April 6, 2020

Ordinary differential equations and mathematical modelling MVE162/MMG511.

1 Prerequisite knowledge for the course.

This relatively difficult course uses the whole scope of linear algebra and analysis that Chalmers students from Technical mathematics group and GU students from the group in mathematics learned during the first year. Students with different backgrounds might lack some of this material. Before starting learning this course it is good to check notions and theorems that are supposed to be known during teaching this course. If you miss some of them, check Appendix 1 and Appendix 2 in the course book by Logemann and Ryan, where all necessary mathematical background is discussed in detail.

Some internatinal students might also need to learn Matlab or use other programming tools to make computations in obligatory modeling projects.

Notions from linear algebra and analysis:

Vector space, normed vector space, norm of a matrix. Eigenvectors and eigenvalues of a matrix. Matrix diagonalization.

Cauchy sequence. Complete vector space (Banach space). Open, closed and compact sets in $\mathbb{R}^n$. Continuous functions and their properties on compact sets. Uniform convergence in the space of continuous functions.

Results from analysis:

Space $C(I)$ of continuous functions on a compact I is a complete vector space (Banach space). Example A.14, p. 272.

Bolzano-Weierstrass theorem. Theorem A.16, p. 273.

Weierstrass criterion for uniform convergence of functional series. Corollary A.23 , p. 277.

2 Introduction. Initial value problem, existence and uniqueness of solutions.

The main subject of the course is systems of differential equations in the form

$$x'(t) = f(t, x(t)) \tag{1}$$

classification and qualitative properties of their solutions. Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function regular enough with respect to time variable t and space variable x. J is an interval, G is an open subset of $\mathbb{R}^n$. Equations where the function f is independent of t are called autonomous:

$$x'(t) = f(x(t))$$

Finding a function $x(t) : L \to \mathbb{R}^n$ satisfying the equation (1) on the interval $L \subset J$ together with the initial condition

$$x(\tau) = \xi \tag{2}$$

for $\tau \in L$ is called the **initial value problem (I.V.P.)**.
The curves $x(t)$ in G have the property that they are tangent to the vector field $f(t, x(t)) \in \mathbb{R}^n$ at each time t and point $x(t) \in G$.
One can reformulate the I.V.P. (1),(2) in the form of the integral equation

$$x(t) = \xi + \int_\tau^t f(\sigma, x(\sigma))\, d\sigma \tag{3}$$

Continuous solutions to the integral equation (3) can be interpreted as generalized solutions to (1),(2) in the case when $f(t, x)$ is only piecewise continuous with respect to t and therefore the integral in (3) does not have derivative in some isolated points. If f is continuous, then these two formulations are equivalent by the Newton-Leibnitz theorem.

More general notions of solutions can be introduced in the case when $f(t, x(t))$ is integrable in the sense of Lebesque, but we do not consider such generalised solutions in this course.

3 Classification of ordinary differential equations and the plan of the course.

1. Equations where the right hand side is independent of time:

$$\begin{aligned} x'(t) &= f(x(t)) \\ f &= f(x), x \in G, \end{aligned}$$

are called **autonomous** as we mentioned before. General differential equations are with $f = f(t, x)$ are called **non-autonomous.**

Autonomous equations have a nice graphical interpretation. One can consider and also draw a picture of the vector field $f : G \to \mathbb{R}^n$. For every point $\xi \in G$ this vector field gives according to the differential equation, the velocity of a possible solution curve $x(t)$ going through the point ξ.

All solutions to an autonomous differential equation have the property that corresponding curves are tangent curves to the vector field $f : G \to \mathbb{R}^n$.

One often calls autonomous differential equations **continuous dynamical systems**.

2. General (**non-autonomous**) linear systems of differential equations in the form

$$x'(t) = A(t)x(t), \quad x(t) \in \mathbb{R}^n, \quad t \in J$$

with a matrix $A(t)$, $A(t) : J \to \mathbb{R}^{n \times n}$ that is a continuous matrix valued function of time t on the interval J. A particular class of non-autonomous linear systems is the class of **periodic linear systems** with periodic matrix $A(t+p) = A(t)$ with some period p.

3. We will also consider linear **non-homogeneous** systems of differential equations in the form

$$x'(t) = A(t)x(t) + g(t), \quad x(t), g(t) \in \mathbb{R}^n, \quad t \in J$$

with a given term $g(t)$ in the right hand side, both autonomous and non-autonomous.

4. Linear autonomous systems of differential equations in the form

$$x'(t) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

with a constant matrix A.

The plan for the course is: to consider after some introductory examples and then all these types of equations in the reverse order, from simpler to more complicated: linear autonomous, linear non-autonomous, linear periodic, nonlinear autonomous. At the very end of the course we will consider the existence of solutions in the most general non-linear non-autonomous case. Many ideas will be introduced and exploated first on the example of linear autonomous ODEs. Later these ideas will be developed further and applied in more complicated situations. This way of studying pursues two goals: to have more material for exercises and to introduce many general mathematical ideas in a more "user friendly" way.

The course is divided into two large qualitatively different parts:

A) one - devoted to linear equations and using and developing some advanced linear algebra, and

B) another one - devoted to non-linear equations and using reasoning based on relatively advanced analysis.

4 Main types of problems posed for systems of ODEs

I) **Existence and uniquness** of solutions to I.V.P. Finding **maximal interval** of existence of solutions to I.V.P.

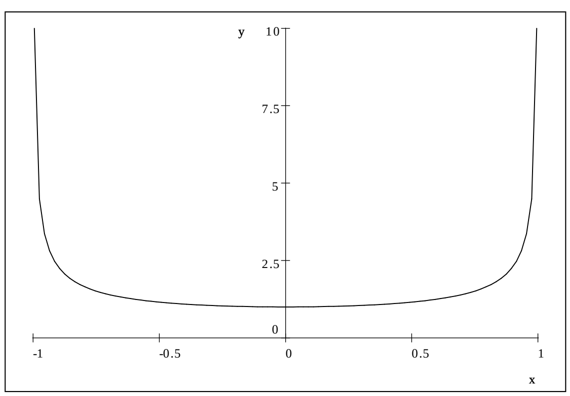
We give here two simple examples illustrating that solutions to a differential equation might exist not on any time interval (solutions can blow up - tend to infinity in finite time), and that solutions do not need to be unique (there can be two different solution curves going through one point (t, x))

Example of bounded maximal interval. (Ex. 1.2, p.14, L.R.) I.V.P.

$$x'(t) = t \cdot x^3; \quad x(0) = 1$$

. By separation of variables we arrive to a solution that exists only on a finite time interval $(-1, 1)$ called later **maximal interval** for these initial conditions.

$$\begin{aligned} \frac{dx}{x^3} &= t\,dt; \quad \int \frac{dx}{x^3} = \int t\,dt; \quad -\frac{1}{2x^2} = \frac{t^2}{2} + \frac{C}{2}; \quad -\frac{1}{x^2} = t^2 + C; \quad C = -1; \\ x &= \frac{1}{\sqrt{1-t^2}} \end{aligned}$$



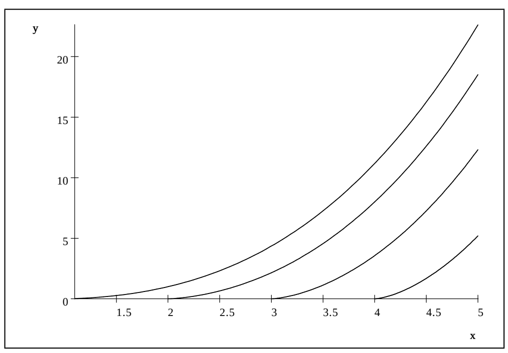
Point out that for another initial conditions the maximal interval can be different.

Example of non-uniqueness. (Ex.1.1, p.13, L.R.) I.V.P.

$$x'(t) = t \cdot x^{1/3}, t \in \mathbb{R}, \quad x(0) = 0.$$

Point out that the right hand side has infinite slope in x variable $\frac{d}{dx}(x^{1/3})$. We will say later, after giving corresponding definition, that this function is **not Lipschitz** with respect to x.

Constant solution $x(t) = 0$ exists. On the other hand for all $c > 0$ functions $x(t) = \frac{\left(t^2 - c^2\right)^{3/2}}{(3)^{3/2}}$, $t \geq c$ are also solutions to the equation. See the calculation below. By extending these solutions by zero to the left from $t = c$ we get a family of different solutions satisfying the same initial conditions $x(0) = 0$.



Calculation of solutions uses separation of variables.

$$\begin{aligned} \frac{dx}{dt} &= tx^{1/3}; \qquad \frac{dx}{x^{1/3}} = tdt \\ \int \frac{dx}{x^{1/3}} &= \int tdt; \qquad \frac{3}{2}x^{\frac{2}{3}} = \frac{1}{2}\left(t^2 - c^2\right) \\ x^{2/3} &= \frac{t^2 - c^2}{3}; \qquad x = \frac{\left(t^2 - c^2\right)^{3/2}}{(3)^{3/2}} \end{aligned}$$

Here c is arbitrary constant $c \leq t$. Check the solution:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(\frac{\left(t^2 - c^2\right)^{3/2}}{(3)^{3/2}}\right) = \frac{1}{3}t\sqrt{3t^2 - 3c^2} = tx^{1/3}$$

II) One can for particular classes of equations pose the problem of finding a reasonable analytical description of all solutions to the above equation. Such an expression is called **general solution**.

III) Find particular types of solutions: **equilibrium points** $\eta \in \mathbb{R}^n$ of autonomous systems (points where $f(\eta) = 0$), **periodic solutions**, such that after some period $T > 0$ the solution comes back to the same point: $x(t) = x(t + T)$ for any starting time t.

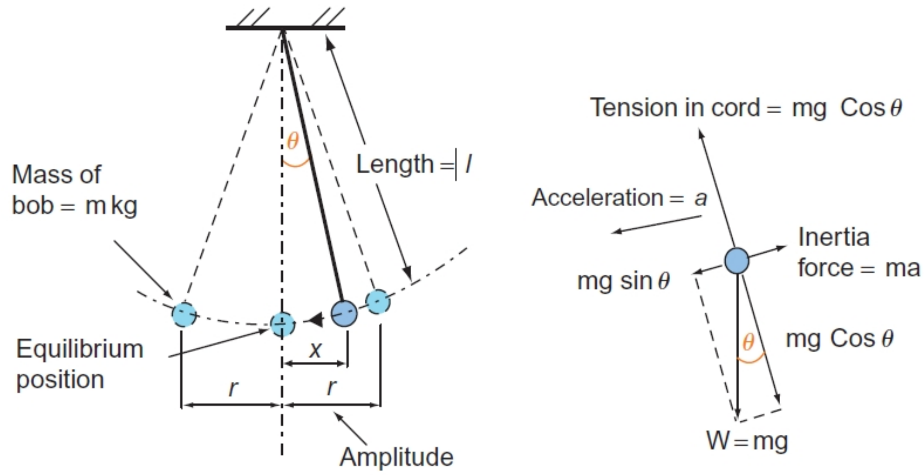
IV) Find how solutions $x(t)$ behave in the vicinity of an equilibrium point η with $t \to \infty$: it is interesting if they stay close to η starting arbitrarily close to it, or solutions can go go out of η with time $t \to \infty$ for some initial points ξ situated arbitrarily close to η (we will call these properties for **stability** or **instability** of the equilibrium point η).

V) Find a geometric description of the set of all trajectories of solutions to an equation. By trajectory we mean here the curve $x(t)$, that the solution goes along, during the time $t \in I$ when it exists. In the case of autonomous systems of dimension 2 we will calle call such a picture *phase portrait*.

VI) Describe geometric properties of so called **limit sets**, or "**attractors**" of a solution: such a set that the solution $x(t)$ "approaches" infinitely close when $t \to \infty$.

Examples

Pendulum is described by the Newton equation: $Force = m \cdot Acceleration$; $Acceleration = l \cdot \theta''(t)$,$Velocity = l \cdot \theta'(t)$.



(a) Simple pendulum (b) Forces acting on bob

$$ml\theta''(t) = -\gamma l\theta'(t) - mg\sin(\theta(t)) = 0$$

Both for theoretical analysis and for numerical solution one always rewrites the second order equation as a system of two equations for $x_1(t) = \theta(t)$ and $x_2(t) = \theta'(t)$:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{aligned}$$

We can rewrite it in general vector form as

$$x'(t) = f(x(t))$$

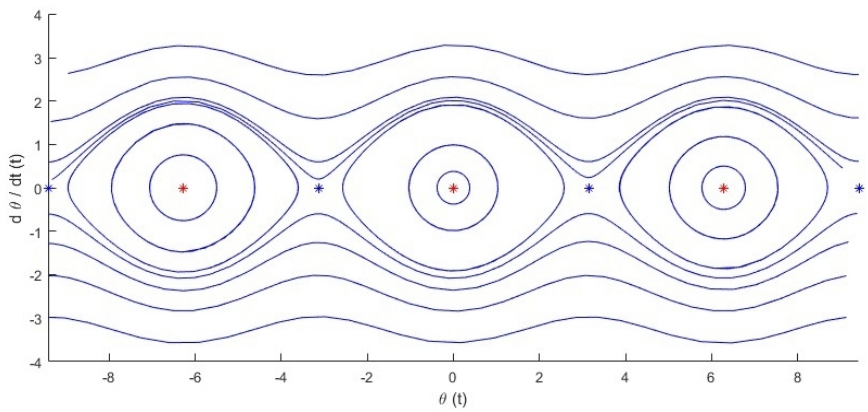
with

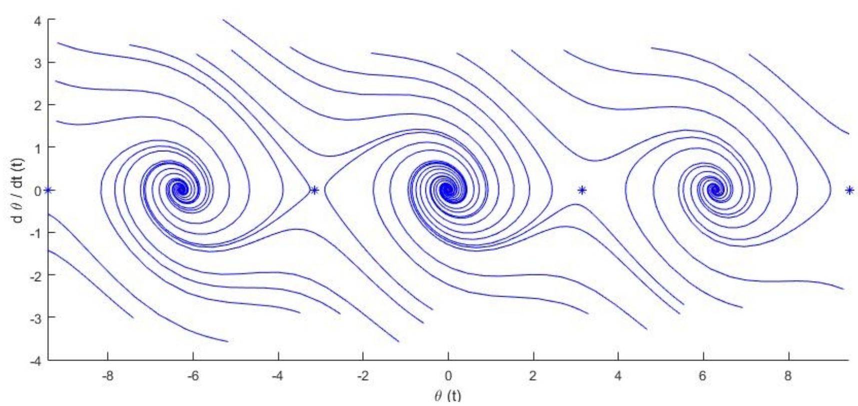
$$f(x) = \begin{bmatrix} x_2 \\ -\frac{\gamma}{m}x_2 - \frac{g}{l}\sin(x_1) \end{bmatrix}$$

This non-linear system of equations cannot be solved analytically. We show below results of numerical solutions of this system in a form of a **phase portrait of the system.**

Phase portrait.

The picture of trajectories - curves $(x_1(t), x_2(t))$ corresponding different solutions to the equation for the pendulum in the **phase plane** of variables x_1 and x_2 looks as the following. Such pictures are called **phase portrait of the system.** We will draw many of them in this course, in particular in modelling projects.





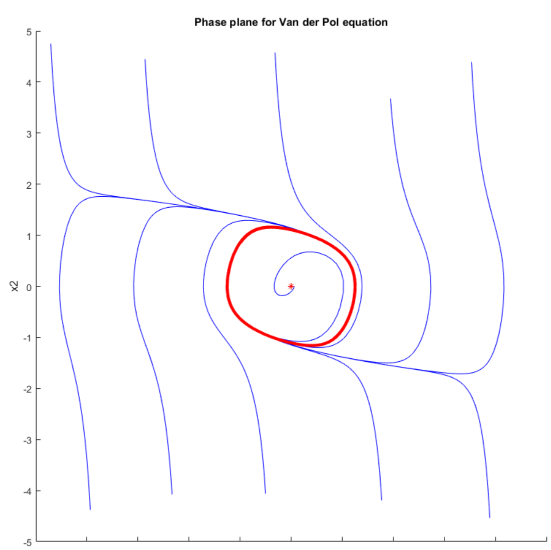
Points $\theta = 0 + 2\pi k$, $\theta' = 0$ and $\theta = \pi + 2\pi k$, $\theta' = 0$ on the first picture are equilibrium points. One can see closed orbits around equilibrium points

$\theta = 2\pi k$, $\theta' = 0$, corresponding to periodic solutions. Points $\theta = \pi + 2\pi k$, $\theta' = 0$ correspond to the upper position of the pendulum that is a non-stable equilibrium point. Higher up and down when the angular velocity is large enough we observe non-bounded solutions corresponding to rotation of the pendulum around the pivot. Orbits for the pendulum without friction can be described by a non-linear equation.

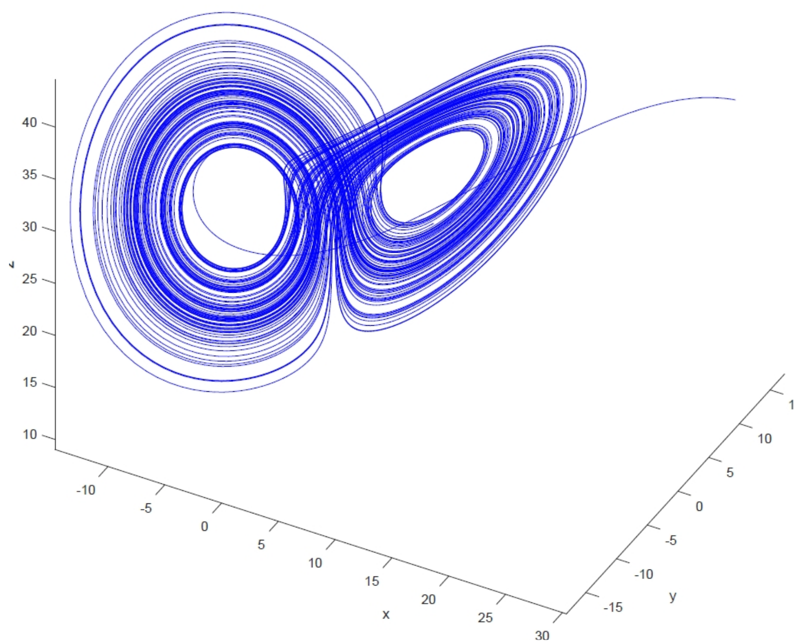
In the case with friction on the second picture one observes the same equilibrium points. But the phase portrait is completely different. Almost all trajectories tend to one of equilibrium points $\theta = 2\pi k$, $\theta' = 0$ when time goes to infinity. No closed orbits and no unbounded solutions are observed in this case.

Van der Pol equation . (Example 1.1.1. p. 2 in Logemann/Ryan)

$$\begin{aligned} x'(t) &= f(x(t)) \\ f(x) &= \begin{bmatrix} x_2 \\ -x_1 + x_2(1 - (x_2)^2) \end{bmatrix} \end{aligned}$$



We see that the equilibrium point in the origin is unstable but all



trajectories tend to a limit set or "attractor" that is a closed curve (depicted in red) that seems to be an orbit corresponding to a periodic solution. For two dimensional systems only stationary points and closed orbits and some chains of stationary points connected with orbits are possible as "attractors". In dimension 3 much more complicated attractors are possible with a classical example being the Lorenz equation.

Lorenz model for turbulence. Strange attractor.

$$
\begin{aligned}
x' &= -\sigma(x-y) \\
y' &= rx - y - xz \\
z' &= xy - bz
\end{aligned}
$$

A trajectory for $\sigma = 10$, $r = 28$, $b = 8/7$.

5 Linear autonomous systems of ODE

We will first consider general concepts in the course in the particular case for linear system of ODEs with constant matrix (linear autonomous systems).

$$x'(t) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R} \tag{4}$$

where A is a constant $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$.
In particular we will find solutions to initial value problem (I.V.P.) with initial condition

$$x(\tau) = \xi, \tag{5}$$

We point out that all general results about linear systems of ODE are also valid in the case of the complex vector space $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^n$ and complex matrix $A \in \mathbb{C}^{n \times n}$. Some of the results are formulated in a more elegant form in the complex case or might be valid only in complex form.
Several general questions that we formulated above will be addressed for this type of systems.
The final goal in this particular case will be to give a detailed analytical description of all solutions and to connect their qualitative properties with specific properties of the matrix A, its eigenvalues and eigenvectors together with more subtle spectral properties such as subspaces of generalised eigenvectors.

5.1 The space of solutions for general non-autonomous linear systems

We make first two simple observations that are valid even for general non-autonomous linear systems with a matrix $A(t)$ that is not constant but is a continuous fu

$$x'(t) = A(t)x(t), \quad x(t) \in \mathbb{R}^n, \quad t \in J \tag{6}$$

nction of time on the interval J.

Lemma. The sets of solutions $\mathcal{S}_{\text{hom}}$ to (4), and to (6) are linear vector spaces.

Proof. $\mathcal{S}_{\text{hom}}$ includes zero constant vector and is therefore not empty. By the linearity of the time derivative $x'(t)$ and of the matrix multiplication $A(t)x(t)$, for a pair of solutions $x(t)$ and $y(t)$ their sum $x(t) + y(t)$ and the product $Cx(t)$ with a constant C are also solutions to the same equation:

$$\begin{aligned} (x(t) + y(t))' &= A(t)(x(t) + y(t)) \\ (Cx(t))' &= A(t)(Cx(t)) \end{aligned}$$

■

5.2 Uniqueness of solutions to autonomous linear systems.

One shows the uniqueness of solutions to (4) by using a simple version of the Grönwall inequality that in general case will be considered later.

Grönwall inequality

Suppose that the I.V.P. (4),(5) for an autonomous linear system has a solution $x(t)$ on an interval I including τ. Consider the case when $\tau \le t$.

We can write an equivalent integral equation for $x(t)$ for $t \in I$, $\tau \leq t$

$$x(t) = \xi + \int_{\tau}^{t} Ax(\sigma)d\sigma \tag{7}$$

We calculate of the norm of the left and right sides in the integral equation (7) and use triangle inequality:

$$\|x(t)\| \leq \|\xi\| + \left\| \int_{\tau}^{t} Ax(\sigma)d\sigma \right\|$$

The triangle inequality for integrals:

$$\left\| \int_{\tau}^{t} x(\sigma)d\sigma \right\| \leq \int_{\tau}^{t} \|x(\sigma)\| \, d\sigma$$

and the definition of the matrix norm:

$$\|A\| = \sup_{\|x\| \neq 0} \left(\|Ax\| \,/\, \|x\| \right) = \sup_{\|x\|=1} \left(\|Ax\| \right)$$

imply that

$$\|x(t)\| \leq \|\xi\| + \int_{\tau}^{t} \|Ax(\sigma)\| \, d\sigma$$

and finally

$$\|x(t)\| \leq \|\xi\| + \int_{\tau}^{t} \|A\| \, \|x(\sigma)\| \, d\sigma$$

We will prove now that this integral inequality for $\|x(t)\|$ implies the famous **Grönwall inequality** giving an estimate for $\|x(t)\|$ in terms of the initial data $\|\xi\|$.

This is a standard argument that will be used within the course again later two more times for more complicated types of equations.

Introducing the notation $G(t) = \|\xi\| + \int_\tau^t \|A\| \|x(\sigma)\| \, d\sigma$ we conclude that $G(\tau) = \xi$, $\|x(t)\| \le G(t)$, and

$$G'(t) = \|A\| \|x(t)\| \le \|A\| G(t)$$

Multiplying the last inequality by the integrating factor $\exp(-\|A\| t)$ we arrive to

$$\begin{aligned} G'(t)\exp(-\|A\| t) - \|A\| \exp(-\|A\| t)G(t) &\le 0 \\ G'(t)\exp(-\|A\| t) + G(t)\left(\exp(-\|A\| t)\right)' &\le 0 \\ \left(G(t)\exp(-\|A\| t)\right)' &\le 0 \end{aligned}$$

Integrating the left and the right hand side from τ to t we get the inequality

$$\begin{aligned} G(t)\exp(-\|A\| (t)) &\le G(\tau)\exp(-\|A\| (\tau)) \\ G(t) &\le \|\xi\| \exp(\|A\| (t-\tau)) \end{aligned}$$

that implies the **Grönwall inequality** in this simple case:

$$\|x(t)\| \le \|\xi\| \exp(\|A\| (t-\tau)) \tag{8}$$

■**(Knowlege of this proof is required at the exam)**

Lemma. The solution to I.V.P. (4),(5) is unique.

Proof. Suppose that there are two solutions $x(t)$ and $y(t)$ to the I.V.P. (4),(5) on a time interval including τ and both are equal to ξ at the initial time $t = \tau$. Consider the vector valued function $z(t) = x(t) - y(t)$ and the case when $\tau \le t$. Then $z(t)$ is also a solution to the same equation (4) and satisfies the initial condition $z(\tau) = 0$.

The estimate (8) applied to $z(t)$ implies that $z(t) = 0$ and therefore the uniqueness of solution to I.V.P. (4),(5). The proof of the case $\tau \le t$ is similar.■

5.3 Exponent of a matrix

Two ideas are used to construct analytical solutions to (4) :

1) One is to find a possibly simple basis $\{v_1(t), ..., v_N(t)\}$ to the solution space.

