OBS! You do not need to solve problems for which you have got bonus points.

1. Lyapunov's functions and stability of stationary points.
   a) Formulate the criterion for asymptotic stability of a stationary point of an ODE using only a weak Lyapunov function. (2p)

   b) Consider the system of equations:
      \[ \begin{align*}
      x' &= -y/3 - x(3x^2 + y^2) \\
      y' &= x - y(3x^2 + y^2)
      \end{align*} \]

      Find a strong Lyapunov function \( V(x, y) \) and show that the stationary point in the origin is asymptotically stable. (2p)

      Hint. Use \( V(x, y) \) in the form \( V(x, y) = ax^2 + y^2 \) and choose parameter \( a \) so that \( V(x, y) \) will be a strong Lyapunov function.

2. Periodical solutions to ODE.
   Formulate the Poincaré-Bendixson theorem. Use Poincare - Bendixsons theorem to show that the system of equations
   \[ \begin{align*}
   x' &= -4y + x(1 - x^2 - y^2); \\
   y' &= 4x + y(1 - x^2 - y^2)
   \end{align*} \]
   has at least one periodical solution in some ring: \( r^2 < x^2 + y^2 < R^2 \).

   Hint. Use polar coordinates and write down an equation for \( r \). (4p)

3. Asymptotic methods for ODE.
   a) Consider the idea of the direct asymptotic expansion in ODE. Which terms are called by secular terms in an expansion? Consider the equation: \( u'' + u = \epsilon(1 - u^2)u' \). Find terms of order zero and one in a direct asymptotic expansion of solutions to the equation for small \( \epsilon \). (2p)

   b) Describe the idea of the averaging method to get an expansion without secular terms and make first steps of the method for the given equation in part a). (2p)

   You do not need to give a complete solution, just describe the ideas and formulate the equations for main steps of the method.

   Useful trigonometric formulas are at the backside.

4. Chemical reactions by Gillespie’s method
   \[ \begin{align*}
   &\begin{alignat*}{3}
   &c_1 &\quad &\quad &c_2 &\quad &\quad &c_3 \\
   &X + Z \quad \rightarrow &\quad &W, &\quad &W &\quad \rightarrow &\quad P
   \end{alignat*}
   \end{align*} \]

   Consider the following reactions: \( X + Z \rightarrow W, \quad W \rightarrow P \) where \( c_i \, dt \) is the probability that during time \( dt \) the reaction with index \( i \) will take place \( i = 1, 2, 3 \).

   a) Write down differential equations for the number of particles for these reactions. (2p)

   b) Give formulas for the algorithm that shell model these reactions stochastically by Gillespie’s method. (2p)

   Max. 16 points;

For GU: VG: 13 points; G: 8 points;
For Chalmers: 5: 14 points; 4: 11 points; 3: 8 points;

Turn !!