}(0) + e^{t \left( -2\sqrt{2} \right)} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \vec{r}(0)
\]

An alternative solution uses the method by Sylvester. One can also use a simpler approach by Sylvester. Define matrices \( Q_1 \) and \( Q_2 \):
\[
Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{1}{(1-2\sqrt{2})-(1+2\sqrt{2})} \left( \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2\sqrt{2} \\ 0 & 1+2\sqrt{2} \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{4} \sqrt{2} & \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} \end{bmatrix}
\]
\[
Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{1}{(1+2\sqrt{2})-(1-2\sqrt{2})} \left( \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2\sqrt{2} \\ 0 & (1-2\sqrt{2}) \end{bmatrix} \right) = \begin{bmatrix} -\frac{1}{4} \sqrt{2} & -\frac{1}{2} \sqrt{2} \\ -\frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} \end{bmatrix}
\]
with properties: $Q_1Q_2 = 0$; $Q_1^2 = Q_1$; $Q_2^2 = Q_2$;

$A = \lambda_1Q_1 + \lambda_2Q_2$;

$e^{tA} = \sum_k \frac{A^k t^k}{k!} = \sum_k (\lambda_1Q_1+\lambda_2Q_2)\frac{t^k}{k!} = \sum_k (\lambda_1)\frac{t^k}{k!}Q_1 + \sum_k (\lambda_2)\frac{t^k}{k!}Q_2 = e^{t(1-2\sqrt{2})}Q_1 + e^{t(1+2\sqrt{2})}Q_2$.

The stationary point at the origin is a saddle point and trajectories are hyperbolas with asymptotic lines parallel to eigenvectors $\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Trajectories tend to the line parallel to $\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ when time goes to $+\infty$.

2. Ljapunovs functions and stability of stationary points.

Consider the system of equations:

\[
\begin{align*}
x' &= -x + 2xy^2 \\
y' &= -(1-x^2)y^3
\end{align*}
\]

Investigate stability of the fixed point in the origin.

**Solution.** $V(x, y) = x^2 + y^2$ is positive definite. We check the sign of $V' = 2x (-x + 2xy^2) - 2y ((1-x^2)y^3) = -2x^2 (1-2y^2) - 2(1-x^2)y^4$.

Therefore $V' \leq 0$ for $(x, y) \neq (0, 0)$, $|x| < 1$, and $|y| < \sqrt{1/2}$ and $V$ is a strong Ljapunov function in the rectangle: $|x| < 1$, $|y| < \sqrt{1/2}$.

It implies that the origin is an asymptotically stable fixed point.

3. Periodic solutions to ODE.

Use Poincare - Bendixsons theory to show that the system of equations

\[
\begin{align*}
x' &= \sin(y) + x(y^2 + 1) \\
y' &= (x-1)^2 + x^2y
\end{align*}
\]

does not have periodic solutions.

**Solution.** For right hand side $\overrightarrow{f}$ we have $\text{div}(\overrightarrow{f}) = (y^2 + 1) + x^2 > 0$. It implies that the system cannot have a periodic solution.


Explain the notion Hopf bifurcation.

Show that the system

\[
\begin{align*}
x' &= \mu x + y - x^3 \cos(x) \\
y' &= -x + \mu^2 y
\end{align*}
\]

has a Hopf bifurcation at $\mu = 0$.

**Solution.** The linearization around the origine gives the matrix:

\[
\begin{bmatrix}
\mu & 1 \\
-1 & \mu^2
\end{bmatrix}
\]

with eigenvalues: $\lambda_{1,2} (\mu) = \frac{1}{2} \mu + \frac{1}{2} \mu^2 \pm \frac{1}{2} \sqrt{\mu^2 - 2\mu^3 + \mu^4 - 4}$. It implies that $\lambda_{1,2} (0) = \pm i$, $\text{Re} \lambda_{1,2} (0) = 0$ and $\frac{d^2}{d\mu^2} (\text{Re} \lambda_{1,2} (\mu)) \bigg|_{\mu=0} = \frac{1}{2} + \mu |_{\mu=0} > 0$.