2) Another one is based on an observation that the matrix exponent

$$e^{A(t-\tau)} \overset{def}{=} I+A(t-\tau)+\frac{1}{2}A^2(t-\tau)^2+...+\frac{1}{k!}A^k(t-\tau)^k ... = \sum_{k=0}^{\infty}\frac{1}{k!}A^k(t-\tau)^k$$

gives an expression of the the unique solution to the I.V.P. (1), (1a):

$$x(t) = e^{A(t-\tau)}\xi$$

One can derive this property of the matrix exponent by considering the integral equation (7) for $x(t)$

$$x(t) = \xi + \int_{\tau}^{t} Ax(\sigma)d\sigma$$

equivalent to the I.V.P. (4),(5). We can try to solve this integral equation by iterations:

$$\begin{aligned} x_{k+1}(t) &= \xi + \int_{\tau}^{t} Ax_k(\sigma)d\sigma \qquad (9) \\ x_0 &= \xi \\ x_k(t) &= \left[I + A(t-\tau) + \frac{1}{2}A^2(t-\tau)^2 + ... + \frac{1}{k!}A^k(t-\tau)^k\right]\xi \end{aligned}$$

Iterations $x_k(t)$ converge uniformly on any finite time interval as $k \to \infty$ and the limit gives the series for $\exp(At)$ formulated above times the initial data ξ.

The series for $\exp(At) = \sum_{k=0}^{\infty}\frac{1}{k!}A^k(t-\tau)^k$ converges uniformly on any finite time interval $[-T, T]$ including initial time point $\tau \in [-T, T]$ by the

Weierstrass criterion. Most of you studied it before. We will remind it's formulation here. It will be used several times in the course.

Weierstrass criterion. Corollary A.23, p. 277 in L.R.

Let X be a normed vector space, Y be a complete normed vector space (Banach space) $K \subset X$ be compact,$\{f_n(x)\}_{n=1}^{\infty}$, $x \in K$ be a sequence of continuous functions $f_n : K \to Y$ and let $\{m_n\}_{n=1}^{\infty}$ a real sequence such that $\|f_n(x)\| \leq m_n$ for all $x \in K$ and all $n \in \mathbb{N}$, where $\|...\|$ is the norm in Y. If $\sum_{n=1}^{\infty} m_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(x)$ is unifomly convergent on K.□

You studied this theorem in the case when $X = \mathbb{R}^N$, $Y = \mathbb{R}^M$. In our situation here K is a closed interval in $\mathbb{R}$ for example $[-T, T]$ in $\mathbb{R}$ and Y is a space of matrices $\mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$).

To prove that our series satisfies the Weierstrass criterion, we will apply the estimate for the norm of the product of two matrices: $\|AB\| \leq \|A\| \, \|B\|$. It implies that $\|A^2\| \leq \|A\| \, \|A\|$, $\|A^3\| \leq \|A\| \, \|A\| \, \|A\|$, ...and $\left\|A^k\right\| \leq \|A\|^k$.

Prove the inequality $\|AB\| \leq \|A\| \, \|B\|$ yourself!

Therefore the norm of each term in the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k (t-\tau)^k$ is estimated by a term from a convergent number series:

$$\begin{aligned} \left\| \frac{1}{k!} A^k (t-\tau)^k \right\| &\leq \frac{1}{k!} \left\|A^k\right\| |t-\tau|^k \leq \\ \frac{1}{k!} \|A\|^k |t-\tau|^k &\leq \frac{1}{k!} \|A\|^k (2T)^k \end{aligned}$$

for the exponential function $\exp(\|A\| (2T))$. We use here that $|t-\tau| \leq 2T$ for each $t \in [-T, T]$.

Application of the Weierstrass criterion to the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k (t-\tau)^k$ leads to the solution of the I.V.P. in the form

$$x(t) = e^{A(t-\tau)}\xi = \exp(A (t-\tau))\xi = \left(\sum_{k=0}^{\infty} \frac{1}{k!} (t-\tau)^k A^k \right) \xi$$

We make this conclusion by tending to the limit $k \to \infty$ in the integral equation (9) defining iterations because the expression under the integral in

(9) converges uniformly and therefore the limit of the integral is equal to the integral of this uniform limit. This solution is unique by the Lemma we proved before.

Corollary 2.9 in L.&R. The function $x(t) = \exp(A(t-\tau))\xi$ is the unique solution to the I.V.P. (4),(5).

This theoretical expression for unique solutions to (1) dispite of it's eleganscе has a huge disadvantage that the series $\exp(At) = \sum_{k=0}^{\infty} \frac{1}{k!} (t-\tau)^k A^k$ is not possible to calculate analytically in a simple way.

We will try instead to find a basis of the vector space $\mathcal{S}_{\text{hom}}$ of all solutions to (1).

5.4 The dimension of the space $\mathcal{S}_{\text{hom}}$ of solutions

Theorem. (Proposition 2.7, p.30, L.R. in the case of non-autonomous systems).

Let $b_1, ..., b_N$ be a basis in $\mathbb{R}^N$(or $\mathbb{C}^N$). Then the functions $y_j : \mathbb{R} \to \mathbb{R}^N$(or $\mathbb{C}^N$) defined as solutions to the I.V.P. (4),(5) with $y_j(\tau) = b_j$, $j = 1, ...N$, by $y_j(t) = \exp(A(t-\tau))b_j$, form a basis for the space $\mathcal{S}_{\text{hom}}$ of solutions to (4). The dimension of the vector space $\mathcal{S}_{\text{hom}}$ of solutions to (4) is equal to N - the dimension of the system (4).

Idea of the proof. This property is a consequence of the linearity of the system and the uniqueness of solutions to the system and is independent of detailed properties of the matrices $A(t)$ and A in (4) and (6).

Proof. Consider a linear combination of $y_j(t)$ equal to zero for some time $\sigma \in \mathbb{R}$: $l(\sigma) = \sum_{j=1}^{N} \alpha_j y_j(\sigma) = 0$. Observe that the trivial constant zero solution $0(t)$ coinsides with l at this time point.

But by the uniqueness of solutions to (4) it implies that $l(t)$ at arbitrary time must coinside with the trivial zero solution for all times and in particular at time $t = \tau$. Therefore $l(\tau) = \sum_{j=1}^{N} \alpha_j b_j = 0$ (point out that $y_j(\tau) = b_j$). It implies that all coefficients $\alpha_j = 0$ because $b_1, ..., b_N$ are linearly independent vectors in $\mathbb{R}^N$(or $\mathbb{C}^N$). It implies that $y_1(t), ..., y_N(t)$ are linearly independent for all $t \in \mathbb{R}$ by definition. Arbitrary initial data $x(\tau) = \xi$ in $\mathbb{R}^N$(or $\mathbb{C}^N$) can be represented as a linear combination of basis vectors $b_1, ..., b_N$: $\xi = \sum_{j=1}^N C_j b_j$. The construction of $y_1(t), ..., y_N(t)$ shows that an arbitrary solution to (4) can be represented as linear combination of $y_1(t), ..., y_N(t)$.

$$\begin{aligned} x(t) &= \exp(A(t-\tau))\xi = \exp(A(t-\tau))\sum_{j=1}^N C_j b_j = \\ &= \sum_{j=1}^N C_j \underbrace{[\exp(A(t-\tau))b_j]}_{=y_j(t)} = \sum_{j=1}^N C_j y_j(t) \end{aligned}$$

Therefore $\{y_1(t), ..., y_N(t)\}$ is the basis in the space of solutions $\mathcal{S}_{\text{hom}}$ and therefore $\mathcal{S}_{\text{hom}}$ has dimension N.$\blacksquare$**(Knowlege of this proof is required at the exam)**

By taking $\xi = e_1, ..., e_n$ we observe that each column in the matrix $\exp(A\,(t-\tau))$ is a solution to the equation (4). We have just shown in the theorem before that these columns are linearly independent and build a basis in the space of solutions.

Properties of the matrix exponent.

We collect in the following Lemma some (may be partially known) properties of the matrix exponent.

For a complex matrix M the notation M^* means transpose and complex conjugate matrix (called also Hermitian transpose)

Lemma (Lemma 2.10 , p. 34 in L.&R.) Let P and Q be matrices in $\mathbb{R}^{N\times N}$ or $\mathbb{C}^{N\times N}$

(1) For a diagonal matrix $P = diag(\lambda_1, \lambda_2, ..., \lambda_n)$

$$\exp(P) = diag(\exp(\lambda_1), ..., \exp(\lambda_n))$$

(2) $\exp(P^*) = (\exp(P))^*$

(3) for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\exp(At) = A\exp(At) = \exp(At)A$$

(4) If P and Q are two **commuting matrices** $PQ = QP$, then $\exp(P)Q = Q\exp(P)$ and

$$\exp(P+Q) = \exp(P)\exp(Q)$$

(5) $\exp(-P)\exp(P) = \exp(P)\exp(-P) = I$ or $\exp(-P) = (\exp(P))^{-1}$

Proof

Proofs of (1),(2) are left as exercises. We proof first (4) by direct calculation.

$$\begin{aligned}
(P+Q)^k &= \sum_{m=0}^{k} \binom{k}{m} P^m Q^{k-m} \quad (for \quad commuting \quad matrices) \\
e^{P+Q} &= \sum_{k=0}^{\infty} \frac{1}{k!}(P+Q)^k = \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \left(\frac{k!}{m!\,(k-m)!}\right) P^m Q^{k-m} \\
&= \sum_{k=0}^{\infty} \sum_{m+p=k} \frac{P^m}{m!}\frac{Q^p}{p!} = \left(\sum_{m=0}^{\infty} \frac{1}{m!}P^m\right)\left(\sum_{p=0}^{\infty} \frac{1}{p!}Q^p\right) = e^P e^Q
\end{aligned}$$

(3) Can be proved in three different ways.

It follows from the definition of $\exp(At)$ by elementwise differentiation of the corresponding uniformly converging series.

It follows also from the observation above that each column in $\exp(At)$ with index k is a solution to the system of equations $x' = Ax$ with initial data $x(0) = e_k$.

A straightforward proof can be given by the definition of derivative and using the relation (4). We use the formula $\exp(P+Q) = \exp(P)\exp(Q)$ for commuting matrices, the fact that At and As commute for any t and s and the Taylor formula applied to for $\exp(Ah) - I$ for small h:

$$\begin{aligned}\exp(A(t+h)) - \exp(At) &= (\exp(Ah) - I)\exp(At) = \\ &\left(Ah + O(h^2)\right)\exp(At)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}(\exp(At)) &= \lim_{h\to 0}\frac{(\exp(A(t+h)) - \exp(At))}{h} = \\ \lim_{h\to 0}\frac{(Ah + O(h^2))\exp(At)}{h} &= A\exp(At)\end{aligned}$$

■

5.5 Analytic solutions. Case when a basis of eigenvectors exists.

An idea that leads to an analytical solution is to find a basis $\{y_1(t), ..., y_N(t)\}$ to the solution space $\mathcal{S}_{\text{hom}}$ by finding a particular basis $\{v_1, ..., v_N\}$ in $\mathbb{C}^N$ or $\mathbb{R}^N$ such that the matrix exponent $\exp(At)$ acts on the elements of this basis in a particularly simple way, so that all $y_k(t) = \exp(A(t-\tau))v_k$ can be calculated explicitely. We will consider mainly the case $\tau = 0$ for autonomous systems.

The simplest example that illustrates this idea is given by eigenvectors to A. These are vectors $v \neq 0$ such that

$$Av = \lambda v$$

for some number λ. Numbers λ are called eigenvalues of A. Eigenvalues must be roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

because rewriting the definition of an eigenvector we arrive to a homogeneous system of linear equations with matrix $(A - \lambda I)$

$$(A - \lambda I)\, v = 0$$

with $v \neq 0$. Using the definition $Av = \lambda v$ for the eigenvalue and the eigenvector k times we conclude that $A^k v = \lambda^k v$. Substituting this formula into the expression $e^{At}v = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k v$ we conclude that

$$e^{At}v = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k v = e^{\lambda t} v.$$

Important idea.

Another more general idea leads to the same formula, but has an advantage that it can be applied in more complicated situations. We use here that the eigenvector v corresponding to the eigenvalue λ makes all powers $(A - \lambda I)^k\, v = 0$ except $k = 0$:

$$\begin{aligned} e^{At}v &= \exp(\lambda t I + (At - \lambda t I))\, v = \exp(\lambda t I) \exp((A - \lambda I)\, t)\, v = \qquad (10) \\ &= \left(e^{\lambda t} I\right) \sum_{k=0}^{\infty} \frac{1}{k!} t^k (A - \lambda I)^k\, v = e^{\lambda t} v. \end{aligned}$$

This observation leads to a simple conclusion that if the matrix A has N linearly independent eigenvectors $\{v_k\}$, then any solution to (4) with initial data $\xi = \sum_{k=1}^{N} C_k v_k$ can be expressed as a linear combination in the form

$$x(t) = \sum_{k=1}^{N} C_k \left(e^{\lambda_k t} v_k\right)$$

with vector functions $\left\{e^{\lambda_k t} v_k\right\}$ building a basis for the space of solutions to (4).

We point out that λ and v can be a complex eigenvalue and a complex eigenvector here. In the case when all these eigenhvalues are real, this basis will be real. In the case if a real matrix A has some complex eigenvalues, they appear as pairs of complex conjugate eigenvalues and corresponding eigenvectors, that still can be used to build a real basis for solutions. We will demonstrate it on a couple of examples later.

Example 1. Consider system $x' = Ax$ with matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix A has characteristic polynomial $p(\lambda) = \lambda^2 - 1$ and two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

Corresponding eigenvectors satisfy homogeneous systems $(A - \lambda_1)\, v_1 = 0$ with matrix $(A - \lambda_1 I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $(A - \lambda_1 I)\, v_2 = 0$ with matrix $(A - \lambda_2 I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and are linearly independent (in particular it follows from the fact that eigenvalues are different). Solutions $y_1(t) = e^t v_1$ and $y_2(t) = e^{-t} v_2$ are linearly independent. Arbitrary real solution to the system of ODEs has the form

$$x(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with arbitrary coefficients C_1 and C_2.Corresponding phase portrait will include particular solutions tending to infinity along the vector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, solutions tending to the origin along the vector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and other solutions filling the rest of the plain having orbits in the form of hyperbolas.

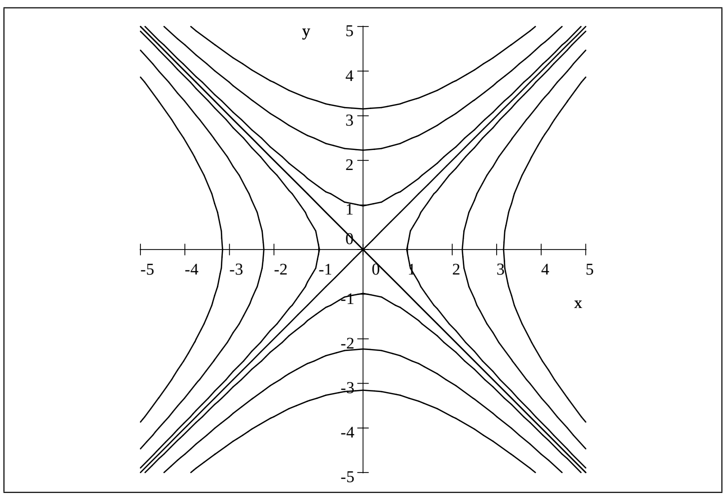
One can observe it by integrating the differential equation

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_1 \end{aligned}$$

$$\frac{dx_2}{dx_1} = \frac{x_1}{x_2}; \quad x_2 dx_2 = x_1 dx_1$$

with separable variables that follows from the system and concluding that

$$x_1^2 - x_2^2 = Const$$



Similar phase portraits will be observed in the arbitrary case when the 2×2 real non-degenerate matrix A has real eigenvalues with different signs but the picture will be rotated and might be less symmetric depending on the directions of the eigenvectors v_1 and v_2 (here they are orthogonal). One can still draw trajectories along eigenvectors and then sketch other trajectories according to the directions of trajectories along eigenvectors.

6 Generalised eigenvectors and eigenspaces.

It is easy to give examples of matrices that cannot be diagonalized. For linear autonomous systems with such matrices the expression of arbitrary solutions in terms of linearly independent eigenvectors is impossible because we just do not have N linearly independent ones.

Example 3.

$$\begin{cases} x_1' = -x_1 \\ x_2' = x_1 - x_2 \end{cases} \quad \text{or} \quad x'(t) = Ax \text{ with } A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \text{ the characteristic}$$

polynomial is $p(\lambda) = (\lambda + 1)^2$.

Matrix A has an eigenvalue $\lambda = -1$ with algebraic multiplicity $m(\lambda) = 2$.

There is only one eigenvector $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfying the equation

$(A - \lambda I)v = 0$.

$$(A - (-1)I) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The function $x(t) = e^{-t}v$ is a solution to the system. One likes to find a basis of solutions to the space $\mathcal{S}_{\text{hom}}$ of all solutions. We need another linearly independent solution for that. Observe that

$$x_1(t) = C_1 e^{-t}$$

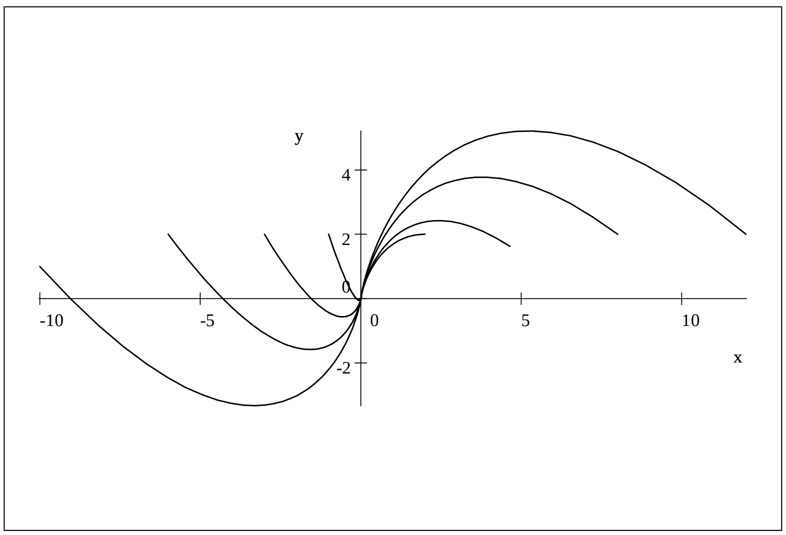
is the solution to the first equation, substitute it into the second equation and solve it explicitely with respect to $x_2(t)$:

$$\begin{aligned} x_2'(t) &= -x_2(t) + C_1 e^{-t} \\ e^t x_2'(t) + e^t x_2(t) &= C_1 \\ \left(e^t x_2(t)\right)' &= C_1 \\ e^t x_2(t) &= C_2 + C_1 t \\ x_2 &= C_2 e^{-t} + C_1 t\, e^{-t} \end{aligned}$$

Therefore the general solution to this particular system has the form

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} 1 \\ t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\ & C_1 e^{-t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= C_1 e^{-t} \left(v^{(1)} + tv \right) + C_2 e^{-t} v \end{aligned}$$

where $v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The phase portrait looks as:



In this particular example we could find an explicit solution using the fact that the matrix A is trianglular. This idea cannot be generalized to the arbitrary case but can be used for linear system with variable coefficients and triangular matrix.

We point out that the initial value for the derived solution $x(t) = C_1 e^{-t} \left(v^{(1)} + tv \right) + C_2 e^{-t} v$ is

$$x(0) = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = C_1 v^{(1)} + C_2 v.$$

Vector $v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is linearly independent of the eigenvector $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and applying the lemma before we conclude that $e^{-t}v$ and $e^{-t}\left(v^{(1)} + tv\right)$ are linearly independent for all $t \in R^N$ and build a basis for the space of solutions to the system.

Observe that $v^{(1)}$ has a remarkable property that $(A - \lambda I)\, v^{(1)} = v$ as $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and therefore $(A - \lambda I)^2\, v^{(1)} = 0$. Such vectors are called **generalised eigenvectors** to A corresponding to the eigenvalue λ.

We point out that the initial data in this explicit solution are represented as a linear combination of an eigenvector and a generalised eigenvector:

$$x(0) = C_1 v^{(1)} + C_2 v.$$

We observe that the general solution we have got could be derived by applying the same idea as in the formula (10) but to the generalised eigenvector $v^{(1)}$:

$$\begin{aligned}
\exp(At)v^{(1)} &= \exp\left(\lambda t I + (At - \lambda t I)\right) v^{(1)} = \exp\left(\lambda t I\right) \exp\left((A - \lambda I)\, t\right) v^{(1)} \\
e^{\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} t^k \left(A - \lambda I\right)^k v^{(1)} &= e^{\lambda t}\left(v^{(1)} + t\,(A - \lambda I)\, v^{(1)}\right) = \\
&= e^{\lambda t}\left(v^{(1)} + tv\right) \\
(A - \lambda I)^k\, v^{(1)} &= 0, \quad k \geq 2
\end{aligned}$$

This reasoning again gives the second basis vector in the space of solutions, that we have got before by the trick with separation of variables, and gives a clue what might be a general way to explicit solution to the linear system with arbitrary constant matrix.■

Definition of generalised eigenvectors.

A non-zero vector $z \in \mathbb{C}^N$(or $\mathbb{R}^N$) is called a generalised eigenvector to the matrix $A \in \mathbb{C}^{N\times N}$ corresponding to the eigenvalue λ with the algebraic multiplicity $m(\lambda)$ if $(A - \lambda I)^{m(\lambda)}\, z = 0$.

If $(A-\lambda I)^r z = 0$ and $(A-\lambda I)^{r-1} z \neq 0$ for some $0 < r < m(\lambda)$ we say that z is a generalised eigenvector of **rank (or height)** r to the matrix A.$\square$

An eigenvector u is a generalised eigenvector of rank 1 because

$$(A-\lambda I)\, u = 0.$$

The set $\ker\left((A-\lambda I)^{m(\lambda)}\right)$ (kernel or nullspace in Swedish) of all generalized eigenvectors of an eigenvalue λ is denoted by $E(\lambda)$ in the course book. $E(\lambda)$ is a subspace in $\mathbb{C}^N$.

Proposition on A - invariance of $E(\lambda)$.

$E(\lambda)$ is A- invariant, namely if $z \in E(\lambda)$, then $Az \in E(\lambda)$.

Proof. We check it by taking $z \in E(\lambda)$ such that $(A-\lambda I)^{m(\lambda)} z = 0$ and calculating $(A-\lambda I)^{m(\lambda)} Az = A\,(A-\lambda I)^{m(\lambda)} z = 0$, the last equality is valid because A and $(A-\lambda I)^{m(\lambda)}$ commute.$\blacksquare$

Proposition on $\exp(At)$ - invariance of $E(\lambda)$.

$E(\lambda)$ is invariant under the action of $\exp(At)$, namely if $z \in E(\lambda)$, then $\exp(At)z \in E(\lambda)$.

Proof. Consider the expression for the $\exp(At)z$ as a series

$$\left.\exp(At)z = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k z = \lim_{m\to\infty} \sum_{k=0}^{m} \underbrace{\frac{1}{k!} t^k A^k z}_{\in E(\lambda)} \right\} \in E(\lambda)$$

All terms $A^k z$ in the sum belong to $E(\lambda)$. One can see it by repeating the argument in the previous proposition.

The expression for $\exp(At)z$ is therefore a limit of linear combinations of elements from the finite dimensional generalized eigenspace $E(\lambda)$ that is a closed and complete set. Therefore $\exp(At)z$ must belong to $E(\lambda)$.$\blacksquare$

A remarkable property of generalised eigenvectors z is that the series for the matrix exponent $\exp(At)$ applied to z can be expressed in such a way that it would include only a finite number of terms and can be calculated analytically.

Theorem (2.11, Part 1), p. 35 in the course book) Let $A \in \mathbb{C}^{N\times N}$.

For an eigenvalue $\lambda \in \sigma(A)$ with algebraic multiplicity $m(\lambda)$ denote the subspace of its associated generalised eigenvectors by $E(\lambda) = \ker(A - \lambda I)^{m(\lambda)}$ and for $z \in E(\lambda)$ denote by $x_z(t) = \exp(At)z$ - the solution of I.V.P. with initial data $x_z(0) = z$. Then for $\lambda \in \sigma(A)$ and $z \in E(\lambda)$ a generalised eigenvector

$$\exp(At)\, z = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!} (A - \lambda I)^k z$$

Proof.

We show it by the following direct calculation:

$$\begin{aligned} x_z(t) &= \exp(At)\, z = \exp(t\lambda I)\exp((A-\lambda I)\, t)\; z = \\ e^{\lambda t}\sum_{k=0}^{\infty} \frac{t^k}{k!}(A-\lambda I)^k z &= e^{\lambda t}\sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A-\lambda I)^k z \end{aligned} \tag{11}$$

because powers $(A - \lambda I)^k z = 0$ - terminate on $z \in E(\lambda)$ for all $k \geq m(\lambda)$ by the definition of generalised eigenvectors.

We also use at the first step of calculations the property (4) from the Lemma about matrix exponents: $\exp(P + Q) = \exp(P)\exp(Q)$ for commuting matrices P and Q. ■

6.1 Analytic solutions. General case using a basis of generalized eigenvectors.

The next theorem gives a theoretical background for a method of constructing analytic solutions to (4) ($x'(t) = Ax(t)$), by representing arbitrary initial data $x(0) = \xi$ using a basis of generalised eigenvectors to A in $\mathbb{C}^N$. We are going to consider initial conditions for autonomous systems only at the point $\tau = 0$, because all other solutions are derived from such ones just by a shift in time, because the right hand side in the equation does not depend on time explicitely and if $x(t)$ is a solution, then $x(t+\tau)$ is also a solution.

Definition The sum $V_1 + V_2 + ... + V_s$ of subspaces V_1,V_2 ...V_s in a vector space is a set of vectors in the form $v_1 + v_2 + ... + v_s$ with vectors $v_j \in V_j$, $j = 1, ..., s$. $\square$

Definition Direct sum $V_1 \oplus V_2 \oplus ... \oplus V_s$ of subspaces V_1,V_2 ...V_s is a usual sum $V_1 + V_2 + ... + V_s$ of these subspaces with a *special additional property* that any vector in $v \in V_1 \oplus V_2 \oplus ... \oplus V_s$ is represented only in a **unique way** as a sum $v = v_1 + v_2 + ... + v_s$ of vectors $v_j \in V_j$, $j = 1, ..., s$. $\square$

It makes in this case any set of vectors $v_j \in V_j$, $j = 1, ..., s$ belonging to different V_j linearly independent.

Subspaces V_j, $j = 1, ..., s$ have only one common point - zero.

Theorem (generalized eigenspace decomposition theorem A.8, p. 268 in the course book, without proof)

Let $A \in \mathbb{C}^{N \times N}$ and $\lambda_1, ..., \lambda_s$ be all distinct eigenvalues of A with multiplicities m_j, $\sum_{j=1}^{s} m_j = N$. Then $\mathbb{C}^N$ can be represented as a direct sum of generalised eigenspaces $E(\lambda_j) = \ker(A - \lambda_j)^{m_j}$ to A having dimensions m_j:

$$\dim\left(\ker(A - \lambda_j)^{m_j}\right) = m_j$$

$$\mathbb{C}^N = \ker(A - \lambda_1)^{m_1} \oplus ... \oplus \ker(A - \lambda_s)^{m_s} \tag{12}$$

$\square$

The formula (11) together with the decomposition of $\mathbb{C}^N$ into direct the sum of generalised eigenspaces gives a recipe for a finite analytic representation of solutions to I.V.P. to (4) and a representation of general solutions to (4).

Theorem (2.11, part 2, p. 35 in the course book) Let $z \in E(\lambda)$ be a generalized eigenvector corresponding to the eigenvalue λ. Denote by $x_z(t) = \exp(At)z$ - the solution of I.V.P. with $x_z(0) = z$.

Let $B(\lambda_j)$ be a basis in $E(\lambda_j)$ having dimesion m_j, and denote $\mathcal{B} = \cup_{j=1}^{s} B(\lambda_j)$ - the union of all bases of generalized eigenspaces $E(\lambda_j)$ for all eigenvalues $\lambda_j \in \sigma(A)$. The set of functions $\{x_z : z \in \mathcal{B}\}$ is a basis of the solution space $\mathcal{S}_{\text{hom}}$ of (4).

Proof. *B*y the generalized eigenspace decomposition theorem $\mathbb{C}^N = \ker(A - \lambda_1)^{m_1} \oplus ... \oplus \ker(A - \lambda_s)^{m_s}$ and therefore all subspaces $E(\lambda_j) = \ker(A - \lambda_j)^{m_j}$ making them linearly independent. The total number of these basis vectors is $\sum_{j=1}^{s} m_j = N$ that is equal to the dimension of $\mathbb{C}^N$. Therefore $\mathcal{B}$ is a basis in $\mathbb{C}^N$.

From the theorem on the dimension of the solution space $\mathcal{S}_{\text{hom}}$ of a linear system it follows that solutions with initial data taken from the basis $\mathcal{B}$ build a basis in the solution space $\mathcal{S}_{\text{hom}}$ of (4).

■

We continue with a description of how this theorem can be used for practical calculation of solutions to I.V.P.

Let the matrix A have s distinct eigenvalues $\lambda_1, ..., \lambda_s$ with corresponding generalised eigenspaces $E(\lambda_j)$. Represent the initial data $x(0) = \xi$ for the solution $x(t)$ as a sum of its components from different generalised eigenspaces:

$$\xi = \sum_{j=1}^{s} x^{0,j}, \quad x^{0,j} \in E(\lambda_j)$$

Here $x^{0,j} \in E(\lambda_j)$ - are components of ξ in the generalized eigenspaces $E(\lambda_j) = \ker(A - \lambda_j)^{m_j}$ of the matrix A. These subspaces intersect only in the origin and are invariant with respect to A and $\exp(At)$. It implies that for the solution $x_z(t)$ with initial data $z \in E(\lambda_j)$, we have $x_z(t) \in E(\lambda_j)$ for all $t \in \mathbb{R}$.

Let m_j be the algebraic multiplicity of the eigenvalue λ_j. We apply the formula (11) to this representation and derive the an expression for solutions for arbitrary initial data as a finite sum (instead of series):

$$x(t) = e^{At}\xi = e^{At}\sum_{j=1}^{s} x^{0,j} = \tag{13}$$

$$\sum_{j=1}^{s}\left(e^{\lambda_j t}\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k \frac{t^k}{k!}\right] x^{0,j}\right) \tag{14}$$

Series expressing $\exp(At)x^{0,j}$ terminates on each of the generalised eigenspaces $E(\lambda_j)$.

The last formula still needs specification to derive to an explicit solution.

General solution can be written explicitely by finding a basis of of eigenvectors v_j and generalized eigenvectors for each generalised eigenspace $E(\lambda_j)$ and expressing all components $x^{0,j}$ of ξ in the generalized eigenspaces $E(\lambda_j)$ in the form

$$x^{0,j} = ...C_p v_j + C_{p+1} v_j^{(1)} + C_{p+2} v_j^{(2)}... \tag{15}$$

including all linearly independent eigenvectors v_j corresponding to λ_j *(it might exist several eigenvectors v_j corresponding to one λ_j*) and enough many linearly independent generalized eigenvectors $v_j^{(1)}$,..., $v_j^{(l)}$.

We will start with examples illustrating this idea in some simple cases.

Example 4. Matrix 3x3 with two linearly independent eigenvectors.

Consider a system of equations $x' = Ax$ with matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ It is easy to see that $\lambda = 1$ is the only eigenvalue with algebraic multyplicity 3. Characteristic polynomial is $p(\lambda) = (1-\lambda)^3$.

The eigenvectors satisfy the equation $(A - I)\,v = 0$: $A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

It has two linearly independent solutions that can be chosen as $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The eigenspace is a plane through the origin orthogonal to the vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We like to find a generalised eigenvector linearly independent of v_1 and v_2. We take the eigenvector v_1 and solve the equation

$$(A - \lambda I)v_1^{(1)} = v_1.$$

We denote it by two indexes to point out that it belongs to a chain with base on v_1. Denoting $v_1^{(1)} = [y_1, y_2, y_3]^T$ we consider the system

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It gives a solution $y_3 = 1,\ y_2 = 0,\ y_1 = 0.$ $v_1^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We point out that if we try to find a chain of generalised eigenvectors starting from the eigenvector v_2, it leads to a system $(A-I)v_2^{(1)} = v_2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

that has no solutions. If we try to extend the chain of generalised eigenvectors with one more: $v_1^{(2)}$by solving the system $(A-I)v_1^{(2)} = v_1^{(1)}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we find that it has no solutions (in fact we know that there cannot be more linearly independent generalised eigenvectors because we have already found 3 of them).

We can write general solution to the system of ODE with matrix A using the general formula (13) and expressing the initial data as a linear combination of eigenvectros v_1 and v_2 and the generalised eigenvector $v_1^{(1)}$:

$$\begin{aligned} x(t) &= e^{\lambda t}\left[\sum_{k=0}^{2}(A-\lambda I)^k\frac{t^k}{k!}\right]\left(C_1v_1 + C_2v_2 + C_3v_1^{(1)}\right) \\ \xi &= C_1v_1 + C_2v_2 + C_3v_1^{(1)} \end{aligned}$$

$m(\lambda) = 3$. It is why we put upper bound in the sum equal to $m(\lambda)-1 = 2$. The expression above simplifies (using that by construction $(A-\lambda I)v_1^{(1)} = v_1$) and therefore $(A-\lambda I)^2 v_1^{(1)} = (A-\lambda I)v_1 = 0$. to

$$\begin{aligned} x(t) &= C_1e^tv_1 + C_2e^tv_2 + C_3e^t\left[I + (A-I)\,t\right]v_1^{(1)} \\ & C_1e^tv_1 + C_2e^tv_2 + C_3e^tv_1^{(1)} + C_3te^tv_1 \end{aligned}$$

Example 5. Matrix 3x3 with one eigenvector.

Consider a system of equations $x' = Ax$ with matrix $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$

It is easy to see that $\lambda = -1$ is the only eigenvalue with multiplicity 3.

Eigenvectors satisfy the equation

$$(A - \lambda I)\, v = 0$$

$A + I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. It has one linearly independent solution that can be chosen as $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We will build a **chain of generalised eigenvectors** starting with this eigenvector. Solve the equation

$$(A - \lambda I)v^{(1)} = v$$

$$(A + I)v = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It implies that $y_2 = -1$, and we are free to choose $y_1 = 0$ and $y_3 = 0$.

$$v^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The next generalised eigenvector $v^{(2)}$ in the chain must satisfy the equation

$$(A - \lambda I)v^{(2)} = v^{(1)}$$

$$(A+I)v^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$y_3 = 1/2,\ y_2 = 0,\ y_1 = 0. \quad v^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

$$\begin{aligned} x(t) &= e^{\lambda t}\left[\sum_{k=0}^{2} (A-\lambda I)^k \frac{t^k}{k!}\right]\left(C_1 v + C_2 v^{(1)} + C_3 v^{(2)}\right) = \\ & C_1 e^{\lambda t} v + C_2 e^{\lambda t} v^{(1)} + C_2 t e^{\lambda t}(A-\lambda I) v^{(1)} + \underbrace{C_2 \left(\frac{t^2}{2}\right) e^{\lambda t}(A-\lambda I)^2 v^{(1)}}_{=0} \\ & + C_3 e^{\lambda t} v^{(2)} + C_3 t e^{\lambda t} \underbrace{(A-\lambda I) v^{(2)}}_{=v^{(1)}} + C_3 \left(\frac{t^2}{2}\right) e^{\lambda t} \underbrace{(A-\lambda I)^2 v^{(2)}}_{=v} \end{aligned}$$

$$\begin{aligned} x(t) &= C_1 e^{\lambda t} v + C_2 e^{\lambda t} v^{(1)} + C_2 t e^{\lambda t} v \\ & + C_3 e^{\lambda t} v^{(2)} + C_3 t e^{\lambda t} v^{(1)} + C_3 \left(\frac{t^2}{2}\right) e^{\lambda t} v \end{aligned}$$

$$\begin{aligned} x(t) &= C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + C_2 t e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ & + C_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} + C_3 t e^{-t} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + C_3 \left(\frac{t^2}{2}\right) e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$x(t) = \begin{bmatrix} C_1 e^{-t} + tC_2 e^{-t} + \frac{1}{2} t^2 C_3 e^{-t} \\ -C_2 e^{-t} - tC_3 e^{-t} \\ \frac{1}{2} C_3 e^{-t} \end{bmatrix}$$

■

6.2 Chains of generalised eigenvectors

A practical method for calculating a basis of linearly independent generalized eigenvectors in the general case is an extension of the approach that we used in the last examples.

We find a basis of the eigenspace to λ consisting of $r(\lambda)$ eigenvectors satisfying the equation $(A - \lambda I)\, u_0 = 0$.Their number $r(\lambda)$ is called **geometric multiplicity** of λ and $r(\lambda) \leq m(\lambda)$. Then for each eigenvector $u_0 \neq 0$ from this basis we find a vector $u_1 \neq 0$ satisfying the equation $(A - \lambda I)\, u_1 = u_0$, and continue this calculation, building a **chain of generalised eigenvectors** $u_1, ..., u_l$ satisfying equations.

$$(A - \lambda I)\, u_k = u_{k-1} \tag{16}$$

up to the index $k = l$ when there will be no solutions to the next equation. The largest possible number l is $(m(\lambda) - r(\lambda) - 1)$, but it can also be smaller if the eigenvalue λ has more than one linearly independent eigenvector.

Claim.

Point out that depending on the range of the operator with matrix $(A - \lambda I)$ (column space of the matrix $(A - \lambda I)$) one might need to be careful choosing non-unique (!) eigenvectors u_0 and generalised eigenvectors u_k in the equations (16) so that they belong to the column space of the matrix $(A - \lambda I)$ (if possible!) to guarantee that the equations (16) have a solution. Alternatively one can start this algorithm from above, solving first the *equation*

$$\begin{aligned} (A-\lambda I)^{l} u_l &= 0 \\ (A-\lambda I)^{l-1} u_l &\neq 0 \end{aligned}$$

for a generalized eigenvector of rank l and then can apply equations (16) to calculate generalized eigenvectors of lower rank that belong to corresponding chain of generalized eigenvectors. The last vector in this calculation will be an eigenvector. Check the solution to the Exercise 864 in the file with exercises, where these observations are important.

Lemma. The chain of generalised eigenvectors constructed in *(16)* is linearly independent. It can be proved by contradiction.(**Exercise!**)

□

Theorem. A set of generalised eigenvectors corresponding to p chains of eigenvectors as in *(16)* is linearly independent if and only if eigenvectors in the bottom of corresponding chains of generalised eigenvectors are linearly independent.

□

In the case when all eigenvalues $\lambda_1, ..., \lambda_s$ to a real matrix $A \in \mathbb{R}^{N\times N}$ are real, the generalized eigenvectors will be also real and therefore

$$\mathbb{R}^N = \ker(A-\lambda_j)^{m_1} \oplus ... \oplus \ker(A-\lambda_j)^{m_s}$$

In this case chains of eigenvectors and generalized eigenvectors build by the procedure as above gives a basis in $\mathbb{R}^N$.

To find a basis in the generalized eigenspace $E(\lambda_j)$ one can start with finding all linearly independent eigenvectors that are linearly independent solutions to the equation $(A-\lambda_j I)\, v = 0$ and collecting them in a set denoted by $\mathcal{E}$. Then find all linearly independent solutions to

$(A-\lambda_j I)^2 v^{(1)} = 0$ (that are not eigenvectors) and adding them $\mathcal{E}$. Next one finds solutions to $(A-\lambda_j I)^3 v^{(2)} = 0$ linearly independent from those in $\mathcal{E}$ and collecting them also in $\mathcal{E}$ e.t.c. Continuing in this way one finishes when the total number of derived linearly independent generalised eigenvectors will be equal to m_j - the algebraic multiplicity of the eigenvalue λ_j.

A more systematic approach to this problem is to calculate such a basis as a chain of generalised eigenvectors corresponding to each of linearly independent eigenvector as it is was suggested in examples before:

$$
\begin{aligned}
(A-\lambda_j I)\, v_j &= 0, \\
(A-\lambda_j I)\, v_j^{(1)} &= v_j \\
(A-\lambda_j I)\, v_j^{(2)} &= v_j^{(1)} \\
& e.t.c. \\
(A-\lambda_j I)\, v_j^{l} &= v_j^{l-1}
\end{aligned}
$$

This approach has also an advantage that using chains of generalised eigenvectors as a basis leads to a particularly simple representation of the system of equations (4) with matrix A in so called Jordan canonical form, that we will learn later.

Substituting the expression (15) for arbitrary initial data ξ in to the general formula above and calculating all matrix $(A-\lambda_j I)$ powers and matrix-vector, multiplications we get a general solution with a set of arbitrary coefficients $C_1, ..., C_N$.

Keep in mind that $(A-\lambda_j I)\, v_j = 0$ and $(A-\lambda_j I)^2 v_j^{(1)} = 0$ e.t.c., so many terms in the general expression for the solution can be zeroes.

Initial value problems. To solve an I.V.P. one needs to express a particular initial data ξ in terms of the basis of generalized eigenvectors soving a linear system of equations for coefficients $C_1, ..., C_N$ in (15)

Look for exercises in a separate file Exercises_3.pdf with exercises on linear autonomous systems of ODE. Check a link in Canvas.

6.3 Real solutions for systems with real matrices having complex eigenvalues.

We will consider an example of a system in plane with real matrix having two simple, conjugate complex eigenvalues (no more because of the small dimension). The idea of solution was to build a complex solution corresponding to one of these eigenvalues and use it's real and imaginary part at two linearly independent solutions to construct a general solution. The same idea works in the general case when a real matrix might have conjugate complex eigenvalues (might be multiple in higher dimensions). We build a basis of eigenvectors and generalized eigenvectors for invariant generalized eigenspaces corresponding to distinct conjugate complex eigenvalues. One can start with one of these eigenvalues and then can just choose the basis for the second one as a complex conjugate (do not need to do it in fact). Then we construct arbitrary complex solutions in the invariant generalized eigenspace corresponding to the first of these conjugate eigenvalues. The real and imaginary parts of these solutions are linearly independent and build a basis of solutions in the corresponding real invariant subspace.

Example 2. Real matrix with complex eigenvalues.

$x' = Ax$ with $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, find a general real solution to the system. In this case we find first a general complex solution and then construct a general real solution based on it.

Solution. $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - 2\lambda + 5 = 0$;

Hint. We point out here that in the case of 2×2 matrices the characteristic polynomial always has a simple representation

$$p(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A)$$

where tr(A) is the sum of diagonal elements in A called trace, and det(A) is determinant. Here tr(A) $=\lambda_1 + \lambda_2$; $\det A = \lambda_1\lambda_2$

■

Eigenvalues are: $\lambda_1 = 1 - 2i$, and $\lambda_2 = 1 + 2i$.
They are complex conjugate:

$$\begin{aligned} \overline{\lambda_1} &= \lambda_2 \\ p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \end{aligned}$$

because the characteriscic polynomial has real coefficients.
Eigenvectors satisfy the equations $(A - \lambda I)v_1 = \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} v_1 = 0$
and
$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} v_2 = 0.$
These eigenvectors muct be also complex conjugate. We see it by considering the equations for v_1 that is
$(A - \lambda_1 I)v_1 = 0$ and its formal complex conjugate $(A - \overline{\lambda_1} I)\overline{v_1} = 0$ that is satisfied because the conjugate of the real matrix A is the matrix A itself. Therefore $\overline{v_1}$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = \overline{\lambda_1}$. We point out that this argument is independent of this particular example and would be valid for any real matrix with complex eigenvalues.
The first and the second equation in each of these systems are equivalent because rows are linearly dependent (homogeneous system has non-trivial solutions and the determinant of the matrix $A - \lambda I$ is zero).
We solve the first equation in the first system by choosing the first component equal to 1. It implies that the second component denoted here

by z satisfies the equation $2+2i-2z=0$ and therefore $z=1+i$. The second eigenvector is just the complex conjugate of the first one.

$v_1 = \left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\} \leftrightarrow \lambda_1 = 1-2i$, and $v_2 = \left\{ \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 1+2i$.

They are linear independent as eigenvectors corresponding to different eigenvalues.

One complex solution is $x^*(t) = e^{(1-2i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$, another one is

$$y^*(t) = e^{(1+2i)t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$x^*(t)$ and $y^*(t)$ are linearly independent at any time as corresponding to linearly independent initial vectors v_1 and v_2 (according to the theorem before) and build a basis of complex solutions to the system. Therefore the matrix $[x^*(t), y^*(t)]$ has determinant $\det([x^*(t), y^*(t)]) \neq 0$.

Two linearly independent real solutions can be chosen as real and imaginary parts of $x^*(t)$ (or $y^*(t)$): $\mathrm{Re}\,[x^*(t)] = \frac{1}{2}(x^*(t) + y^*(t))$ and $\mathrm{Im}\,[x^*(t)] = \frac{1}{2i}(x^*(t) - y^*(t))$ that are linearly independent because the the matrix $T = \frac{1}{2}\begin{bmatrix} 1 & 1/i \\ 1 & -1/i \end{bmatrix}$ of the transformation

$$\begin{aligned} [x^*(t), y^*(t)]\, T &= \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} \begin{bmatrix} 1/2 & 1/(2i) \\ 1/2 & -1/(2i) \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2}x_1^* + \frac{1}{2}y_1^* & \frac{1}{2i}x_1^* - \frac{1}{2i}y_1^* \\ \frac{1}{2}x_2^* + \frac{1}{2}y_2^* & \frac{1}{2i}x_2^* - \frac{1}{2i}y_2^* \end{bmatrix} &= [\mathrm{Re}\,[x^*(t)]\,, \mathrm{Im}\,[x^*(t)]] \end{aligned}$$

is invertible: $\det T = -\frac{1}{2i} \neq 0$ and therefore, by the property of the determinant for the product of matrices,

$$\det[x^*(t), y^*(t)]\det(T) = \det([\mathrm{Re}\,[x^*(t)]\,, \mathrm{Im}\,[x^*(t)]]) \neq 0$$

and $\mathrm{Re}\,[x^*(t)]$ and $\mathrm{Im}\,[x^*(t)]$ are linearly independent.

Therefore real valued vector functions $\operatorname{Re}[x^*(t)]$ and $\operatorname{Im}[x^*(t)]$ can be used as a basis for representing the general real solution to the system:

$$x(t) = C_1 \operatorname{Re}[x^*(t)] + C_2 \operatorname{Im}[x^*(t)].$$

We express $x^*(t)$ with help of Euler formulas and separate real and imaginary parts

$$x^*(t) = e^{(1-2i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = e^t (\cos 2t - i \sin 2t) \begin{bmatrix} 1 \\ 1+i \end{bmatrix} =$$

$$e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ (1+i)\cos 2t + (1-i)\sin 2t \end{bmatrix} = e^t$$

$$\begin{bmatrix} \cos 2t - i \sin 2t \\ \cos 2t + \sin 2t + i(\cos 2t - \sin 2t) \end{bmatrix} =$$

$$e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - i\, e^t \begin{bmatrix} \sin 2t \\ (\sin 2t - \cos 2t) \end{bmatrix}$$

The answer follows as a linear combination of real and imaginary parts:

$$x(t) = C_1 \operatorname{Re}[x^*(t)] + C_2 \operatorname{Im}[x^*(t)].$$

Answer: $x(t) = C_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}.$

We will transform this expression to clarify its geometric meaning and the shape of orbits in the phase plane. We observe first that if we drop exponents e^t, in the expression for $x(t)$ and consider the expression $x(t)e^{-t} = C_1 \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$,we will observe that it represents a movement along ellipses in the plane.

We use an elementary trick that makes that any linear combination of

$\sin(\gamma)$ and $\cos(\gamma)$ is $C\sin(\gamma+\beta)$ or $C\cos(\gamma-\beta)$ with some constants C, β.

$$
\begin{aligned}
x_1(t)e^{-t} &= C_1\cos(2t) + C_2\sin(2t) = \\
& \sqrt{C_1^2+C_2^2}\left(\left(\frac{C_1}{\sqrt{C_1^2+C_2^2}}\right)\cos 2t + \left(\frac{C_2}{\sqrt{C_1^2+C_2^2}}\right)\sin 2t\right) \\
&= \sqrt{C_1^2+C_2^2}\left(\cos(\theta)\cos 2t + \sin(\theta)\sin 2t\right) \\
&= \sqrt{C_1^2+C_2^2}\cos(2t-\theta) \\
\theta &= \arccos\left(\left(\frac{C_1}{\sqrt{C_1^2+C_2^2}}\right)\right)
\end{aligned}
$$

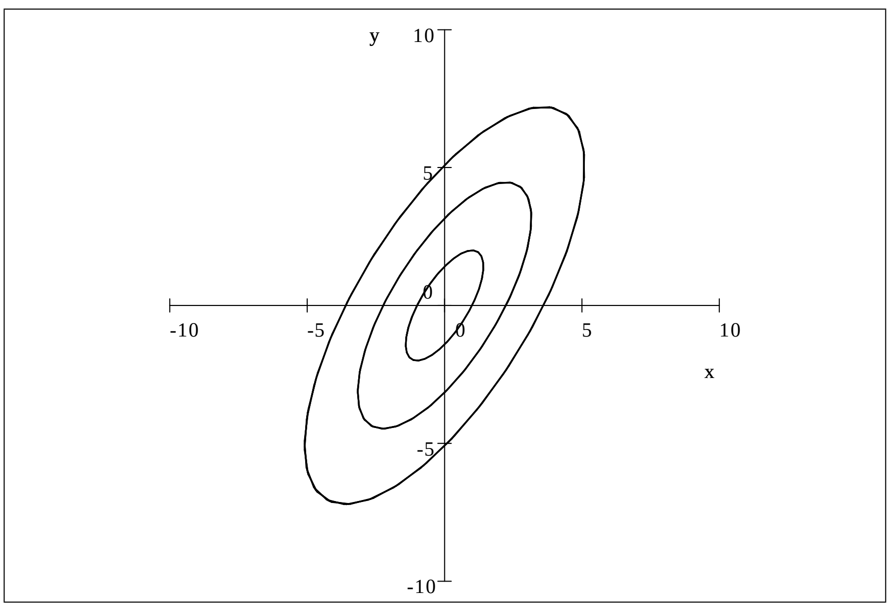
Similarly

$$
\begin{aligned}
\left[x_2(t)-x_1(t)\right]e^{-t} &= C_1\sin(2t) - C_2\cos(2t) = \\
& \sqrt{C_1^2+C_2^2}\left(\left(\frac{C_1}{\sqrt{C_1^2+C_2^2}}\right)\sin 2t - \frac{C_2}{\sqrt{C_1^2+C_2^2}}\cos 2t\right) \\
&= \sqrt{C_1^2+C_2^2}\left(\cos(\theta)\sin 2t - \sin(\theta)\cos 2t\right) \\
&= \sqrt{C_1^2+C_2^2}\sin(2t-\theta)
\end{aligned}
$$

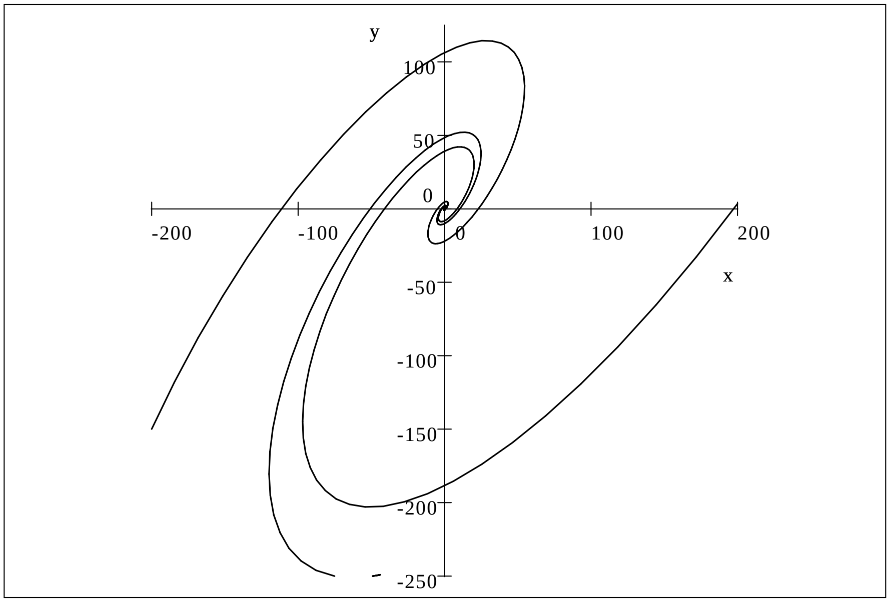
Finally we arrive to a parametric expression for a periodic movement along ellipses with size depending on C_1 and C_2.

$$
\begin{aligned}
x(t)e^{-t} &= C_1 \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} \\
&= \sqrt{C_1^2 + C_2^2} \begin{bmatrix} \cos(2t - \theta) \\ \cos(2t - \theta) + \sin(2t - \theta) \end{bmatrix} \\
&= \sqrt{C_1^2 + C_2^2} \begin{bmatrix} \cos(2t - \theta) \\ \sqrt{2}\left[\sin(\pi/4)\cos(2t - \theta) + \cos(\pi/4)\sin(2t - \theta)\right] \end{bmatrix} \\
&= \sqrt{C_1^2 + C_2^2} \begin{bmatrix} \cos(2t - \theta) \\ \sqrt{2}\left[\sin(2t - \theta + \pi/4)\right] \end{bmatrix}
\end{aligned}
$$

illustrated in the next picture:



This movement is modulated in our solution $x(t)$ by the exponential term e^t giving orbits as spirals going to infinity out of the origin that is an unstable equilibrium point for this system.