Consider the system for $\mu = 0$:

\[
\begin{align*}
x' &= y - x^3 \cos(x) \\
y' &= -x
\end{align*}
\]

The function $V(x, y) = x^2 + y^2$ has the derivative along trajectories: $V'(x, y) = 2x (y - x^3 \cos(x)) + 2y (-x) = -2x^4 \cos(x) \leq 0$ for $|x| < \pi/2$ and $x \neq 0$.

Therefore $V$ - is a weak Ljapunov function. On the other hand for $x = 0$ we have $x' = y \neq 0$ for $y \neq 0$. It means that the line $x = 0$ ($y$ - axis) does not include whole trajectories except the
origin. All trajectories coming to the y-axis outside the origin leave it immediately. It implies that the origin is an asymptotically stable fixpoint. Therefore Hopf bifurcation takes place at $\mu = 0$. When $\mu$ changes from negative values to positive values the stable focus becomes unstable focus surrounded by a limit cycle, that increases with $\mu$.

5. Chemical reactions by Gillespies method

Consider the following reactions:

\[ X + Z \xrightarrow{c_1} W, \quad W + Z \xrightarrow{c_2} P \]

where $c_i dt$ is the probability that during time $dt$ the reaction with index $i$ will take place $i = 1, 2, 3, 4$.

a) Write down differential equations for the number of particles for these reactions. (2p)

b) Give formulas for the algorithm that shell model these reactions stochastically by Gillespies method. (2p)

Solution

Equations for the numbers of particles are:

\[
\begin{align*}
X' &= -c_1 XZ + c_2 W \\
Z' &= -c_1 XZ + c_2 W - c_3 WZ + c_4 P \\
W' &= c_1 XZ - c_2 W - c_3 WZ + c_4 P \\
P' &= -c_4 P + c_3 WZ
\end{align*}
\]

b) Gillespies method.

$P(\tau, \mu)d\tau$ is the probability that the reaction of type $\mu$ will take place during the time interval $d\tau$ after the time $\tau$ when no reactions were observed.

$P(\tau, \mu) = P_0(\tau) h_\mu c_\mu d\tau$.

Here $P_0(\tau)$ is the probability that no reactions will be observed during time $\tau$.

$h_\mu c_\mu d\tau$ is the probability that only the reaction $\mu$ will be observed during the time $d\tau$.

$h_\mu$ is the number of combinations of particles necessary for the reaction $\mu$. For reaction 1 in the example $h_1 = X \cdot Z$, for reaction 2 $h_2 = W$, for reaction 3 $h_3 = WZ$, for reaction 4 $h_4 = P$.

For $P_0(\tau) = \exp(-a \tau)$ with $a = \sum_{i=1}^{4} h_i c_i$.

Algorithm to model reactions:

0) initialize variables $X$, $Z$, $W$, $P$ for time $t = 0$.
1) Compute $h_i$, $a$ for actual values of variables.
2) Generate two random numbers $r$ and $p$ uniformly distributed over the interval $(0, 1)$.

Random time $\tau$ before the next reaction is $\tau = 1/a \ln(1/r)$.

Choose the next reaction $\mu$ so that $\sum_{i=1}^{\mu-1} h_i c_i \leq p a \leq \sum_{i=1}^{\mu} h_i c_i$.

3) Add $\tau$ to the time variable $t$. Change the numbers of particles after the chosen reaction:

$\mu = 1 \quad \rightarrow \quad X = X - 1, \quad Z = Z - 1, \quad W = W + 1$.

$\mu = 2 \quad \rightarrow \quad X = X + 1, \quad Z = Z + 1, \quad W = W - 1$.

$\mu = 3 \quad \rightarrow \quad P = P + 1, \quad W = W - 1, \quad Z = Z - 1$.

$\mu = 4 \quad \rightarrow \quad P = P - 1, \quad W = W + 1, \quad Z = Z + 1$.
3) If time is larger then the maximal time we are interested in - finish computation, otherwise go to the step 1.

Max. 20 points;

For GU: VG: 15 points; G: 10 points. For Chalmers: 5: 17 points; 4: 14 points; 3: 10 points;
Total points for the course will be an average of points for the project (60%) and for this exam (40%).