Example. It is good to consider here the solution to the exercise 858. Ideas about solutions to systems with complex eigenvalues demonstrated in exercises can in the general situation expressed by the following Theorem.

Theorem 2.14. p. 38 on real solutions to autonomous systems with real matrix and complex eigenvalues (without proof)

Let $A \in R^{N \times N.}$ for λ an eigenvalue, let $m(\lambda)$ be the algebraic multiplicity of λ, $E(\lambda) = \ker(A - \lambda I)^{m(\lambda)}$ denote it's generalised eigenspace. Let $B(\lambda)$ be a basis in $E(\lambda)$ chosen to be real for real λ.

For all $z \in C^N$, we denote x_z, $y_z : R \to R^N$ real solutions to the equation $x' = Ax$ as

$$x_z = \exp(At)\,\mathrm{Re}\,z, \quad y_z = \exp(At)\,\mathrm{Im}\,z$$

Then

1) Let B_0 (respectively B_+) denote the union of all $B(\lambda)$ for all real eigenvalues λ to A (correspondingly for all λ with $\mathrm{Im}\,\lambda > 0$) The set of real

functions given by

$$\{x_z, \quad z \in B_0 \cup B_+\} \cup \{y_z : \quad z \in B_+\}$$

forms a basis of the solutuion space to $x' = Ax$.

2) If λ is a real eigenvalue to A, then for every generalized eigenvector $z \in E(\lambda)$, the solution x_z is expressed as

$$x_z(t) = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!} (A - \lambda I)^k \operatorname{Re} z$$

3) If $\lambda = \alpha + i\beta$ with $\beta \neq 0$, is an eigenvalue of A,then for every generalized eigenvector $z \in E(\lambda)$,

solutions $x_z = \exp(At) \operatorname{Re} z$ and $y_z = \exp(At) \operatorname{Im} z$ with initial data $\operatorname{Re} z$ and $\operatorname{Im} z$ are expressed as

$$\begin{aligned} x_z(t) &= e^{\alpha t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!} \left[\cos(\beta t) \operatorname{Re}\left((A - \lambda I)^k z\right) - \sin(\beta t) \operatorname{Im}\left((A - \lambda I)^k z\right)\right] \\ y_z(t) &= e^{\alpha t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!} \left[\cos(\beta t) \operatorname{Re}\left((A - \lambda I)^k z\right) + \sin(\beta t) \operatorname{Im}\left((A - \lambda I)^k z\right)\right] \end{aligned}$$

$\square$

The theorem shows the how $m(\lambda)$ real linearly independent solutions can be obtained for a real matrix A with complex eigenvalues λ. The part 1) of the theorem shows that such solutions build a real basis of the solution space for $x' = Ax$ with a real matrix.

7 Jordan canonical form of matrix. Functions of matrices.

7.1 Change of variables. Properties of similar matrices. Block matrices.

We tried in previous lectures to find a basis $\left\{v_1, v_1^{(1)}, ...\right\}$ in C^N or in R^N such that expressing initial data ξ in I.V.P.

$$x'(t) = Ax(t), \quad x(0) = \xi$$

in terms of this basis led to a particularly simple expression of the solution as an explicit linear combination including polynomials of $t(A - \lambda_i I)$ acting on basis vectors. We can interpret these results by introducing a linear change of variables

$$x = Vy; \quad y = V^{-1}x$$

with matrix V of this transformation having columns consisting of N linearly independent vectors.

In terms of the new variable y the system has the form

$$y'(t) = V^{-1}AVy, \quad y(0) = V^{-1}\xi$$

In the case when the matrix A has N linearly independent eigenvectrosthe matrix $V^{-1}AV = D$ is diagonal with eigenvalues $\{\lambda_1..., \lambda_j, ...\}$ of the matrix A standing on the diagonal $m(\lambda_j)$ times equal to the algebraic multiplicity of λ_j. The number $r(\lambda_j)$ of linearly independent eigenvectors belonging to λ_j is equal to $m(\lambda_j)$ in this case.

Definition. Matrices A and $V^{-1}AV$ are called similar.

They have several characteristics the same: determinant, and characteristic polynomials. It is a simple consequence of properties of determinants of products of matrices.

Prove it as an exercise using: $\det(AB) = \det(A)\det(B)$; $\det(B^{-1}) = (\det B)^{-1}$ if $\det B \neq 0$.

Using the associative property of matrix multiplication we arrive to the property

Theorem. If matrices A and B are similar through $B = V^{-1}AV$, $A = VBV^{-1}$ then

$$\begin{aligned} B^k &= V^{-1}(A^k)V; \\ \exp(B) &= V^{-1}(\exp A)V \\ A^k &= V(B^k)V^{-1} \\ \exp(A) &= V(\exp B)V^{-1} \end{aligned}$$

Prove it as an exercise.

Corollary. If the matrix A is diagonalisable, then $\exp(A) = V\exp(D)V^{-1}$ where V matrix of linearly independent eigenvectors and the matrix D is diagonal matrix of eigenvalues λ_j and $\exp(D)$ is a diagonal matrix with $\exp(\lambda_j)$ on the diagonal. In this case the system in new variables $y(t) = V^{-1}x(t)$ consists of independent differential equations $y_j'(t) = \lambda_j y_j(t)$ for he components $y_j(t)$ of $y(t)$ that have simple solutions $y_j(t) = C_j e^{\lambda_j t}$

Definition. Block - diagonal matrices

Block-diagonal matrices are square matrices that have a number of square blocks $\mathbb{B}_1$,... along diagonal and other terms all zero. For example:

$$B = \begin{bmatrix} \mathbb{B}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{B}_3 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{B}_4 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} \mathbb{B}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{B}_3 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{B}_4 \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{B}_3 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{B}_4 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1^2 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2^2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{B}_3^2 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{B}_4^2 \end{bmatrix}$$

$$=$$

These matrices have a property that their powers lead to block diagonal matrices of the same structure with powers of original blocks on the diagonal:

$$B^k = \begin{bmatrix} (\mathbb{B}_1)^k & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & (\mathbb{B}_2)^k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & (\mathbb{B}_3)^k & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & (\mathbb{B}_4)^k \end{bmatrix}$$

This simple observation leads immediately to the formula for the exponent of a block diagonal matrix.

$$\exp(B) = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = \begin{bmatrix} \exp(\mathbb{B}_1) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \exp(\mathbb{B}_2) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \exp(\mathbb{B}_3) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \exp(\mathbb{B}_4) \end{bmatrix}$$

In fact the same relation would be valid even for an arbitrary analytical function f with power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, converging in the whole $\mathbb{C}$:

$$f(B) = \begin{bmatrix} f(\mathbb{B}_1) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & f(\mathbb{B}_2) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & f(\mathbb{B}_3) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & f(\mathbb{B}_4) \end{bmatrix}$$

Claim. Let the space $\mathbb{C}^N$ or $\mathbb{R}^N$ be represented as a direct sum of subspaces $V_1, ..., V_s$, invariant under the action of operator Ax:

$$\mathbb{C}^N = V_1 \oplus V_2 \oplus ... \oplus V_s$$

Then there is a basis $\{u_1, , , , , u_N\}$ in $\mathbb{C}^N$, correspondingly $\mathbb{R}^N$ such that the operator Ax in this basis has matrix B similar to $A : B = U^{-1}AU$, or

$$UB = AU$$

that is block diagonal, with blocks of size equal to dimensions of subspaces $V_1, ..., V_s$ and matrix U that has columns u_1,...,u_N.

The basis $\{u_1, ..., u_N\}$ is easy to choose as a union of bases for each invariant subspace V_j. It is evident that this construction leads to a block diagonal matrix for the operator Ax because columns with index j in the matrix B are equal to $U^{-1}Au_j$ that are coordinates of vectors Au_j in terms of the basis $\{u_1, ..., u_N\}$ and belong to the same invariant subspace as u_j.

We illustrate this fact on a simple example with two invariant subspaces.

Consider a decomposition of the space $\mathbb{C}^N$ into two subspaces V and W , $\dim V = m$, $\dim W = p$, $m + p = N$ invariant with respect to the operator defined by the multiplication Ax. Choose base vectors in each of these subspaces: $\{u_1, ..., u_m\}$ and $\{w_1, ..., w_p\}$. They constitute a basis $\{u_1, ..., u_m, w_1, ..., w_p\}$ for the whole space $\mathbb{C}^N$.

Introduce a matrix $T = [u_1, ..., u_m, w_1, ..., w_p]$ with basis vectors of the whole $\mathbb{C}^N$ collected according to the invariant subspace they belong to. Represent a vector x in terms of this basis: $x = Ty$ where

$$y = [y_1, ..., y_m, y_{m+1}, ..., y_{p+m}]$$

is a vector of coordinates of x in the basis consisting of columns in T. The operator Ax acting on the vector x is expressed in terms of these coordinates y as

$$Ax = ATy$$

We express now the image of this operation also in terms of the basis $\{u_1, ..., u_m, w_1, ..., w_p\}$:

$$T\left(T^{-1}Ax\right) = ATy$$

Here $(T^{-1}Ax)$ gives coordinates of the vector Ax in terms of the basis $\{u_1, ..., u_m, w_1, ..., w_p\}$ that are columns in the matrix T. It implies that

$$T^{-1}Ax = \left(T^{-1}AT\right)y$$

So the matrix $(T^{-1}AT)$ is a standard matrix of the original mapping Ax in terms of the basis $\{u_1, ..., u_m, w_1, ..., w_p\}$ associated with invariant subspaces V and W. Now observe that taking vector of y - coordinates with only components $y_1, ..., y_m$ non-zero we get vectors that belong to the invariant subspace V, namely vectors having only y - coordinates $1, ..., m$ non-zero. It means that first m columns in $(T^{-1}AT)$ must have elements $m+1, ...m+p$ equal to zero because A maps V into itself. If we choose y coordinates with only components $y_{m+1}, ...y_{m+p}$ non-zero, we get a vector that belongs to the subspace W, namely vectors that have only coordinates $m+1$, ..., $m+p$ non-zero. It means that last p columnst in $(T^{-1}AT)$ must have elements $1, ...m$ equal to zero because A maps W into itself. It means finaly that $(T^{-1}AT)$ has a block diagonal structure with blocks of size $m \times m$ and $p \times p$ corresponding to the invariant subspaces V and W.

7.2 Jordan canonical form of matrix and it's functions.

We will observe now that a basis of generalised eigenvectors

$$\mathbb{C}^N = E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E(\lambda_s)$$

build with help of chains of generalised eigenvectors as we discussed before, leads to a particular "canonical" matrix J similar to the matrix A by the transformation

$$V^{-1}AV = J$$

or $A = VJV^{-1}$ with the matrix

$$V = \left[...v, v^{(1)}, ..., v^{(r-1)}...\right]$$

where columns are generalised eigenvectors from different chains of generalised eigenvectors corresponding to linearly independent eigenvectors put in the same order as in (17).

Consider first an $m \times m$ matrix A in $\mathbb{C}^{m\times m}$ that has one eigenvalue λ from characteristic polynomial $p(z) = (z-\lambda)^m$, of multiplicity m and only one linearly independent eigenvector v.Corresponding chain of generalised eigenvectors $\left\{v, v^{(1)}, ..., v^{(m-1)}\right\}$ has rank m equal to the dimension of the space and satisfies equations:

$$\begin{aligned}
(A-\lambda I)\, v &= 0, \\
(A-\lambda I)\, v^{(1)} &= v \\
(A-\lambda I)\, v^{(2)} &= v^{(1)} \\
& \quad e.t.c. \\
(A-\lambda I)\, v^{(m-1)} &= v^{(m-2)}
\end{aligned} \tag{17}$$

$$(A-\lambda I)^m v^{(m-1)} = 0.$$

We rewrite this chain of equations as

$$\begin{aligned}
Av &= \lambda v \\
Av^{(1)} &= \lambda v^{(1)} + v \\
Av^{(2)} &= \lambda v^{(2)} + v^{(1)} \\
& \quad e.t.c. \\
Av^{m-1} &= \lambda v^{(m-1)} + v^{(m-2)}
\end{aligned}$$

Using the definition of the matrix product and the matrix V defined as

$$V = \left[v, v^{(1)}, ..., v^{(m-1)}\right]$$

we observe that vector equations for the chain of generalised eigenvectors are equivalent to the matrix equation

$$AV = VD + V\mathcal{N} = V(D + \mathcal{N})$$

where D is the diagonal matrix with the eigenvalue λ on the diagonal and the matrix $\mathcal{N}$ has all elements zero except elements over the diagonal that are equal to one:

$$\mathcal{N} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix};$$

Shifting property of the right multiplication by the matrix $\mathcal{N}$.

The specific structure of $\mathcal{N}$ makes that the product $B\mathcal{N}$ of an arbitrary square matrix B by the matrix $\mathcal{N}$ from the right is a matrix where each column k is a column $k-1$ from the matrix B shifted one step to the right, except the first one that consists of zeroes. It follows from the definition of the matrix product and the observation that elements from the column k in the matrix B in the product $B\mathcal{N}$ meet exactly one non zero element 1 in the column $k+1$ in the matrix $\mathcal{N}$:

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & ... & B_{1(m-1)} & B_{1m} \\ B_{21} & B_{22} & B_{23} & ... & B_{2(m-1)} & B_{2m} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ B_{(m-1)1} & B_{(m-1)2} & B_{(m-1)3} & ... & B_{(m-1)(m-1)} & B_{(m-1)m} \\ B_{m1} & B_{m2} & B_{m3} & ... & B_{m(m-1)} & B_{mm} \end{bmatrix}; \quad \mathcal{N} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix};$$

We observe this transformation in equations for the chain of generalized eigenvectors with the matrix V instead of an arbitrary matrix B.

Observe also that $\mathcal{N}^m = 0$, m is the size of $\mathcal{N}$.

Therefore

$$\begin{aligned} AV &= V(D+\mathcal{N}) \\ V^{-1}AV &= (D+\mathcal{N}) = J \end{aligned}$$

Definition of the Jordan block. The matrix $J = D + \mathcal{N}$

$$J = \begin{bmatrix} \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda \end{bmatrix}$$

is called Jordans block. Here D is a diagonal matrix with the eigenvalue λ on the diagonal and the matrix $\mathcal{N}$ defined above, consists of zeroes except for the diagonal above the main one consisting of ones.

We have proved the following theorem.

Theorem (special case of Theorem A.9 , p. 268) Let $m \times m$ matrix A have one eigenvalue of multiplicity m (characteristic polynomial $p(z) = (z - \lambda)^m$) and only one linearly independent eigenvector v. Then the matrix A is similar to the Jordans block J with the similarity relations:

$$\begin{aligned} A &= VJV^{-1} \\ J &= V^{-1}AV \end{aligned}$$

where the matrix V has columns $V = \left[v, v^{(1)}, ..., v^{(m-1)}\right]$ that are elements from the chain of generalized eigenvectors built as solutions to the equations (17).

The "shifting" property of the matrix $\mathcal{N}$ implies that $\mathcal{N}^2$ consists of zeroes except the second diagonal over the main one filled by 1, $\mathcal{N}^3$ consists of zeroes except the third diagonal over the main one filled by 1, and finally $\mathcal{N}^m = 0$.

Definition A matrix with such property that for some integer r we have $\mathcal{N}^r = 0$ is called nilpotent.

Corollary

$$\exp(J) = e^{\lambda} \sum_{k=0}^{m-1} \frac{1}{k!} (\mathcal{N})^k \tag{18}$$

$$\exp(J) = e^{\lambda} \begin{bmatrix} 1 & 1 & 1/2 & ... & \frac{1}{(m-2)!} & \frac{1}{(m-1)!} \\ 0 & 1 & 1 & ... & \frac{1}{(m-3)!} & \frac{1}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 1/2 \\ 0 & 0 & 0 & ... & 1 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

because $\exp(J) = \exp(\lambda I + \mathcal{N}) = \exp(\lambda I)\exp(\mathcal{N}) = e^{\lambda} \sum_{k=0}^{m-1} \frac{1}{k!} (\mathcal{N})^k$ and each term with index k in the sum is a matrix with k -th diagonal over the main one, filled by $\frac{1}{k!}$■

Similarly

$$\exp(Jt) = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (\mathcal{N})^k \tag{19}$$

$$\exp(Jt) = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 & ... & \frac{t^{m-2}}{(m-2)!} & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & ... & \frac{t^{m-3}}{(m-3)!} & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & t & t^2/2 \\ 0 & 0 & 0 & ... & 1 & t \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

By properties of similar matrices we arrive to the

Corollary. See proof of the spectral theorem 2.19 on page 60-61 in Logemann Ryan.

For an $m \times m$ matrix A having one eigenvalue of multiplicity m and only one linearly independent eigenvector v it follows the following expression for $\exp(At)$:

$$\exp(At) = V \exp(Jt) V^{-1} = V \left(e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (\mathcal{N})^k \right) V^{-1}$$

Remark.

If instead of the exponential function we like to calculate an arbitrary analytical function that has converging in $\mathbb{C}$ Maclorain series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

then the same reasoning and the Maclorain series for the function f lead to an expression for the matrix function $f(J)$

$$f(J) = \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!} (\mathcal{N})^k \tag{20}$$

Theorem A.9 , on Jordan canonical form of matrix p. 268 in Logemann Ryan.

Let $A \in \mathbb{C}^{N \times N}$,. There is an invertible matrix $T \in \mathbb{C}^{N \times N}$ and an integer $k \in \mathbb{N}$ such that

$$J = T^{-1} A T$$

has the block diagonal structure

$$\mathbb{J} = \begin{bmatrix} J_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & J_2 & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & J_k \end{bmatrix}$$

where J_j has dimension $r_j \times r_j$ and is a Jordan block. Furthermore, $\sum_{j=0}^{k} r_j = N$ and if $r_j = 1$ them $J_j = \lambda$ for some eigenvalue $\lambda \in \sigma(A)$. Every eigenvalue λ occurs at least at one block; the same λ can occur in more than one block. The number of blocks with the same eigenvalue λ on the diagonal is equal to the number of linearly independent eigenvectors corresponding to this eigenvalue λ (it's geometric multiplicity $g(\lambda)$).

Specification of detailes for Theorem A.9 with a sketch of the proof.

1) Our considerations about chains of generalised eigenvectors and the **special case of Theorem A.9** considered above imply that the matrix T in the general theorem A.9 on Jordan canonical form can be chosen in such a way that it's columns are elements from chains of generalised eigenvectors built on the maximal number of linearly independent eigenvectors to the matrix A.

2) The matrix $J = T^{-1}AT$ has a block diagonal structure with one block corresponding to each linearly independent eigenvector. It follows from the fact that generalised eigenspaces are invariant with respect to the transformation A and from the fact that linear envelopes of the chains of generalised eigenvectors are linearly independent of each other and are also invariant with respect to A.

3) Each block corresponding to a particular eigenvector is a Jordan block with corresponding eigenvalue on diagonal, because of the special case of Theorem A.9 considered above. The size of a particular Jordan block in the Jordan canonical form depends on the length of the corresponding chain of

generalised eigenvectors, that is the smallest integer r such that the equations $(A-\lambda I)^r v^{(r)} = 0$ and $(A-\lambda I)^{r-1} v^{(r)} \neq 0$ are satisfied.

4) It follows from the structure of the canonical Jordan form that the algebraic multiplicity $m(\lambda)$ of an eigenvalue λ is equal to the sum of sizes r_j of Jordan blocks corresponding to λ and coinsides with the dimension of it's generalised eigenspace $E(\lambda) = \ker\left((A-\lambda)^{m(\lambda)}\right)$.

Definition. An eigenvalue is called semisimple if it's generalised eigenspace consists only of eigenvectors and its algebraic multiplicity is equal to its geometric multiplicity: $m(\lambda) = r(\lambda)$. In this case corresponding the Jordan blocks will all have size 1×1.

Jordan blocks in the Jordan canonical form are unique but can be combined in various orders. The position of Jordan blocks within a canonical Jordan form depends on positions of the chains of generalised eigenvectors in the transformation matrix T and is not unique in this sense.

Example of calculating the Jordan canonical form of a matrix.

(Try to solve yourself exercises from the file with exercises on linear autonomous systems, where all answers and some solutions are given)

Consider matrix $C = \begin{bmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, Find its canonical Jordan's form and corresponding basis.

Find first the characteristic polynomial.

$$\det(C-\lambda I) = \det \begin{bmatrix} 1-\lambda & -1 & -2 & 3 \\ 0 & -\lambda & -2 & 3 \\ 0 & 1 & 1-\lambda & -1 \\ 0 & 0 & -1 & 2-\lambda \end{bmatrix} =$$

$$(1-\lambda)\det \begin{bmatrix} -\lambda & -2 & 3 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} =$$

$$(1-\lambda)(-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} - (1-\lambda)\det\begin{bmatrix} -2 & 3 \\ -1 & 2-\lambda \end{bmatrix} =$$
$$(1-\lambda)(-\lambda)\left(\lambda^2 - 3\lambda + 1\right) - (1-\lambda)(2\lambda - 1) =$$
$$(1-\lambda)\left(3\lambda^2 - \lambda - \lambda^3\right) + (1-\lambda)(1-2\lambda) =$$
$$(1-\lambda)\left(3\lambda^2 - 3\lambda - \lambda^3 + 1\right) = (1-\lambda)(1-\lambda)^3 = (1-\lambda)^4.$$

Matrix C has one eigenvalue $\lambda = 1$ with multiplicity 4. Consider the equation for eigenvectors $(C-I)x = 0$ with matrix

$$(C-I) = \begin{bmatrix} 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$ Gauss elimination gives

$$\Longrightarrow \begin{bmatrix} 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & -1 & -2 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with two free variables: x_1 and x_4. Therefore the dimension of the eigenspace is 2. There are two linearly independent eigenvectors that can be chosen as

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Each of these eigenvectors might generate a chain of generalised eigenvectors.

We check the equation $(C - \lambda I\ )v_1^{(1)} = v_1$ with extended matrix

$$\begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$ and carry out the same Gauss elimination as

before: $\Longrightarrow \begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$. The second equation is not compatible and the system has no solution.

For the second eigenvector v_2 we solve similar system $(C - \lambda I\ )v_2^{(1)} = v_2$ with the extended matrix $\begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & -1 & -2 & 3 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$

Gauss elimination implies the echelon matrix

$\begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ that has a two-dimensional set of solutions. We choose one as $v_2^{(1)} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and build up the chain of generalized eigenvectors by solving one more equation $(C - \lambda I\ )v_2^{(2)} = v_2^{(1)}$ with the extended matrix

$\begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & -1 & -2 & 3 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \Longrightarrow$

$\begin{bmatrix} 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ leading to a generalized eigenvector (not unique)

$v_2^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Finally we conclude that the Jordan canonic form of the matrix C in the basis v_1, v_2, $v_2^{(1)}$, $v_2^{(2)}$ is $J = T^{-1}CT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

with transformation matrix $T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, inverse:

$$T^{-1} = \begin{bmatrix} 1 & 1 & 2 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 2 \end{bmatrix};$$

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9 Theorem about conditions for the exponential decay and for the boundedness of the norm $\|\exp(At)\|$ (Corollary 2.13)

Theorem.

Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix. Let $\mu_A = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ where $\sigma(A)$ is the set of all eigenvalues to A. μ_A is the maximal real part of all egenvalues to A.

Then three following statements are valid.

1. $\|\exp(At)\|$ decays exponentially if and only if $\mu_A < 0$. (It means that there are $M_\beta > 0$ and $\beta > 0$ such that $\|\exp(At)\| \le M_\beta e^{-\beta t}$)
2. $\lim_{t\to\infty} \|\exp(At)\xi\| = 0$ for every $\xi \in \mathbb{C}^N$ (it means that all solutions to the ODE $x' = Ax$ tend to zero) if and only if $\mu_A < 0$.
3. if $\mu_A = 0$ then $\sup_{t\ge 0} \|\exp(At)\| < \infty$ if and only if all purely imaginary eigenvalues and zero eigenvalues are semisimple meaning that $m(\lambda) = g(\lambda)$.

Remark. One can prove this theorem in two slightly different but essentially equivalent ways.

1) Using the similarity of the matrix A and it's Jordan matrix J

$$J = T^{-1}AT; \quad A = TJT^{-1}$$

corresponding expression of $\exp(At)$ in terms of $\exp(Jt)$ that is known explicitely:

$$\exp(At) = T\exp(Jt)T^{-1}$$

2) Using the expression for general solution to a linear autonomous system

in terms of eigenvectors and generalized eigenvectors to A :

$$x(t) = \exp(At)x_0 = \sum_{j=1}^{s} \left(\left[\sum_{k=0}^{m_j-1} (A - \lambda_j I)^k \frac{t^k}{k!} \right] x^{0,j} e^{\lambda_j t} \right)$$

for solutions with initial data $x_0 = \sum_{j=1}^{s} x^{0,j}$ with $x^{0,j} \in E(\lambda_j)$ - components of x_0 in the generalized eigenspaces $E(\lambda_j) = \ker(A - \lambda_j)^{m_j}$ of the matrix A, where λ_j, $j = 1..., s$ are all distinct eigenvalues to A with algebraic multiplicities m_j.

The first method is shorter and more explicit.

In the course book the second method is used for proving Theorem 2.12 that is formulated in a slightly unfriendly style.

The Corollary 2.13 almost equivalent and can be proven in exactly the same way as Theorem 2.12 but a bit simpler.

We give here a proof based on the expression $\exp(At) = T \exp(Jt) T^{-1}$ using Jordan matrix.

Proof.

We point out that any matrix $A \in \mathbb{C}^{N \times N}$ can be represented with help of its Jordan matrix J as $A = TJT^{-1}$ where T is an invertible matrix with columns that are linearly independent eigenvectors and generalized eigenvectors to A ordered as in chains of generalised eigenvectors. The Jordan matrix J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & \mathbb{O} & ... & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & J_2 & ... & \mathbb{O} & \mathbb{O} \\ ... & ... & ... & ... & ... \\ \mathbb{O} & \mathbb{O} & ... & J_{p-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & ... & \mathbb{O} & J_p \end{bmatrix}$$

where the number of blocks p is equal to the number of linearly independent eigenvectors to A. The symbol $\mathbb{O}$ denotes zero block.

Each Jordan block J_k has the structure as the following:

$$J_k = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 \\ 0 & 0 & \lambda_i & 1 & 0 \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

with possibly some blocks of size 1×1 being just one number λ_i . The sum of sizes of blocks is equal to N.

We use the expression

$$\exp(At) = T \exp(Jt) T^{-1}$$

that reduces analysis of the boundedness and limits of the norm $\|\exp(At)\|$ to the similar analysis for the matrix $\exp(Jt)$ because for two matrices A and B the estimate $\|AB\| \leq \|A\| \, \|B\|$ and therefore

$$\|\exp(At)\| \leq \|T\| \left\|T^{-1}\right\| \|\exp(Jt)\|$$

For $\exp(Jt)$ we have the following explicit expression in terms of eigenvalues and their algebraic and geometric multiplicities:

$$\exp(Jt) = \begin{bmatrix} \exp(J_1 t) & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \exp(J_2 t) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \exp(J_{p-1} t) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \exp(J_p t) \end{bmatrix} \qquad (21)$$

where for example the block of size 5×5 looks as

$$\exp(J_k t) = \exp(\lambda_i t) \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \frac{t^4}{4!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

For a block of the size 1×1 we will get $\exp(J_k t) = \exp(\lambda_i t)$. If an eigenvalue λ_i is semisimple, that means it has the number of linearly independent eigenvectors (geometric multiplicity) $r(\lambda_i)$ equal to the algebraic multiplicity $m(\lambda_i)$ of λ_i. In this case all blocks corresponding to this eigenvalue and corresponding blocks in the exponent $\exp(Jt)$ all have size 1×1 and have this form $\exp(J_k t) = \exp(\lambda_i t)$.

Matrices $N \times N$ build a finite dimentional linear space with dimension $N \times N$. All norms in a finite dimensional space are equivalent. It means that for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in the space of matrices, there are constants C_1, $C_2 > 0$ such that for any matrix A

$$C_1 \|A\|_1 \leq \|A\|_2 \leq C_2 \|A\|_1$$

It is easy to observe that the expression $\max_{i,j=1\ldots N} |A_{ij}| = \|A\|_{\max}$ is a norm in the space of matrices and therefore can be used instead of the standart eucledian norm. There are constants B_1 and $B_2 > 0$ such that

$$B_1 \|A\|_{\max} \leq \|A\| \leq B_2 \|A\|_{\max}$$

I makes that to show the boundedness of the matrix norm $\|\exp(Jt)\|$ for $\exp(Jt)$, it is enough to show boundedness of all elements in $\exp(Jt)$. Similarly, to show that $\|\exp(Jt)\| \to 0$ when $t \to \infty$ it is enough to show that all elements in $\exp(Jt)$ go to zero when $t \to \infty$

To prove the statements in the theorem we need just to check how elements *in the explicit expressions (22) for blocks in* $\exp(Jt)$ *see (21), behave depending on the maximum of the real part of eigenvalues:* $\max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ *and check situations when blocks of size* 1×1 *not including powers* t^p *can appear.*

• We observe in *(22)* that all elements in $\exp(Jt)$ have the form: $\exp(\lambda_i t)$ or $C\exp(\lambda_i t)t^p$ with some constants $C > 0$ and some $p > 0$ with possibly similar λ_i in different blocks.

• Absolute values of the elements in $\exp(Jt)$ have the form: $\exp((\operatorname{Re}\lambda_i)\, t)$ or $C\exp((\operatorname{Re}\lambda_i)\, t)t^p$ where all $\operatorname{Re}\lambda_i \leq \mu_A$. because $|\exp(i \operatorname{Im}\lambda_j)|$ according to the Euler formula.

We prove first sufficiency of the conditions in the statement **1.** for the formulated conclusions.

1. If $\mu_A < 0$ then maximum of absolute values of all elements $[\exp(Jt)]_{ij}$ in $\exp(Jt)$ satisfy the inequality

$$\max_{i,j}\left|[\exp(Jt)]_{ij}\right| \leq M\exp\left[(\mu_A + \delta)t\right] \underset{t\to\infty}{\longrightarrow} 0$$

and tends to zero exponentially for some constant $M > 0$ and δ so small that $-\beta = \mu_A + \delta < 0$. It follows because

$$\begin{aligned}\exp(\operatorname{Re}\lambda_i t)t^p &\leq \exp(\mu_A t)t^p = \exp\left[(\mu_A + \delta - \delta)\, t\right] t^p \\ &= \exp\left[(\mu_A + \delta)\, t\right] \underbrace{\left(t^p \exp\left[-\delta t\right]\right)}_{\leq M} \leq M\exp\left[-\beta t\right]\end{aligned}$$

Therefore $\|\exp(Jt)\| \leq M_\beta \exp\left[-\beta t\right] \underset{t\to\infty}{\longrightarrow} 0$ with another constant M_β and therefore $\|\exp(At)\| \leq (\|T\|\,\|T^{-1}\|\, M_\beta)\exp\left[-\beta t\right]$ decays exponentially.

Now we prove the sufficiency of the conditions in the statement **2.** for the formulated conclusion.

2. The definition of the matrix norm implies immediately that if $\mu_A < 0$

then by the result for the matrix norm $\|\exp(At)\|$ that $\lim_{t\to\infty} \|\exp(At)\xi\| \leq \|\xi\| \lim_{t\to\infty} \|\exp(At)\| = 0$ for every $\xi \in \mathbb{C}^N$.

Now we prove the sufficiency and necessity of the conditions in the statement **3.** for the uniform boundedness of the transition matrix $\exp(At)$: $\sup_{t\geq 0} \|\exp(At)\| < \infty$.

3. If $\mu_A = 0$ and then there are purely imaginary or zero eigenvalues λ. Then elements in the blocks of $\exp(Jt)$ corresponding to purely imaginary or zero eigenvalues will have the form $\exp(i \operatorname{Im} \lambda_i t)$ or $C \exp(i \operatorname{Im} \lambda_i t) t^p$. The absolute values of these elements will be 1 or Ct^p because $|\exp(i \operatorname{Im} \lambda_i t)| = 1$. Therefore absolute values of these elements will be bounded if and only if corresponding blocks are of size 1×1 and therefore elements Ct^p with powers of t are not present. This situation takes place if and only if purely imaginary and zero eigenvalues are **semisimple** (have geometric and algebraic multiplicities equal: $m(\lambda) = g(\lambda)$). Elements in $\exp(Jt)$ in the blocks corresponding to eigenvalues with negative real parts will be exponentially decreasing by the arguments in the proof of statement **1.**

   Finally we prove necessity of the condition in the Statement **1.** We observe that if $\mu_A = 0$ then referring to the analysis in **3.** absolute values of the elements corresponding to purely imaginary or zero λ_i in $\exp(Jt)$ are be bounded in the case if the conditions in **3.** are satified, or otherwise they have the form Ct^p and go to infinity when $t \to \infty$. Therefore the norm $\|\exp(At)\|$ does not decay exponentially in this case. If $\mu_A > 0$ the matrix $\exp(Jt)$ will include terms that are exponentially rising and the norm $\|\exp(At)\|$ can not decay exponentially in this case.

   The nessecity of the conditions in the statement **2** follows from the behaviour of the elements in $\exp(Jt)$ considered before or from the formula for general solution to the linear autonomous system.

The condition $\mu_A \geq 0$ means that there are eigenvalues λ with real part $\operatorname{Re}\lambda$ positive or zero. In the first case choosing vector ξ equal to a generalized eigenvector or an eigenvector corresponding to λ_i with $\operatorname{Re}\lambda_i > 0$ we get a solution $\exp(At)\xi$ represented as a sum with terms including exponents $\exp(\lambda_i t)$ such that $|\exp(\lambda_i t)| = |\exp(\operatorname{Re}\lambda_i t)| \to \infty$. In the second case there are eigenvalues $\lambda_i = i \operatorname{Im}\lambda_i$. Choosing ξ equal to one of corresponding generalized eigenvectors we obtain a solution $\exp(At)\xi$ represented as a sum including terms with constant absolute value or an absulute value that rises as some power t^p with $t \to \infty$. It implies the necessity of conditions in **2**. for having $\lim_{t\to\infty} \exp(At)\xi = 0$ for every $\xi \in \mathbb{C}^N$.■

The proof of the Corollary 2.13 in the book uses the explicit expression of solutions that we discussed at the beginning of this chapter of lecture notes and is a bit more complicated.

10 Definition of stable equilibrium points.

Definition. A point $x_* \in G$ is called an equilibrium point to the equation $x' = f(x)$ if $f(x_*) = 0$.

The corresponding solution $x(t) \equiv x_*$ is called an equilibrium solution.

Definition. (5.1, p. 169, L.R.)

The equilibrium point x_* is said to be stable if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any **maximal solution** $x : I \to G$ to the I.V.P.

$$\begin{aligned} x' &= f(x) \\ x(0) &= \xi \end{aligned}$$

such that $0 \in I$ and $\|x(0) - x_*\| \leq \delta$ we have $\|x(t) - x_*\| \leq \varepsilon$ for any $t \in I \cap \mathbb{R}_+$ for all "future times".

Below a picture is given in the case $x_* = 0$.

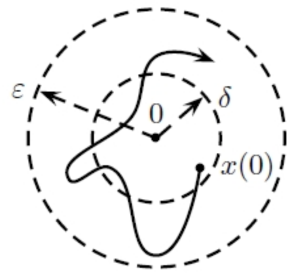


Figure 5.1 Stable equilibrium

Definition. (5.14, p. 182, L.R.)
The equilibrium point x_* of (**??**) is said to be *attractive* if there is $\delta > 0$ such that for every $\xi \in G$ with $\|\xi - x_*\| \leq \delta$ the following properties hold: the solution $x(t) = \varphi(t, \xi)$ to I.V.P. with $x(0) = \xi$ exists on $\mathbb{R}_+$and $\varphi(t, \xi) \to x_*$ as $t \to \infty$.

Definition. We say that the equilibrium x_* is **asymptotically stable** if it is both stable and attractive.

In the analysis of stability we will always choose a system of coordinates so that the origin coinsides with the equilibrium point. In the course book this agreement is applied even in the definition of stability.

Definition. The equilibrium point x_* is said to be *unstable* if it is not stable. It means that there is a $\varepsilon_0 > 0$, such that for any $\delta > 0$ there is point $x(0) : \|x(0) - x_*\| \leq \delta$ such that for some $t_0 \in I$ we have $\|x(t_0) - x_*\| > \varepsilon_0$.(a formal negation to the definition of stability)

11 Classification of phase portraits of autonomous linear systems in the plane.

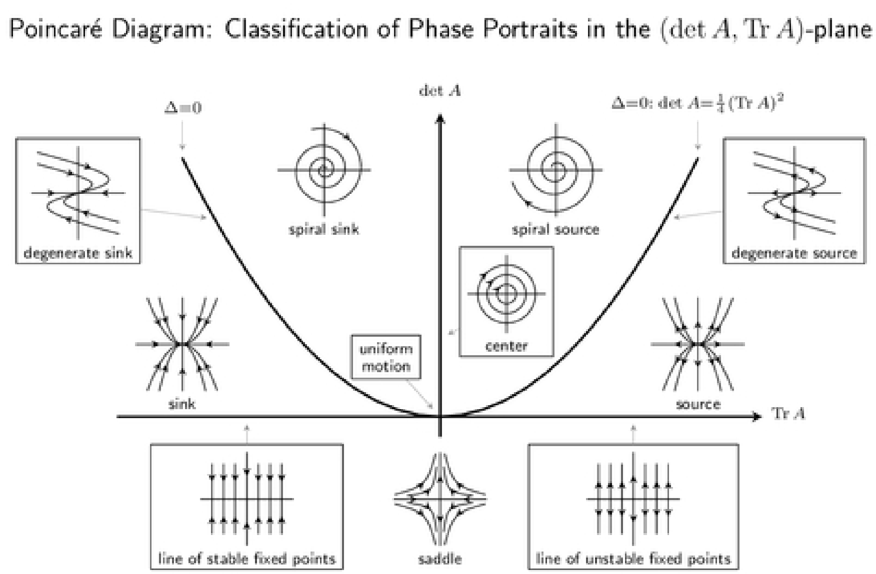
Characteristic polynomial for a 2×2 matrix A is

$$p(\lambda) = \lambda^2 - \lambda \mathrm{Tr} A + \det A$$

Eigenvalues are:

$$\lambda_{1,2} = \frac{\mathrm{Tr}A}{2} \pm \sqrt{\frac{(\mathrm{Tr}A)^2}{4} - \det A}$$

The line $\det A = \frac{(\mathrm{Tr}A)^2}{4}$ separates points in the plane $(\mathrm{Tr}A, \det A)$ corresponding to real and complex eigenvalues of the matrix A.
For $\mathrm{Tr}A, \det A$ in the first and second quadrants in the plane $(\mathrm{Tr}A, \det A)$ both $\mathrm{Re}\,\lambda_{1,2}$ are correspondingly positive and negative.
In the half plane where $\det A < 0$ eigenvalues $\lambda_{1,2}$ are real but have different signs.
These observations imply the following classification of phase portraits for linear autonomous systems in plane.



A classification of phase portraits for non-degenerate linear autonomous systems in plane in terms of the determinant and the trace of the matrix A.

Stable (unstable) nodes when eigenvalues λ_1, λ_2 are real, different, negative (positive). $\det(A) < \frac{1}{4}(tr(A))^2$; $\det(A) > 0$; $tr(A) < 0$, ($tr(A) > 0$).

Saddle (always unstable) when eigenvalues λ_1, λ_2 are real, with different signs. $\det(A) < 0$.

Stable (unstable) focus - spiral when λ_1, λ_2 are complex, with negative (positive) real parts. $\det(A) > \frac{1}{4}(tr(A))^2 \neq 0$, $tr(A) < 0$ ($tr(A) > 0$).

Stable (unstable) improper - degenerate node when eigenvalue λ_1 is real negative (positive) with multiplicity 2 having only one linearly independent eigenvector. $\det(A) = \frac{1}{4}(tr(A))^2$, $tr(A) < 0$ ($tr(A) > 0$).

Center (stable but not asymptotically stable) when λ_1, λ_2 are complex purely imaginary. $tr(A) = 0$; $\det(A) > 0$

Stable (unstable) star, when eigenvalue λ_1 is real negative (positive) with multiplicity 2 as for improper node, but having two linearly independent eigenvectors (diagonal matrix A)

□

Example.

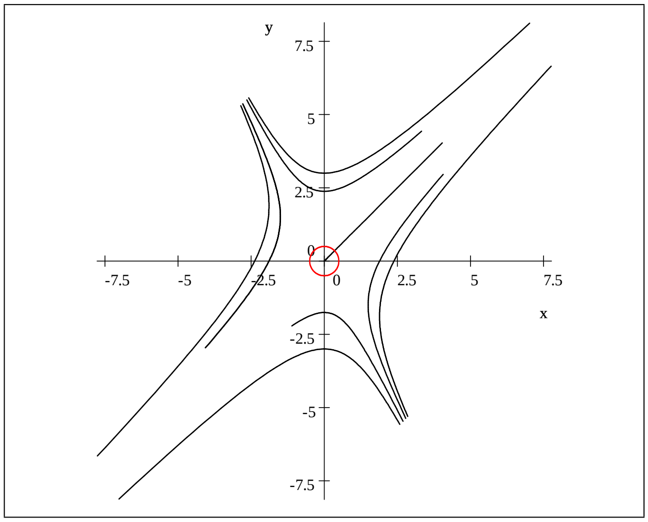
An example on instability: saddle point. There are trajectories (not all) that leave a neighbourhood $\|x\| < d$ of the origin for initial conditions ξ arbitrary close to the origin: for any $\varepsilon > 0$ and $0 < \|\xi\| \leq \varepsilon$ after some time T_ε.

$$r' = Ar \text{ with } A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \text{ characteristic polynomial: } \lambda^2 - \lambda - 2 = 0;$$

$$\text{eigenvectors: } \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \leftrightarrow \lambda_1 = -1, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 2$$

$$r = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ - the general solution}$$

choosing a ball $\|x\| \leq 1$, and for arbitrary $\varepsilon > 0$, $\xi = \varepsilon \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\|\xi\|$ we see that the corresponding solution $x(t) = e^{2t}\varepsilon \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ will leave this ball $\|x\| \leq 1$,after time $2T_\varepsilon = -\ln \varepsilon$.



Exercise.

Consider the following system of equations:

$$\begin{cases} x' = & 2y - x \\ y' = & 3x - 2y \end{cases}$$

1. a) can the system have a trajectory going from the point $(-a^2-1, -1)$ to the point $(1, a^2+1)$?

   b) which type of fixed point is the origin?

   c) draw a sketch of the phase portrait. **(4p)**

Solution

Matrix of the system is $A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$. Characteristic polynomila is $\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} = \lambda^2 + 3\lambda - 4$. Eigenvalues and eigenvectors are $\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$, eigenvalues: $\lambda_1 = -4, \lambda_2 = 1$.

Eigenvectors $v_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \leftrightarrow \lambda_1 = -4;$ satisfies the equation

$(A - \lambda_1)\, v_1 = 0$ with $(A - \lambda_1) = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 1$,satisfies the equation $(A - \lambda_2)\, v_2 = 0$ with $(A - \lambda_2) =$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$$

Origin is a saddle point and is unstable. Trajectories are hyperbolas asymptotically approaching with $t \to \infty$ or $t \to -\infty$ trajectories L_1, L_2, L_3, L_4, that are straight lines through the origin and are parallell to the eigenvectors.

Checking points $(-a^2 - 1, -1)$ and $(1, a^2 + 1)$ we observe that they are separated by the above mentioned straight trajectories L_1, L_2, L_3, L_4. Therefore no one trajectory can go between these two points because such a trajectory should cross one of L_1, L_2, L_3, L_4 that is impossible because of the uniquness of solutions to linear systems.■

Exercise 868. Exponent of a matrix with complex eigenvalues.

Calculate $\exp(A)$ for the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; with eigenvalues $\pm i$.

The set of matrices of the structure $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *have the same properties with respect to matrix multiplication and addition as complex numbers of the form* $a + ib$.

In particular matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ behave as real numbers and matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ behave as imaginary unit i.

We check that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$

and $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}$ and observe that the diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ commute.

It makes that we can apply the Euler formula!!!!!!

$$\exp(a + ib) = \exp(a)(\cos(b) + i\sin(b))$$

for computing the exponentof a matrix of such structure:

$$\exp\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \exp\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right)\exp\left(\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right) =$$

$$\exp(a)I\left[\cos(b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(b)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right] = \exp(a)\begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

$$\exp\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \exp(a)\begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

It implies immediately that

$$\exp(A) = \exp\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$$

Example of a stable but NOT asymptotically stable equilibrium point.

Consider the system $x'(t) = Ax(t)$ with $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Eigenvalues of the matrix A are $\lambda = \pm 2i$ are purely imaginary (and non-zero). Therefore there are no other equilibrium points except the origin. The $\exp(At) = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$. The solution to the initial value problem with initial data $[\xi_1, \xi_2]^T$ is

$$x(t) = \begin{bmatrix} \xi_1 \cos(2t) - \xi_2 \sin(2t) \\ \xi_1 \sin(2t) + \xi_2 \cos(2t) \end{bmatrix} = |\xi| \begin{bmatrix} \frac{\xi_1}{|\xi|} \cos(2t) - \frac{\xi_2}{|\xi|} \sin(2t) \\ \frac{\xi_1}{|\xi|} \sin(2t) + \frac{\xi_2}{|\xi|} \cos(2t) \end{bmatrix} =$$

$$= |\xi| \begin{bmatrix} \cos(\theta)\cos(2t) - \sin(\theta)\sin(2t) \\ \cos(\theta)\sin(2t) + \sin(\theta)\cos(2t) \end{bmatrix} = |\xi| \begin{bmatrix} \cos(\theta + 2t) \\ \sin(\theta + 2t) \end{bmatrix}$$

with $\cos(\theta) = \frac{\xi_1}{|\xi|}$. Therefore orbits of solutions are circles around the origin with the radius equal to $|\xi|$. It implies that the equlibrium point in the origin is stable. $\delta_\varepsilon > 0$ in the definition of stability can be chosen equal to $\varepsilon > 0$.■

Exercise.

Calculate $\exp(At)$ for the constant matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ and sketch phase portrait for the system $x' = Ax$.

Solution.

$\exp(At)$ is a **fundamental matrix** to the system of differential equations $x' = Ax$. It means that columns in $\exp(At)$ are solutions to the system above with initial data $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The plan is to find first the general solution, and then these two particular solutions.

The characteristic polynom for A is $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $X^2 - 3X + 2 = (X-1)(X-2) = 0$, so eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$. Eigenvectors are

$$v_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \leftrightarrow \lambda_1;\ v_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \lambda_2$$

General solution is $x(t) = C_1 v_1 e^t + C_2 v_2 e^{2t}$. To satisfy the initial data

$$x(0) = C_1 v_1 e^t + C_2 v_2 e^{2t} = e_1$$

we solve a system of two equations for C_1 and C_2:

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or in matrix form } \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow C_1 = -1$ and $C_2 = 2$. Therefore the first columnt in $\exp(At)$ is: $-v_1 e^t + 2v_2 e^{2t} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^t + 2e^{2t} \\ -2e^t + 2e^{2t} \end{bmatrix}$

Similarly we find the second column:

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Longrightarrow C_1 = 1$ and $C_2 = -1$.

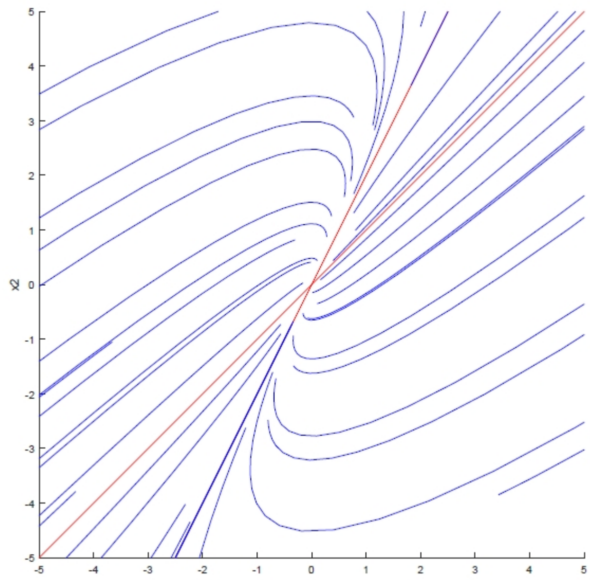
The second column in $\exp(At)$ is: $v_1 e^t - v_2 e^{2t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^t - e^{2t} \\ 2e^t - e^{2t} \end{bmatrix}$

and finally $\exp(At) = \begin{bmatrix} -e^t + 2e^{2t} & e^t - e^{2t} \\ -2e^t + 2e^{2t} & 2e^t - e^{2t} \end{bmatrix}$

An alternative but more complicated solution would be to represent $\exp(At)$ as $\exp(At) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}$, where the matrix P has columns of eigenvectors: $P = (v_1, v_2) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and the inversion of P can be calculated by Cramer's formulas: $P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$. We derive the final expression by multiplication of the three matrices:

$$\exp(At) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} e^t & e^{2t} \\ 2e^t & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{2t} & e^t - e^{2t} \\ -2e^t + 2e^{2t} & 2e^t - e^{2t} \end{bmatrix}$$



11.1 A general way to calculate exponents of matrices. (particularly useful for matrices having complex eigenvalues)

We use here general solution to the equation $x' = Ax$.

We clarify first in which way it can be used.

- For any matrix B the product Be_k gives the column k in the matrix B.
- Therefore the column k in $\exp(A)$ is the product $\exp(A)e_k$, where vector e_k is a standard basis vector, or colum with index k from the unit matrix I.
- On the other hand $\exp(At)\xi$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = \xi$

- The expressions $x_k(t) = \exp(At)e_k$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = e_k$
- Therefore the value of the solution in time $t = 1$: $x_k(1) = \exp(A)e_k$ gives the column k in the matrix $\exp(A)$
- Having the general solution for example in the case of dimension 3:

$$x(t) = C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t)$$

  in terms of linearly independent solutions $\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$, we can for every k find a set of constants $C_{1,k}$,$C_{2,k}$,$C_{3,k}$, corresponding to each of the initial data e_k. Namely we solve equations $C_{1,k}\Psi_1(0)+C_{2,k}\Psi_2(0)+C_{3,k}\Psi_3(0) = e_k$, $\quad k = 1, 2, 3$
- that are equivalent to the matrix equation

$$[\Psi_1(0), \Psi_2(0), \Psi_3(0)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [e_1, e_2, e_3] = I$$

- Values at $t = 1$ of corresponding solutions:

$$x_k(1) = C_{1,k}\Psi_1(1) + C_{2,k}\Psi_2(1) + C_{3,k}\Psi_3(1) = \exp(1 \cdot A)e_k$$

  will give us columns $\exp(1 \cdot A)e_k$ in $\exp(A)$.
- In the matrix form this result can be expressed as

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1}$$

$$\begin{aligned}
\exp(A) &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \\
&= [\Psi_1(1), \Psi_2(1), \Psi_3(1)]\,[\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1}
\end{aligned}$$

We demonstrate this idea using the result on the general solution from the problem 859.

We can calculate $\exp\left(\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}\right)$, eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1 - i$, $\lambda_3 = 1 + i$

General solution to the system $x' = Ax$ is:

$$\begin{aligned}
x(t) &= C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t) \\
&= C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^{t} \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix}
\end{aligned}$$

introducing shorter notations for each term:

$x(t) = C_1\Psi_1(t) + C_2\Psi_3(t) + C_3\Psi_3(t)$.

We calculate initial data for arbitrary solution by

$$x(0) = C_1\Psi_1(0) + C_2\Psi_3(0) + C_3\Psi_3(0) = C_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$x(0) = [\Psi_1(0), \Psi_3(0), \Psi_3(0)] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$\exp(A)$ has columns that are values of $x(1)$ for solutions that satisfy initial conditions $r(0) = e_1,\ e_2, e_3$ and therefore

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1;$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2;$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3;$$

We solve all three of these systems for $\begin{bmatrix} C_{1,k} \\ C_{2,k} \\ C_{3,k} \end{bmatrix}$ in one step as a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = I$$

It is equivalent to the Gauss elimination of the following extended matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$. The result at the rigth half will be the inverted matrix:

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It can also found by applying Cramer's rule.

We arrive to the expression of the matrix exponent by collecting these results through the matrix multiplication:

$$\exp(At) = [\Psi_1(t), \Psi_2(t), \Psi_3(t)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}$$

$$\exp(At) = \begin{bmatrix} e^{-t} & e^t(\cos t - \sin t) & e^t(\cos t + \sin t) \\ e^{-t} & e^t \cos t & e^t \sin t \\ -e^{-t} & e^t \sin t & -e^t \cos t \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^t(\cos t + \sin t) - e^{-t} + e^t(\cos t - \sin t) & -e^t(\cos t + \sin t) + e^{-t} & -e^{-t} + e^t(\cos t - \sin t) \\ (\cos t)\, e^t + (\sin t)\, e^t - e^{-t} & -(\sin t)\, e^t + e^{-t} & (\cos t)\, e^t - e^{-t} \\ -(\cos t)\, e^t + (\sin t)\, e^t + e^{-t} & (\cos t)\, e^t - e^{-t} & (\sin t)\, e^t + e^{-t} \end{bmatrix}$$

and finally for $t = 1$ we get $\exp(A)$

$$\exp(A) = e \begin{bmatrix} (\cos 1 + \sin 1) - e^{-2} + (\cos 1 - \sin 1) & -(\cos 1 + \sin 1) + e^{-2} & -e^{-2} + (\cos 1 - \sin 1) \\ (\cos 1) + (\sin 1) - e^{-2} & -(\sin 1) + e^{-2} & (\cos 1) - e^{-2} \\ -(\cos 1) + (\sin 1) + e^{-2} & (\cos 1) - e^{-2} & (\sin 1) + e^{-2} \end{bmatrix}$$

A general way to calculate exponents of matrices. (particularly useful for matrices having complex eigenvalues)

We use here general solution to the equation $x' = Ax$.

We clarify first in which way it can be used.

- For any matrix B the product Be_k gives the column k in the matrix B.
- Therefore the column k in $\exp(A)$ is the product $\exp(A)e_k$, where vector e_k is a standard basis vector, or colum with index k from the unit matrix I.
- On the other hand $\exp(At)\xi$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = \xi$
- The expressions $x_k(t) = \exp(At)e_k$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = e_k$
- Therefore the value of the solution in time $t = 1$: $x_k(1) = \exp(A)e_k$ gives the column k in the matrix $\exp(A)$
- Having the general solution for example in the case of dimension 3:

  $$x(t) = C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t)$$

  in terms of linearly independent solutions $\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$, we can for every k find a set of constants $C_{1,k}$,$C_{2,k}$,$C_{3,k}$, corresponding to each of the initial data e_k. Namely we solve equations $C_{1,k}\Psi_1(0)+C_{2,k}\Psi_2(0)+C_{3,k}\Psi_3(0) = e_k\ , \quad k = 1, 2, 3$
- that are equivalent to the matrix equation

  $$[\Psi_1(0), \Psi_2(0), \Psi_3(0)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [e_1, e_2, e_3] = I$$

- Values at $t = 1$ of corresponding solutions:

  $$x_k(1) = C_{1,k}\Psi_1(1) + C_{2,k}\Psi_2(1) + C_{3,k}\Psi_3(1) = \exp(1 \cdot A)e_k$$

  will give us columns $\exp(1 \cdot A)e_k$ in $\exp(A)$.

- In the matrix form this result can be expressed as

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1}$$

$$\begin{aligned} \exp(A) &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \\ &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)] \, [\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1} \end{aligned}$$

We demonstrate this idea using the result on the general solution from the problem 859.

We can calculate $\exp\left(\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}\right)$, eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1 - i$, $\lambda_3 = 1 + i$

General solution to the system $x' = Ax$ is:

$$\begin{aligned} x(t) &= C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t) \\ &= C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^{t} \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix} \end{aligned}$$

introducing shorter notations for each term: $x(t) = C_1\Psi_1(t) + C_2\Psi_3(t) + C_3\Psi_3(t)$.

We calculate initial data for arbitrary solution by

$$x(0) = C_1\Psi_1(0)+C_2\Psi_3(0)+C_3\Psi_3(0)=C_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$x(0) = [\Psi_1(0), \Psi_3(0), \Psi_3(0)] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$\exp(A)$ has columns that are values of $x(1)$ for solutions that satisfy initial conditions $r(0) = e_1,\ e_2, e_3$ and therefore $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \end{bmatrix} =$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3;$$

We solve all three of these systems for $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ in one step as a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = I$$

It is equivalent to the Gauss elimination of the following extended matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$. The result at the rigth half will be the inverted matrix:

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It can also found by applying Cramer's rule.

We arrive to the expression of the matrix exponent by collecting these results through the matrix multiplication:

$$\exp(At) = [\Psi_1(t), \Psi_2(t), \Psi_3(t)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}$$

$$\exp(At) = \begin{bmatrix} e^{-t} & e^t(\cos t - \sin t) & e^t(\cos t + \sin t) \\ e^{-t} & e^t \cos t & e^t \sin t \\ -e^{-t} & e^t \sin t & -e^t \cos t \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^t(\cos t + \sin t) - e^{-t} + e^t(\cos t - \sin t) & -e^t(\cos t + \sin t) + e^{-t} & -e^{-t} + e^t(\cos t - \sin t) \\ (\cos t)e^t + (\sin t)e^t - e^{-t} & -(\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} \\ -(\cos t)e^t + (\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} & (\sin t)e^t + e^{-t} \end{bmatrix}$$

and finally for $t = 1$ we get $\exp(A)$

$$\exp(A) = e \begin{bmatrix} (\cos 1 + \sin 1) - e^{-2} + (\cos 1 - \sin 1) & -(\cos 1 + \sin 1) + e^{-2} & -e^{-2} + (\cos 1 - \sin 1) \\ (\cos 1) + (\sin 1) - e^{-2} & -(\sin 1) + e^{-2} & (\cos 1) - e^{-2} \\ -(\cos 1) + (\sin 1) + e^{-2} & (\cos 1) - e^{-2} & (\sin 1) + e^{-2} \end{bmatrix}$$

n	$f(z)$	R–H criterion
2	$a_0z^2+a_1z+a_2$	$a_2>0,\ a_1>0$
3	$a_0z^3+a_1z^2+a_2z+a_3$	$a_3>0,\ a_1>0$ $a_1a_2>a_0a_3$
4	$a_0z^4+a_1z^3+a_2z^2+a_3z+a_4$	$a_4>0,\ a_2>0,$ $a_1>0,$ $a_3(a_1a_2-a_0a_3)>a_1^2a_4$

3.4 Two-Dimensional Linear Autonomous Systems

In this section we shall apply Theorem 3.3.6 to classify the behavior of the solutions of two-dimensional linear systems [H1]

$$\dot{x}=Ax,\ A=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\ \det A\neq 0 \tag{3.10}$$

where a,b,c,d are real constants. Then $(0,0)$ is the unique rest point of (3.10). Let λ_1,λ_2 be the eigenvalues of A, consider the following cases:

Case 1: λ_1,λ_2 are real and $\lambda_2<\lambda_1$.
Let v^1,v^2 be unit eigenvectors of A associated with λ_1,λ_2 respectively. Then from (3.9), the general real solution of (3.10) is

$$x(t)=c_1e^{\lambda_1 t}v^1+c_2e^{\lambda_2 t}v^2.$$

Case 1a (Stable node) $\lambda_2<\lambda_1<0$.
Let L_1,L_2 be the lines generated by v^1,v^2 respectively. Since $\lambda_2<\lambda_1<0$, $x(t)\approx c_1e^{\lambda_1 t}v^1$ as $t\to\infty$ and the trajectories are tangent to L_1. The origin is a stable node (see Fig. 3.1).

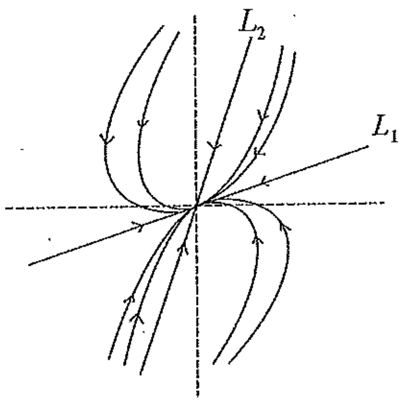


Fig. 3.1

Case 1b (Unstable node) $0 < \lambda_2 < \lambda_1$.

Then $x(t) \approx c_1 e^{\lambda_1 t} v^1$ as $t \to \infty$. The origin is an unstable node (see Fig. 3.2).

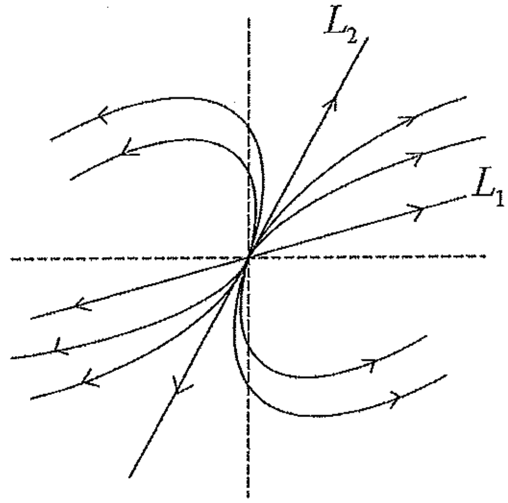


Fig. 3.2

Case 1c (Saddle point) $\lambda_2 < 0 < \lambda_1$. In this case, the origin is called a saddle point and L_1, L_2 are called unstable manifold and stable manifold of the rest point $(0, 0)$ respectively (see Fig. 3.3).

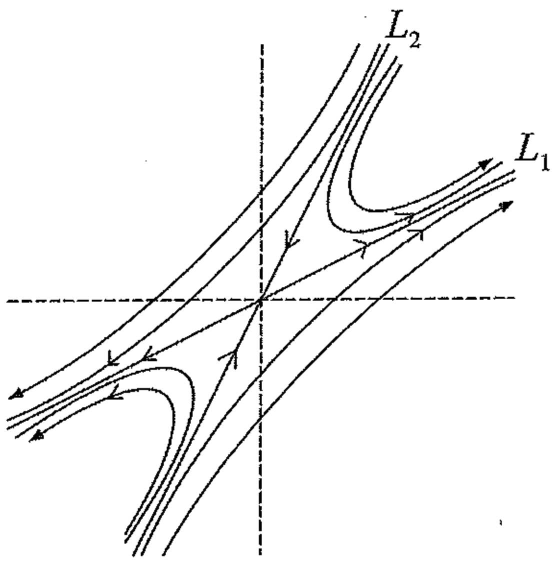


Fig. 3.3

Case 2: λ_1, λ_2 are complex.

Let $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and $v^1 = u + iv$ and $v^2 = u - iv$ be complex eigenvectors. Then

$$x(t) = ce^{(\alpha+i\beta)t}v^1 + \bar{c}e^{(\alpha-i\beta)t}\overline{v^1} = 2Re\ \left(ce^{(\alpha+i\beta)t}v^1\right).$$

Let $c = ae^{i\delta}$. Then

$$x(t) = 2ae^{\alpha t}\left(u\cos(\beta t+\delta) - v\sin(\beta t+\delta)\right).$$

Let U and V be the lines generated by u, v respectively.

Case 2a (Center) $\alpha = 0,\ \beta \neq 0$. The origin is called a center (see Fig. 3.4).

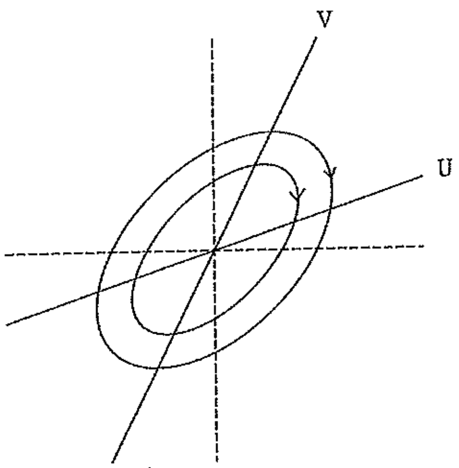


Fig. 3.4

Case 2b (Stable focus, spiral) $\alpha < 0,\ \beta \neq 0$. The origin is called a stable focus or stable spiral (see Fig. 3.5).

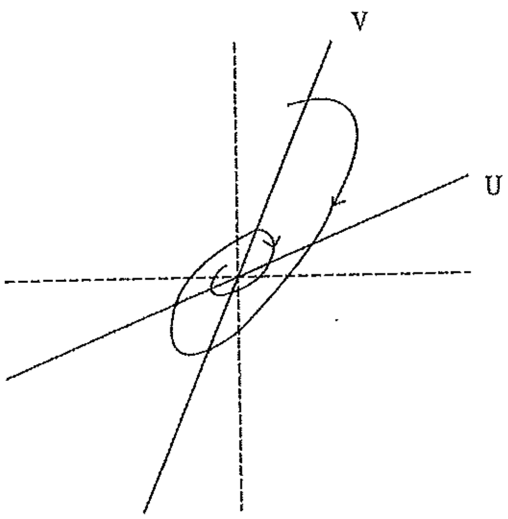


Fig. 3.5

Case 2c (Unstable focus, spiral) $\alpha > 0$, $\beta \neq 0$. The origin is called an unstable focus or unstable spiral (see Fig. 3.6).

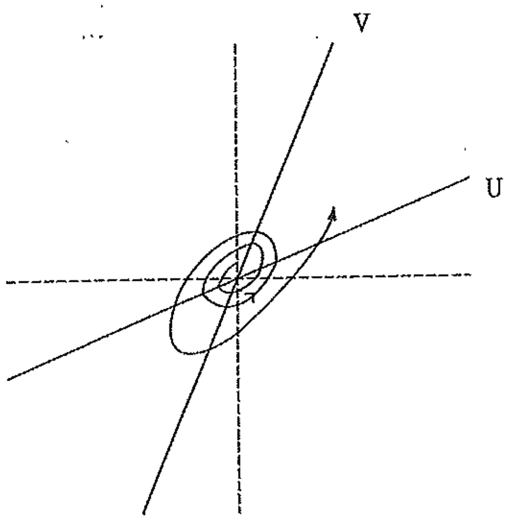


Fig. 3.6

Case 3 (Improper nodes) $\lambda_1 = \lambda_2 = \lambda$

Case 3a: There are two linearly independent eigenvectors v^1 and v^2 of the eigenvalue λ. Then,

$$x(t) = \left(c_1 v^1 + c_2 v^2\right) e^{\lambda t}.$$

If $\lambda > 0$ ($\lambda < 0$) then the origin 0 is called an unstable (stable) improper node (see Fig. 3.7).

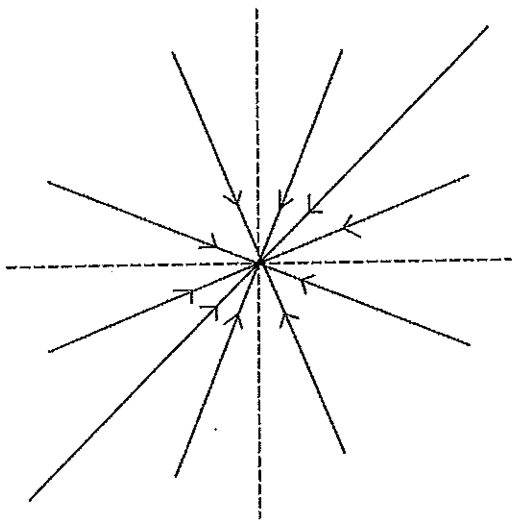


Fig. 3.7

Case 3b: There is only one eigenvector v^1 associated with eigenvalue λ. Then from (3.9). v^2 - generalized eigenvector.

$$x(t) = (c_1 + c_2 t) e^{\lambda t} v^1 + c_2 e^{\lambda t} v^2$$

where v^2 is ~~any~~ vector independent of v^1 (see Fig. 3.8).

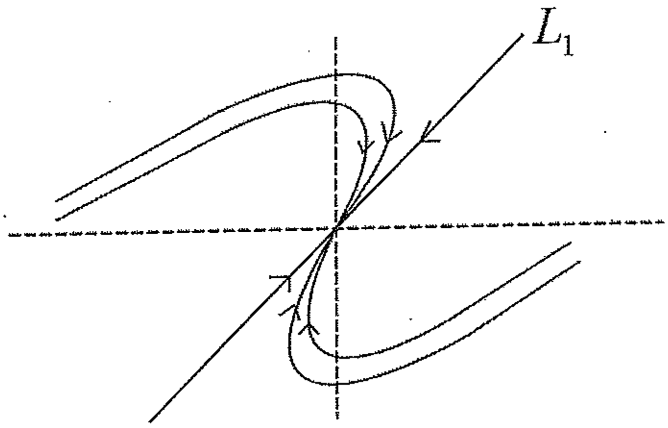


Fig. 3.8

April 25, 2020

1 Stability of equilibrium points by linearization.

We consider in this chapter of the course properties of solutions of I.V.P to nonlinear autonomous systems of ODEs

$$x' = f(x), \quad x(0) = \xi \tag{1}$$

where $f : G \to \mathbb{R}^N$ is locally Lipschitz with respect to x. J is and interval and $G \subset \mathbb{R}^N$ is a non-empty open set.

We will consider in this chapter of the course the stability of equilibrium points x_* of such nonlinear systems ($f(x_*) = 0$) in connection with properties of corresponding linearized systems in the form

$$y'(t) = Ay \tag{2}$$

where A is a Jacoby matrix of the function f calculated in an equilibrium point of interest.

Definition. (p. 115, L.R.) A function f is called locally Lipschitz in G if for any point $y \in G$ there is a neighborhood $V(y)$ and a number $L > 0$

(depending on $V(y)$) such that for any $v, w \in V(y)$

$$\|f(v) - f(w)\| \leq L \|v - w\|$$

Example. Functions having continuous partial derivatives are locally Lipschitz function. (Exercise)

Definition. A solution $x(t) : I \to \mathbb{R}^N$ is called **maximal solution** to an I.V.P. if it cannot be extended to a larger time interval.

1.1 Peano existence theorem.

The theorem by Peano, states that if $f : G \to \mathbb{R}^N$ is continuous, the the I.V.P. (1) above has a solution (not unique!!!) for any $\xi \in G$ on some, might be small time interval $(-\delta, \delta)$. (Theorems 4.2, p. 102;)

We will consider Peano theorem it at the end of the course.

1.2 Picard and Lindelöf's existence and uniqueness theorem.

The theorem by Picard and Lindelöf, states that if $f : G \to \mathbb{R}^N$ is locally Lipschitz, then the I.V.P. (1) above has a unique solution for any $\xi \in G$ on some, might be small time interval $(-\delta, \delta)$. (Theorems 4.17, p. 118; Theorem 4.22, p.122.)

We will formulate it in a more general form and will prove it at the end of the course.

1.3 Definition of stable equilibrium points (repetition).

Definition. A point $x_* \in G$ is called an equilibrium point to the equation (1) if $f(x_*) = 0$.

The corresponding solution $x(t) \equiv x_*$ is called an equilibrium solution.

Definition. (5.1, p. 169, L.R.)

The equilibrium point x_* is said to be stable if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any maximal solution $x : I \to G$ to (1) such that $0 \in I$ and $\|x(0) - x_*\| \leq \delta$ we have $\|x(t) - x_*\| \leq \varepsilon$ for any $t \in I \cap \mathbb{R}_+$. Below a picture is given in the case $x_* = 0$.

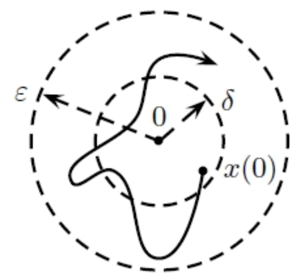


Figure 5.1 Stable equilibrium

Definition. (5.14, p. 182, L.R.)

The equilibrium point x_* of (1) is said to be *attractive* if there is $\delta > 0$ such that for every $\xi \in G$ with $\|\xi - x_*\| \leq \delta$ the following properties hold: the solution $x(t) = \varphi(t, \xi)$ to I.V.P. with $x(0) = \xi$ exists on $\mathbb{R}_+$and $\varphi(t, \xi) \to x_*$ as $t \to \infty$.

Definition. We say that the equilibrium x_* is **asymptotically stable** if it is both stable and attractive.

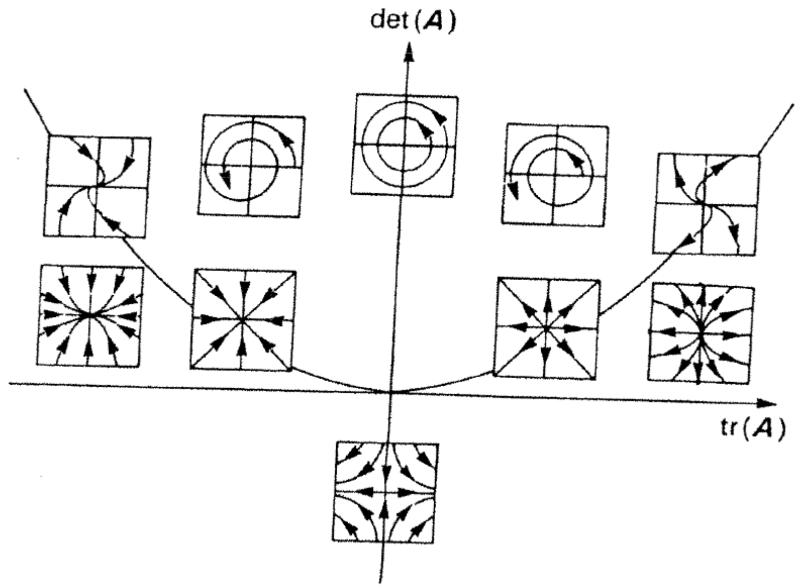
In the analysis of stability we will always choose a system of coordinates so that the origin coincides with the equilibrium point. In the course book this agreement is applied even in the definition of stability.

Definition. The equilibrium point x_* is said to be *unstable* if it is not stable. It means that there is a $\varepsilon_0 > 0$, such that for any $\delta > 0$ there is point $x(0) : \|x(0) - x_*\| \leq \delta$ such that for some $t_0 \in I$ we have $\|x(t_0) - x_*\| > \varepsilon_0$.(a formal negation to the definition of stability)

1.4 Stability and instability of the equilibrium point in the origin for autonomous linear systems.

Origin is an equilibrium point for all linear systems of ODE. If the matrix A is degenerate namely if $\det(A) = 0$, there can appear lines or hyperplanes of equilibrium points except the origin, corresponding to the non-trivial kernel of the matrix A.

Reminder of classification of phase portraits for autonomous systems of ODEs in the plane:



Summary of phase portraits for the system x'=Ax depending on tr(A) and det(A).
The division line is det(A)= ¼ (tr(A))^2.

1.5 Stability and instability of the equilibrium in the origin for arbitrary autonomous linear systems of ODEs

General statement about stability and instability of the equilibrium in the origin for arbitrary autonomous linear systems of ODEs follow immediately from the Corollary 2.13 in L.&R.

Theorem. (Propositions 5.23, 5.24, 5.25, pp. 189-190, L.R.)
Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix.
Then three following statements are valid for the system $x'(t) = Ax(t)$

1. The origin is asymptotically stable equilibrium point if and only if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

2. The equilibrium point in the origin is stable if and only if $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma(A)$ and all eigenvalues λ with $\operatorname{Re}\lambda = 0$ are semisimple (the number of linearly independent eigenvectors to λ is equal to the algebraic multiplicity of λ)

3. The equilibrium point in the origin is unstable if and only if there is at least one eigenvalue λ with $\operatorname{Re}\lambda > 0$ or an eigenvalue λ with $\operatorname{Re}\lambda = 0$ that is not semisimple.

   (**3.** is a direct consequence of the **1.** and **2.**)

   Prof is a simple exrecise based on the definition and the Corollary 2.13 about the properties of $\|\exp(At)\|$

   Definition. Matrix A with the property $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ is called Hurwitz matrix.

1.6 Inhomogeneous linear systems with constant coefficients.

Corollary. Duhamel formula, autonomous case. (Corollary 2.17, p. 43)

Consider the inhomogeneous system

$$x'(t) = Ax + g(t)$$

with continuous or piecewise continuous function $g : \mathbb{R} \to \mathbb{R}^N$. Then the unique solution to the I.V.P. with initial data

$$x(0) = \xi$$

is represented by the Duhamel formula:

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))g(\sigma)d\sigma \tag{3}$$

Proof of the Corollary: check that the formula gives a solution and show that it is unique.

$$\begin{aligned}
x(t) &= \exp(At)\xi + \int_0^t \exp(A(t-\sigma))g(\sigma)d\sigma \\
&= \exp(At)\xi + \exp(At)\int_0^t \exp(-A\sigma)g(\sigma)d\sigma \\
&= \exp(At)\left[\xi + \int_0^t \exp(-A\sigma)g(\sigma)d\sigma\right] \\
x'(t) &= A\exp(At)\left[\xi + \int_0^t \exp(-A\sigma)g(\sigma)d\sigma\right] + \exp(At)\exp(-At)g(t) \\
&= Ax(t) + g(t)
\end{aligned}$$

for all points t where $g(t)$ is continuous. Difference $z(t) = x(t) - y(t)$ between two solutions $x(t)$ and $y(t)$ satisfies the homogeneous systems $z'(t) = Az(t)$ and zero initial condition $z(0) = 0$ and the integral equation: $z(t) = \int_0^t Az(\sigma)d\sigma$. The same reasoning as before, using the Grönwall inequality, or just a reference to the uniqueness of solutions to homogeneous systems implies that $z \equiv 0$.

1.7 Stability of equilibrium points to linear systems perturbed by a small right hand side.

Theorem (Theorem 5.27, p. 193, L.R.) Let $G \subset \mathbb{R}^N$ be a non-empty open subset with $0 \in G$. Consider the differential equation

$$\begin{aligned} x'(t) &= Ax + h(x) \qquad (4)\\ x(0) &= \xi \qquad (5) \end{aligned}$$

where $A \in \mathbb{R}^{N\times N}$ and $h : G \to \mathbb{R}^N$ is a continuous function satisfying

$$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0. \qquad (6)$$

If A is Hurwitz, that is $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$, then 0 is an asymptotically stable equilibrium of 4.

Moreover, there is $\Delta > 0$ and $C > 0$ and $\alpha > 0$ such that for $\|\xi\| < \Delta$ the solution $x(t)$ to the initial value problem with initial data

$$x(0) = \xi$$

satisfies the estimate

$$\|x(t)\| \leq C\,\|\xi\|\,e^{-\alpha t}$$

Proof. (This proof is required at the exam)

If $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ then there is $\beta > 0$ such that $\operatorname{Re}\lambda < -\beta$

(strictly smaller!) for all $\lambda \in \sigma(A)$ and

$$\|\exp(At)\| \leq Ce^{-\beta t} \tag{7}$$

for some constant $C > 0$.

We can choose $\varepsilon > 0$ such that $C\varepsilon < \beta$ and using (6) choose δ_ε such that for $\|z\| < \delta_\varepsilon$, $z \in G$

$$\begin{aligned} \frac{\|h(z)\|}{\|z\|} &< \varepsilon && (8) \\ \|h(z)\| &< \varepsilon \|z\| && (9) \end{aligned}$$

It follows from properties of h :$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0$.

We know from Peano theorem or from Picard - Lindelöf theorem in case f is Lipschitz, that the solution to the equation (4) exists on some time interval $t \in [0, \delta)$ (another δ!!!)

We apply Duhamel formula (3) for solutions to (4):

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))d\sigma$$

As long as $x(\sigma)$ under the integral, belongs to the ball $\{x : \|x\| < \delta_\varepsilon\} \subset G$, we apply the triangle inequality for integrals and estimates (7) and (8):

$$\begin{aligned} \|x(t)\| &= \|\exp(At)\| \|\xi\| + \int_0^t \|\exp(A(t-\sigma))\| \|h(x(\sigma))\| d\sigma \\ \|x(t)\| &\leq Ce^{-\beta t} \|\xi\| + \int_0^t Ce^{-\beta(t-\sigma)}\varepsilon \|x(\sigma)\| d\sigma \end{aligned}$$

Introduce the function $y(t) = \|x(t)\| e^{\beta t}$. Then multiplying the last inequality by $e^{\beta t}$ we arrive to

$$y(t) \le C\,\|\xi\| + \int_0^t (C\varepsilon)\, y(\sigma) d\sigma$$

The Grönwall inequality implies that

$$\|y(t)\| \le C\,\|\xi\|\, e^{(C\varepsilon)t}$$

and

$$\|x(t)\| \le C\,\|\xi\|\, e^{-(\beta - C\varepsilon)t} \tag{10}$$

It is valid as long as $\|x(t)\| < \delta_\varepsilon$. Now we can choose $\alpha = \beta - C\varepsilon > 0$, by choosing ε small enough, $\Delta = \frac{1}{2}\delta_\varepsilon / C$ and $\|\xi\| < \Delta$. This choice of initial conditions implies that

$$\|x(t)\| \le \delta_\varepsilon, \tag{11}$$

as long as this solution exists (!!!)

(Important theoretical argument! Check similar argument in Lemma 4.9, p. 110 in LR)

The last estimate implies in fact an important conclusion that the solution must exist in fact on the whole $\mathbb{R}_+$, because supposing the opposite, namely that there is some maximal existence time $t_{\max}$, leads to a contradiction.

Let consider this important argument. It consists of two steps.

1) We use the continuity and boundedness of the solution $x(t)$ on $[0, t_{\max})$ together with the integral form of the equation

$$x(t) = \xi + \int_0^t Ax(\sigma) d\sigma + \int_0^t h(x(\sigma)) d\sigma$$

The set $\{x(t) : t \in [0, t_{\max})\}$ (that is the orbit of the solution), is bounded according to (11). The closure C of this set is therefore compact. The function $h(x)$ is continuous on G and is therefore bounded on the compact set C.

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to t_{\max}$ the sequence of integrals $\{x(t_k)\}_{k=1}^{\infty}$is a Cauchy sequence and therefore has a limit $\lim_{k\to\infty} x(t_k) = \eta$, because

$$\|x(t_m) - x(t_k)\| \leq \left\| \int_{t_k}^{t_m} Ax(\sigma)d\sigma + \int_{t_k}^{t_m} h(x(\sigma))d\sigma \right\| \leq$$

$$\left| \int_{t_k}^{t_m} \|A\| \, \|x(\sigma)\| \, d\sigma \right| + \left| \int_{t_k}^{t_m} \|h(x(\sigma))\| \, d\sigma \right| \leq C \, |t_m - t_k| \to 0, \quad m, k \to \infty$$

This limit is unique and independent of the sequence $\{t_k\}_{k=1}^{\infty}$ by a similar estimate. Therefore we can extend $x(t)$ up to the point $t_{\max}$ as

$$x(t_{\max}) = \eta = \lim_{t \to t_{\max}} x(t)$$

2) Now using an existence theorem (Peano or Picard-Lindelöf) for non-linear systems of ODEs, we conclude that there is a solution $y(t)$ to the equation

$$y'(t) = Ay + h(y)$$

on the time interval $[t_{\max}, t_{\max} + \delta)$ with the initial condition $y(t_{\max}) = \eta$ at time $t_{\max}$. This solution is evidently an extension of the original solution $x(t)$ to a larger time interval, that contradicts the our supposition.

Therefore the solution $x(t)$ can be extended to the whole $\mathbb{R}_+$ and satisfies the estimate (11). It in turn implies that this solution must satisfy the desired estimate

$$\|x(t)\| \leq C \, \|\xi\| \, e^{-\alpha t}$$

and implies the asymptotic stability of the equilibrium point in the origin.■

This theorem implies immediately the following result on the stability of equilibrium points by linearization.

Theorem.On stability of equilibrium points by linearization. (Corollary 5.29, p. 195)

Let $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ be a non empty open set with $0 \in G$, f be continuous and $f(0) = 0$. Let f be differentiable in 0 and A be the Jacoby matrix of f in the point 0, $A = D(f)(0)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(0), \quad i, j = 1, ...N$$

If A is a Hurwitz matrix (all eigenvalues $\lambda \in \sigma(A)$ have $\operatorname{Re}\lambda < 0$), then the equilibrium point of the system

$$x'(t) = f(x(t))$$

in the origin is asymptotically stable.

Proof. Consider the function $h(z) = f(z) - Az$. Then by the definition of derivatives $h(z)/\|z\| \to 0$ as $z \to 0$. An application of the theorem about stability of a small perturbation of a linear system to the function $f(z) = Az + h(z)$ proves the the claim. ■

The following general theorem by Grobman and Hartman that we formulate without proof is a strong result on connection between solutions to a nonlinear system

$$\begin{aligned} x'(t) &= f(x(t)), \qquad (12) \\ x(0) &= \xi \qquad (13) \end{aligned}$$

with right hand side $f(x)$ close to an equilibrium point x_*, $f(x_*) = 0$ and solutions to the linearized system

$$\begin{aligned} y'(t) &= Ay \qquad (14) \\ y(0) &= \zeta - x_* \qquad (15) \end{aligned}$$

with constant matrix A that is Jacobi matrix of the right hand side f in the equilibrium point x_*, $A = D(f)(x_*)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(x_*), \quad i, j = 1, ...N$$

■

Definition. An equilibrium point x_* of the system (12) is called hyperbolic if for all eigenvalues $\lambda \in \sigma(A)$ it is valid that $\mathrm{Re}\,\lambda \neq 0$.

Theorem. (Grobman-Hartman) A formulation and a (difficult!) proof can be found as Th. 9.9 at the page 266, in the book by Teschl: http://www.mat.univie.ac.at/%7Egerald/ftp/book-ode/index.html

Consider an I.V.P. for a autonomous system of differential equations

$$\begin{aligned} x'(t) &= f(x(t)), && (16) \\ x(0) &= \xi && (17) \end{aligned}$$

Let $f \in C^1(B)$, in $B_R(x_*) = \{\xi : \|\xi - x_*\| < R \quad \} \subset G$ and $x_* \in G$ be a hyperbolic equilibrium point of (12): $f(x_*) = 0$.

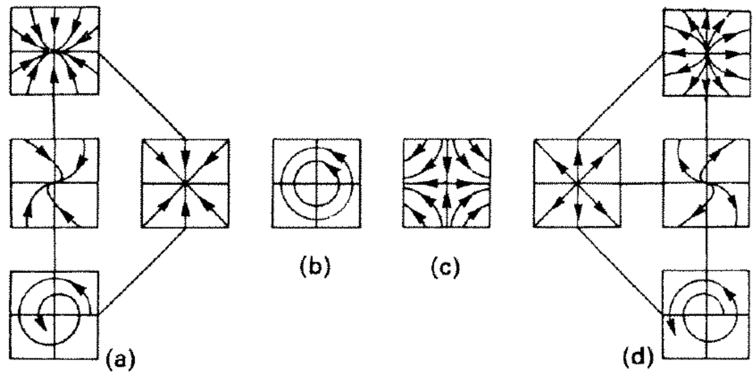
Then there are neighborhoods $U_1(x_*)$ and $U_2(x_*)$ of x_* and an invertible continuous mapping $R : U_1(x_*) \to U_2\,(x_*)$ such that R maps shifted solutions $x_* + e^{At}(\zeta - x_*)$ to the linearized system (14) onto solutions $x(t) = \varphi(t, R(\zeta))$ of the non-linear system (12) with initial data

$$\xi = R(\zeta), \zeta = R^{-1}\,(\xi)$$

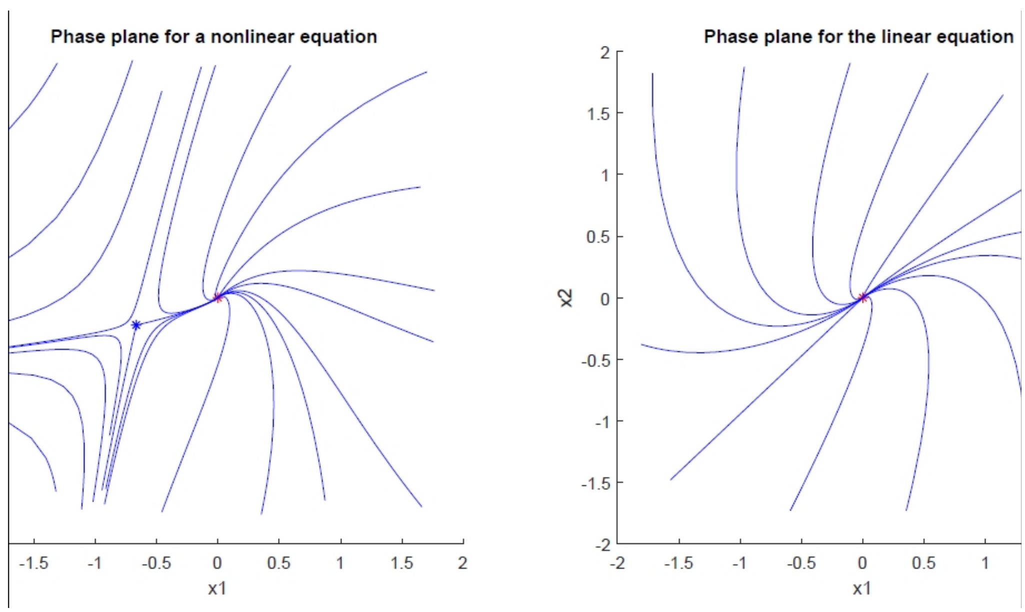
$$R\left(x_* + e^{At}(\zeta - x_*)\right) = \varphi(t, R(\zeta))$$

and back

$$R^{-1}\,(\varphi(t,\xi)) = x_* + e^{At}(R^{-1}\,(\xi) - x_*)$$



as long as $x_* + e^{At}(R^{-1}(\xi) - x_*) \in U_1(x_*)$.$\square$



Various classes of topologically equivalent equilibrium points in the plane: a) asymptotically stable, b) center, c) saddle point, d) unstable:

In higher dimensions there is a larger variety of topologically different configurations of phase protraits around equilibrium points.

Example on application of the Grobman - Hartman theorem

Consider the system

$$\begin{aligned} x_1' &= -\tfrac{1}{2}(x_1 + x_2) - x_1^2 \\ x_2' &= \tfrac{1}{2}(x_1 - 3x_2) \end{aligned}$$

It has two equilibrium points: one in the origin $(0,0)$ and the second one is $(-2/3, -2/9)$.We find them by expressing $x_1 = 3x_2$, from the equation $\frac{1}{2}(x_1 - 3x_2) = 0$, substituting to the equation $-\frac{1}{2}(x_1 + x_2) - x_1^2 = 0$, and solving the quadratic equation $-\frac{1}{2}(3x_2 + x_2) - 9x_2^2 = 0$ for x_2.

$$-\frac{1}{2}(3x_2 + x_2) - 9x_2^2 = -x_2\,(9x_2 + 2) = 0.$$

and its linearization in the origin:

$$\begin{aligned} x_1' &= -\tfrac{1}{2}(x_1 + x_2) \\ x_2' &= \tfrac{1}{2}(x_1 - 3x_2) \end{aligned}$$

The linearized system has matrix $A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$, characteristic polynomial: $\lambda^2 + 2\lambda + 1 = 0$, eigenvalues: $\lambda_{1,2} = -1$. The only eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The origin is a stable for both systems. This equilibrium point is asymptotically stable.

On the other hand we see that another equilibrium $(-2/3, -2/9)$ of the non-linear system seems to be a saddle point.

We check it now. For an arbitrary point we need first to calculate the Jacoby matrix of the right hand side in the system $x' = f(x)$ in an arbitrary point $x \in \mathbb{R}^2$

$$
\begin{aligned}
[Df]_{ij}(x) &= \frac{\partial f_i}{\partial x_j}(x) \\
[Df](x) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix} = \begin{bmatrix} -1/2 - 2x_1 & -1/2 \\ 1/2 & -3/2 \end{bmatrix}
\end{aligned}
$$

Calculating the Jacoby matrix in the second equilibrium point $(-2/3, -2/9)$ we get the matrix for the linearization of the right hand side in this point:

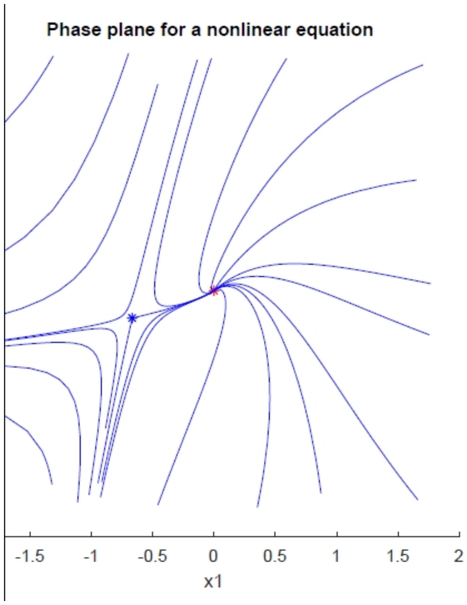
$$
A = \begin{bmatrix} -1/2 - 2(-2/3) & -1/2 \\ 1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}
$$

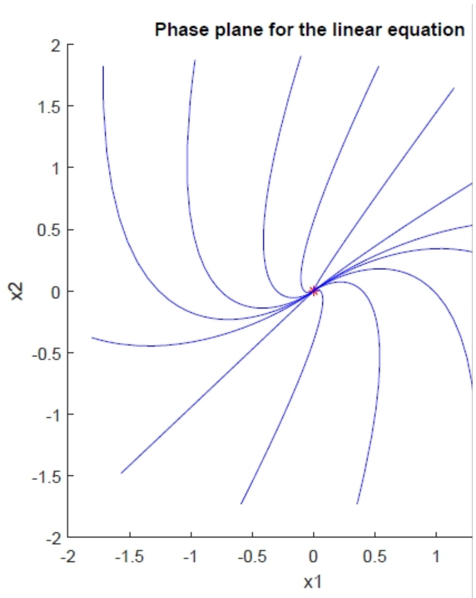
The characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A)$. $\mathrm{tr}(A) = 5/6 - 3/2 = -\frac{2}{3}$. $\det(A) = \frac{5}{6}\left(-\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right) = -1$. Therefore $p(\lambda) = \lambda^2 + \frac{2}{3}\lambda - 1$. Eigenvalues are real and have different signs because the determinant of A is negative. We do not need to calculate them to make these conclusions.

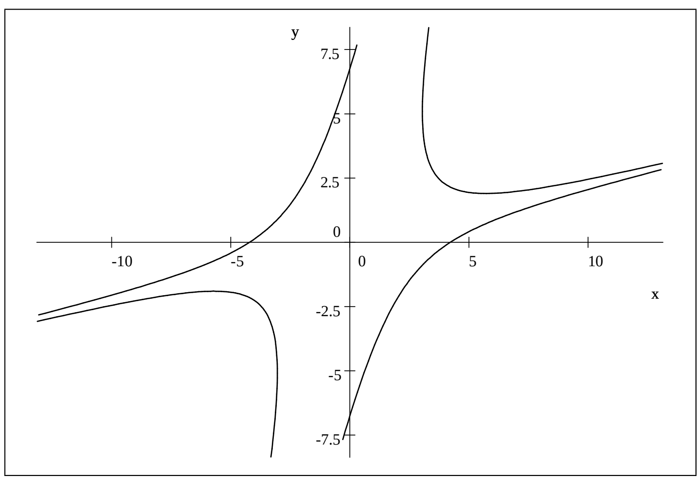
Therefore the linearized system

$$
y' = Ay
$$

has a saddle point in the origin. The non-linear system also has a saddle point configuration in the phase portrait close to the equilibrium point $(-2/3, -2/9)$ according to the Grobman-Hartman theorem. This equilibrium point is unstable. If we like to sketch a more precise phase portrait for the linearized system we can calculate eigenvalues and eigenvectors. But we can only guess the global phase portrait for the non-linear system (how local phase portraits connect with each other). We give below phase portraits for the non-linear system and for the linearized system around each of equilibrium points.







Phase plane for the linearized system around the equilibrium point $(-2/3, -2/9)$

Counterexample to the Grobman - Hartman theorem.

A system such that the linearized system has a center (stable) but the non-linear has an unstable equilibrium point.

Consider the system

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 + (x_1^2 + x_2^2)x_1 \\ \frac{dx_2}{dt} &= -x_1 + (x_1^2 + x_2^2)x_2\end{aligned}$$

The origin $(0,0)$ is an equilibrium point and the linearized system in this point has the form

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

The origin is a center that is a stable equilibrium point.

Consider the equation for $r^2(t) = x_1^2(t) + x_2^2(t)$.We derive it by multiplying the first equation by x_1 and the second by x_2 and considering the sum of the equations leading to

$$x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = \frac{1}{2}\frac{d\,(x_1)^2}{dt} + \frac{1}{2}\frac{d\,(x_2)^2}{dt} = \frac{1}{2}\frac{d}{dt}\left(r^2(t)\right) = \left(r^2(t)\right)^2$$

We see that the solution to this equation $z = r^2$

$$\begin{aligned}\frac{1}{2}\frac{dz}{dt} &= z^2 \\ \frac{dz}{z^2} &= 2dt \\ \int \frac{dz}{z^2} &= \int 2dt \\ -\frac{1}{z} &= 2t + C \\ \frac{-1}{z(0)} &= C \\ -\frac{1}{z} &= 2t + \frac{-1}{z(0)} \\ z &= r^2\end{aligned}$$

with separable variables with arbitrary initial data $r(0)$ is

$$r^2(t) = \frac{r^2(0)}{1 - 2r^2(0)t}$$

The solution $r^2(t)$ is increasing with time and tends to infinity with t rising and blows up in finite time.

The equilibrium $(0, 0)$ to the nonlinear system is unstable. The phase portraits of the nonlinear system and the linearized system are qualitatively different in this example when eigenvalues to the Jacoby matrix of the right hand side of the nonlinear system in the equilibrium point have real parts equal to zero.

Example on application of the Grobman - Hartman theorem

Find for which values of the parameter a the origin is an asymptotically stable equilibrium, stable equilibrium, unstable equilibrium of the following system:

$$\begin{cases} x' = y \\ y' = -ay - x^3 - a^2 x \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the Jacoby matrix of the right hand side in the equatiuon.

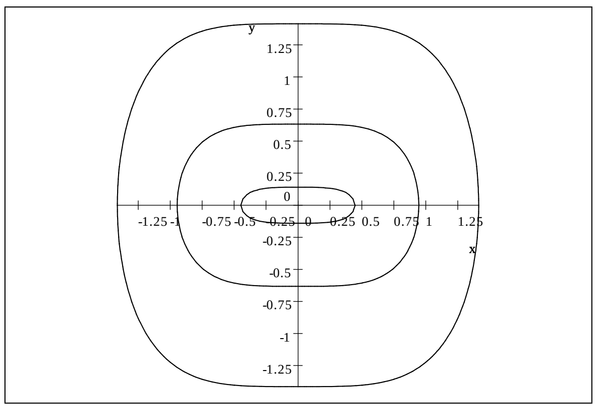
$A(x, y) = \begin{bmatrix} 0 & 1 \\ -a^2 - 3x^2 & -a \end{bmatrix}$. It's value in the origin is $A(0, 0) = \begin{bmatrix} 0 & 1 \\ -a^2 & -a \end{bmatrix}$, with characteristic polynomial: $p(\lambda) = \lambda^2 + a\lambda + a^2$.

Eigenvalues are $\lambda_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - a^2} = -\frac{a}{2} \pm i\sqrt{\frac{3a^2}{4}}$

The Grobman - Hartman theorem about stability by linearization imples that the origin is asymptotically stable when $a > 0$ and is unstable when $a < 0$. For $a = 0$ linearization does not give any information about stability because in this case $\operatorname{Re} \lambda = 0$. In this case the system is reduced to $\begin{cases} x' = y \\ y' = -x^3 \end{cases}$ and we can find an equation for orbits of the system from an

ODE with separable variables:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{-x^3}{y} \\
ydy &= -x^3 dx \\
\int ydy &= -\int x^3 dx \\
\frac{y^2}{2} &= -\frac{x^4}{4} + C \\
\frac{x^4}{4} + \frac{y^2}{2} &= C
\end{aligned}$$



Example. Stability by linearization for the pendulum with friction.

$$\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m} x_2(t) - \frac{g}{l}\sin(x_1(t))
\end{aligned}$$

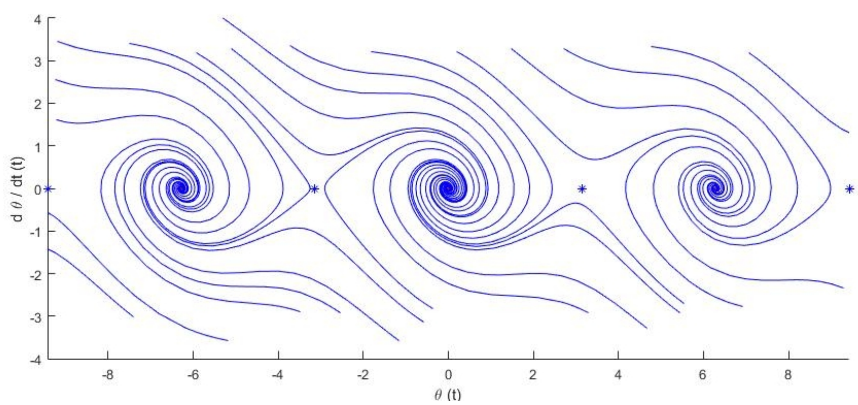
Linearized equation around $(0,0)$ is

$$
\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}x_1(t)
\end{aligned}
$$

The matrix of the system is

$$
A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}
$$

$\mathrm{tr}(A) = -\frac{\gamma}{m} < 0$; $\det(A) = \frac{g}{l} > 0$. Therefore the $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A)$. For small friction coefficient γ the equilibrium will be focus, for large friction it will be a stable node. An intermediate case with stable improper node is also possible.



Point out that the case with zero friction: $\gamma = 0$ cannot be treated by linearization, because the linearized system has a center in the origin. The non-linear system has in fact also a center in the origin, but we cannot prove it by means of linearization. We will consider this case later by different means.

The linearization of the equation around $(\pi, 0)$.

Linear approximation for sin around π. Let $(x_1 - \pi) = y_1(t)$.

$$\sin(x_1) = \sin(\pi) + \cos(\pi)(x_1 - \pi) + O(x_1 - \pi)^2 \approx -(x_1 - \pi) = -y_1(t)$$

$$y_1(t) = x_1(t) - \pi; y_1'(y) = x_1'(t)$$

therefore

$$\begin{aligned} x_1(t) &= y_1(t) + \pi;\ x_1'(y) = y_1'(t) \\ x_2(t) &= x_1' = y_1'(t) \end{aligned}$$

Introducing $y_2 = y_1' = x_2$; we get $x_2 = y_2$

$$\sin(x_1) = \sin(\pi) + \cos(\pi)y_1 + O(\pi - x_1)^2$$

;

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1) \end{aligned}$$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m}y_2(t) - \frac{g}{l}(-y_1) \end{aligned}$$

The linearized equation around $(\pi, 0)$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= \frac{g}{l}y_1(t) - \frac{\gamma}{m}y_2(t) \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

Characteristic polynomial: $p(\lambda) = \lambda^2 - \left(-\frac{\gamma}{m}\right)\lambda - \frac{g}{l}$.

$\operatorname{tr}(A) = -\frac{\gamma}{m} < 0$; $\det(A) = -\frac{g}{l} < 0$. The equilibrium is always a saddle point (instable).

Example on application of the Grobman - Hartman theorem

Find all stationary points of the system of ODE $\begin{cases} x' = e^y - e^x \\ y' = \sqrt{3x + y^2} - 2 \end{cases}$ and investigate their stability by linearization.

1. **Solution.**

   We find stationary points by pointing out that the first equation implies $y = x$ and then $\sqrt{3x + x^2} - 2 = 0$ implies $3x + x^2 - 4 = (x+4)(x-1)$ $= 0$ and therefore two roots $x_1 = 1$ and $x_2 = -4$ follow.

   We have two stationary points: $(1,1)$ and $(-4,-4)$.

   The Jacobi matrix is $J(x,y) = \begin{bmatrix} -e^x & e^y \\ \frac{3}{2\sqrt{3x+y^2}} & \frac{y}{\sqrt{3x+y^2}} \end{bmatrix}$

   $J(1,1) = \begin{bmatrix} -e & e \\ \frac{3}{2\sqrt{3+1}} & \frac{1}{\sqrt{3+1}} \end{bmatrix} = \begin{bmatrix} -e & e \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$ The trace of $J(1,1)$ is $\operatorname{tr}(J(1,1)) = 1/2 - e < 0$

   $\det(J(1,1)) = e(-1/2 - 3/4) = -\frac{5}{4}e < 0$ it implies that the stationary point $(1,1)$ is has one negative and one postive eigenvalue and therefore is a saddle point and is unstable by the Grobman Hartman theorem.

   The characteristic equation for a 2x2 matrix A is $\lambda^2 - tr(A)\lambda - \det(A) = 0$.

   In this particular situation it is $\lambda^2 + \left(e - \frac{1}{2}\right)\lambda - \frac{5}{4}e = 0$.

Eigenvalues are: $\lambda_1 = -\frac{1}{2}e + \frac{1}{4} - \frac{1}{4}\sqrt{16e + 4e^2 + 1}$, $\lambda_2 = -\frac{1}{2}e + \frac{1}{4} + \frac{1}{4}\sqrt{16e + 4e^2 + 1}$.

$J(-4,-4) = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & \frac{-4}{2} \end{bmatrix} = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & -2 \end{bmatrix}$.

The trace of $J(-4,-4)$ is $\operatorname{tr}(J(-4,-4)) = -2 - e^{-4} < 0$.

$\det(J(-4,-4)) = e^{-4}\left(2 - \frac{3}{4}\right) = \frac{5}{4}e^{-4} > 0$. Therefore the the real parts of eigenvalues are negative and the stationary point $(-4,-4)$ is an asymptotically stable node by the Grobman Hartman theorem.

The characteristic equation is $\lambda^2 + (e^{-4} + 2)\lambda + \frac{5}{4}e^{-4} = 0$.

Eigenvalues are : $\lambda_1 = -\frac{1}{2}e^{-4} - 1 - \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$, $\lambda_2 = -\frac{1}{2}e^{-4} - 1 + \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$

Example on the application of the Grobman - Hartman theorem

$$\begin{cases} x' = & y \\ y' = & -y - x - x^2 \end{cases}$$

Equilibrium points are $(0,0)$ and $(-1,0)$.

Jacobi matrix is $A(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2x & -1 \end{bmatrix}$

Jacobi matrix at the origin is $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$,

characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda + 1$, eigenvalues are $-\frac{1}{2}i\sqrt{3} - \frac{1}{2}, \frac{1}{2}i\sqrt{3} - \frac{1}{2}$. Real parts of eigenvalues are negative and therefore the origin is stable focus, asymptotically stable equilibrium.

Jacobi matrix at the point $(-1,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$,

characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 1$,eigenvalues are $-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}$. One is negative, another is positive, the equilibrium point is a saddle point and is unstable.

1 Stability by linearization for the pendulum with friction.

$$\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t))
\end{aligned}$$

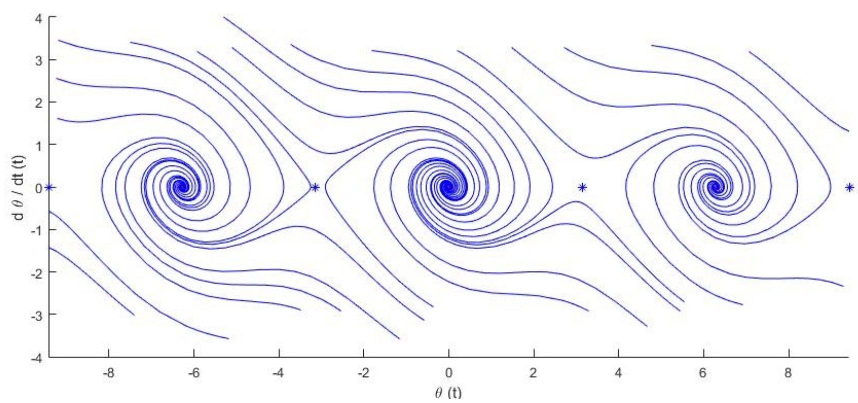
Linearized equation around $(0,0)$ is

$$\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}x_1(t)
\end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

$tr(A) = -\frac{\gamma}{m} < 0$; $\det(A) = \frac{g}{l} > 0$. Therefore the $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. For small friction coefficient γ the equilibrium will be focus, for large friction it will be a stable node. An intermediate case with stable improper node is also possible.



Point out that the case with zero friction: $\gamma = 0$ cannot be treated by linearization, because the linearized system has a center in the origin. The non-linear system has in fact also a center in the origin, but we cannot prove it by means of linearization. We will consider this case later by different means.

The linearization of the equation around $(\pi, 0)$.

Linear approximation for sin around π. Let $(x_1 - \pi) = y_1(t)$.

$$\sin(x_1) = \sin(\pi) + \cos(\pi)(x_1 - \pi) + O(x_1 - \pi)^2 \approx -(x_1 - \pi) = -y_1(t)$$

$$y_1(t) = x_1(t) - \pi; y_1'(y) = x_1'(t)$$

therefore

$$\begin{aligned} x_1(t) &= y_1(t) + \pi;\ x_1'(y) = y_1'(t) \\ x_2(t) &= x_1' = y_1'(t) \end{aligned}$$

Introducing $y_2 = y_1' = x_2$; we get $x_2 = y_2$

$$\sin(x_1) = \sin(\pi) + \cos(\pi) y_1 + O(\pi - x_1)^2$$

;

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m} x_2(t) - \frac{g}{l} \sin(x_1) \end{aligned}$$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m} y_2(t) - \frac{g}{l}(-y_1) \end{aligned}$$

The linearized equation around $(\pi, 0)$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m} y_2(t) + \frac{g}{l} y_1 \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

Characteristic polynomial: $p(\lambda) = \lambda^2 - \left(\frac{g}{l}\right)\lambda + \left(\frac{1}{m}\gamma\right)$.

$tr(A) = -\frac{\gamma}{m} < 0$; $\det(A) = -\frac{g}{l} < 0$. The equilibrium is always a saddle point (unstable).

May 5, 2020

Lecture notes on general and periodic linear ODEs

Plan

1. Transition matrix function, existence and equations. Lemma 2.1, p.24, Cor. 2.3, p.26.

2. Grönwall inequality. General case. Lemma. 2.4, p. 27 (skiped)

3. Uniqueness of solutions and dimenstion of solution space. Th. 2.5, p. 28, Prop. 2.7(1), p.30

4. Group properties of transition matrix function and transition mapping.
(Chapman - Kolmogorov relations) Cor. 2.6, p. 29

5. Fundamental matrix solution and its connection with transition matrix function. Prop. 2.8, p.33

6. Inhomogeneous linear systems. Variation of constant formula (Duhamels formula), general case. Th. 2.15, p.41,Cor. 2.17.

7. Transition matrix function for periodic linear systems. Formula. 2.31, p. 45.

8. Monodromy matrix and properties of transition matrix function for periodic systems. Th. 2.30, p. 53.

9. Logarithm of a matrix. Prop. 2.29, p. 53.

10. Floquet multipliers and exponents.

11. Boundedness and zero limits for solutons to periodic linear systems. Th. 2.31, p.54. Cor. 2.33, p-59

12. Existence of periodic solutions to periodic linear systems. Prop. 2.20, p. 45

13. Abel-Liouville's formula. Prop. 2.7(2)., p.30.

14. Conditions for existence of unbounded solutions based on the Abel-Liouvilles formula

15. Hill equation and Kapitza pendulum. pp. 55-57.

0.1 Transition matrix function, existence and equations.

The subject of this chapter of lecture notes is general non - autonomous linear systems of ODEs and in particular systems with periodic coefficients and Floquet theory for them.

The general theory for non - autonomous linear systems (linear systems with variable coeffcients) is very similar to one for systems with constant coefficients. The existence is established through the solution of the integral form of equations by iterations. Uniqueness is based on a general form of the Grönwall inequality that is also proved here in a very similar fashion. These results lead to the fundamental result on the dimension of the space of solutions that is based on the uniqueness result similarly to the proof for systems with constant coefficients. The essential difference from the case with constant coefficients is that in the case with variable coefficients one cannot find analytical solutions except some particular cases as systems with triangular matrices.

We consider the I.V.P. in the differential

$$x' = A(t)x(t), \quad x(\tau) = \xi \tag{1}$$

or in the integral form

$$x(t) = \xi + \int_{\tau}^{t} A(s)x(s)ds \tag{2}$$

with matrix valued function $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) that is continuous or piecewise continuous on the interval J. Here it is important that the initial time τ is an arbitrary real number from J, not just zero. The solution is defined as a continuous function $x(t)$ on an interval I that includes point τ acting into $\mathbb{R}^N$ or $\mathbb{C}^N$ satisfying the integral equation (2). By a version of Calculus main theorem (Newton-Leibnitz theorem) the solution defined in such a way will satisfy the differential equation (1) in points t where $A(t)$ is continuous.

We remind the following lemma considered in the beginning of the course.

Lemma. The set of solutions $\mathcal{S}_{\text{hom}}$ to (2) is a linear vector space.

□

It motivates us to search solution in the form $\Phi(t,s)\xi$ where $\Phi(t,s)$ is a continuous matrix valued function on $J \times J$ and ξ is an arbitrary initial data at $t = s : x(s) = \xi$. It implies also that $\Phi(s,s) = I$. Substituting the expression $x(t) = \Phi(t,s)\xi$ into the integral form of the i.V.P. we arrive to the vector equation

$$\begin{aligned}
\Phi(t,s)\xi &= \xi + \int_s^t A(\sigma)\Phi(\sigma,s)\xi d\sigma \Longrightarrow \\
\Phi(t,s)\xi &= \left(I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma \right)\xi
\end{aligned}$$

with arbitrary $\xi \in \mathbb{R}^N$ that implies the matrix equation for $\Phi(t,s)$:

$$\Phi(t,s) = I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma \tag{3}$$

or the same equation in differential form valid outside points of disconituity of $A(t)$:

$$\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s); \quad \Phi(s,s) = I.$$

We will solve the equation (3) by means of iterational approximations $M_n(t,s)$ to $\Phi(t,s)$ introduced in the following way:

$$M_1(t,s) = I; \quad M_{n+1}(t,s) = I + \int_s^t A(\sigma)M_n(\sigma,s)d\sigma, \quad \forall n \in \mathbb{N} \tag{4}$$

Lemma 2.1 , p. 24 and **Corollary 2.3,** p. 26 in L&R

For any closed and bounded interval $[a, b] \subset J$ the sequence $\{M_n(t, s)\}$ converges uniformly on $[a, b] \times [a, b]$ to a continuous on $[a, b] \times [a, b]$ matrix valued function $\Phi(t, s)$ that satisfies the integral equation (3).

Proof.

The classical idea of the proof is instead of considering $M_n(t, s)$ to consider **telescoping series** with elements $f_{n+1}(t, s) = M_{n+1}(t, s) - M_n(t, s)$, $f_1 = M_1 = I$, with partial sum that is equal to M_n:

$$M_n = \sum_{k=1}^{n} f_k$$

where $f_k(t, s)$ is represented as a repeated integral operator from (4):

$$\begin{aligned}
M_1(t,s) &= I; \quad M_2(t,s) = I + \int_s^t A(\sigma) M_1(\sigma, s) d\sigma, \\
M_3(t,s) &= I + \int_s^t A(\sigma_1) M_2(\sigma_1, s) d\sigma_1 = \\
&= I + \int_s^t A(\sigma_1) \left[I + \int_s^{\sigma_1} A(\sigma_2) M_1(\sigma_2, s) d\sigma_2 \right] d\sigma_1 \\
& \quad I + \int_s^t A(\sigma_1) I d\sigma_1 + \int_s^t A(\sigma_1) \int_s^{\sigma_1} A(\sigma_2) M_1(\sigma_2, s) d\sigma_2 d\sigma_1 \\
f_3 &= M_3 - M_2 = \int_s^t A(\sigma_1) \int_s^{\sigma_1} A(\sigma_2) M_1(\sigma_2, s) d\sigma_2 d\sigma_1
\end{aligned}$$

$$f_{n+1}(t,s) = M_{n+1}(t,s) - M_n(t,s) = \int_s^t A(\sigma_1) \int_s^{\sigma_1} A(\sigma_2) ... \int_s^{\sigma_{n-1}} A(\sigma_n) d\sigma_n ... d\sigma_2 d\sigma_1$$

for all $(t, s) \in J \times J$, $\forall n \in \mathbb{N}$. Since $A(t)$ is piecewise continuous on J, it's norm is bounded on any compact subinterval $[a, b] \subset J$:

$$\|A(t)\| \leq K \quad \forall t \in [a, b]$$

We observe using triangle inequality for integrals several times, and the last estimate, that

$$\|f_{n+1}(t,s)\| = \|M_{n+1}(t,s) - M_n(t,s)\| \leq K^n \int_s^t \int_s^{\sigma_1} ... \int_s^{\sigma_{n-1}} d\sigma_n ... d\sigma_2 d\sigma_1$$

and after calculating the integral $\int_s^t \int_s^{\sigma_1} ... \int_s^{\sigma_{n-1}} d\sigma_n ... d\sigma_2 d\sigma_1 = \frac{1}{n!}(t-s)^n$, based essentially on $\int s^k ds = \frac{s^{k+1}}{k+1}$.

$$\|f_{n+1}(t,s)\| = \|M_{n+1}(t,s) - M_n(t,s)\| \leq \frac{K^n}{n!}(t-s)^n \leq \frac{K^n}{n!}(b-a)^n$$

The number series $\sum_{n=0}^{\infty} \frac{K^n}{n!}(b-a)^n$ is convergent to $\exp(K(b-a))$. Therefore by the Weierstrass' criterion the functional series $\sum_{n=1}^{\infty} f_n(t, s)$ converges uniformly on $[a, b] \times [a, b]$ to the limit denoted here by $\Phi(t, s)$. It implies by construction, that the sequence $M_n(t, s)$ converges uniformly on $[a, b] \times [a, b]$ to the limit denoted by $\Phi(t, s)$. Going to the limit in the relation defining iterations (4), we observe that the limit functional matrix $\Phi(t, s)$ satisfies the equation (3).■

Since the interval $[a, b] \in J$ is arbitrary we may define the function $\Phi : J = J \times J \to \mathbb{R}^{N \times N}$ (or $\mathbb{C}^{N \times N}$)

as the (pointwise) limit:

$$M_n(t,s) \to \Phi(t,s), \quad n \to \infty$$

for all $(t,s) \in J \times J$.

Definition. The matrix $\Phi(t,\tau)$ is called **transition matrix function.**

Point out that $\Phi(t,t) = I$. The product $x(t) = \Phi(t,\tau)\xi$ gives the solution to I.V.P. to the equation $x'(t) = A(t)x(t)$ with initial data $x(\tau) = \xi$. In the case when $A(t)$ is only piecewise continuous, $x(t)$ will be continuous and satify the corresponding integral equation. It will satify the differential equation outside discontinuities of $A(t)$.

Example. For an autonomous linear system with constant matrix A the transition matrix function is $\Phi(t,\tau) = \exp(A(t-\tau))$.

0.2 Grönwalls inequality. Uniqueness of solutions.

Grönwall 's lemma. Lemma 2.4., p. 27 in L&R.

(We skip it for now. A simpler version was considered before)

Let $I \subset \mathbb{R}$, be an interval, let $\tau \in I$, and let $g, h : I \to [0,\infty)$ be continuous nonnegative functions. If for some positive constant $c > 0$,

$$g(t) \le c + \left| \int_\tau^t h(\sigma) g(\sigma) d\sigma \right| \quad \forall t \in I$$

then

$$g(t) \le c \exp\left(\left| \int_\tau^t h(\sigma) d\sigma \right| \right) \quad \forall t \in I$$

Proof.

The proof uses the idea of integrating factor similar to the simpler case with constant $h = \|A\|$ considered before. Introduce $G, H : I \to [0,\infty)$ by

$$\begin{aligned} G(t) &= c + \left| \int_\tau^t h(\sigma) g(\sigma) d\sigma \right| \\ H(t) &= \left| \int_\tau^t h(\sigma) d\sigma \right| \end{aligned}$$

By the hypothesis in the lemma, $0 \le g(t) \le G(t)$.

We consider first the case $\tau < t$. Then integrals in the expressions for G and H are nonnegative:

$$G(s) = c + \int_\tau^s h(\sigma) g(\sigma) d\sigma; \quad H(s) = \int_\tau^s h(\sigma) d\sigma, \quad \forall s \in [\tau, t]$$

Differentiation and the Newton - Leibnitz theorem imply

$$\begin{aligned} G'(s) &= h(s)g(s) \le h(s)G(s) = H'(s)G(s), \quad \forall s \in [\tau, t] \\ G'(s) - H'(s)G(s) &\le 0, \quad \forall s \in [\tau, t] \end{aligned}$$

Multiplying the inequality by $\exp(-H(s))$ and observing that

$$(G'(s) - H'(s)G(s))\exp(-H(s)) = (G(s)\exp(-H(s)))'$$

we arrive to

$$(G(s)\exp(-H(s)))' \leq 0, \quad \forall s \in [\tau, t]$$

Integrating the last inequality from τ to t we arrive to

$$(G(t)\exp(-H(t))) \leq (G(\tau)\exp(-H(\tau))) = c$$

Therefore we arrive to the Grönwalls inequality:

$$(G(t)) \leq c\exp(H(t)) = c\exp\left(\int_\tau^t h(\sigma)d\sigma\right)$$

The case when $t < \tau$ **is considered similarly** by observing that for $t < \tau$

$$G(t) = c + \int_s^\tau h(\sigma)(\sigma)d\sigma; \quad H(t) = \int_s^\tau h(\sigma)d\sigma, \quad \forall s \in [t, \tau]$$

Do it as an exercise!

Uniqueness of solutions to I.V.P.

Theorem 2.5, p. 28 L&R

Let $(\tau, \xi) \in J \times \mathbb{R}^N(J \times \mathbb{C}^N)$. The function $x(t) = \Phi(t, \tau)\xi$ is a unique solution to the I.V.P. (1).If $y : J_y \to \mathbb{R}^N$ or $(\mathbb{C}^N)$ is a another solution to (1). then $y(t) = x(t)$ for all $t \in J_y$.

Proof.

The fact that $x(t) = \Phi(t, \tau)\xi$ is a solution to I.V:P. follows by construction and from the properties of the transition matrix function. Only uniqueness must be proved. Consider function $e(t) = x(t) - y(t)$ on the interval $J_y \subset J$. By linearity it satisfies the equation

$$e(t) = \int_\tau^t A(\sigma)e(\sigma)d\sigma, \quad \forall t \in J_y$$

Applying the triangle inequality for integrals we conslude that

$$\|e(t)\| \leq \int_\tau^t \|A(\sigma)\| \, \|e(\sigma)\| \, d\sigma, \quad \forall t \in J_y$$

Point out that on an arbitrary bounded closed (compact) interval $[a, b] \subset J_y$ the piecewise continuous $A(\sigma)$ matrix valued function has a bounded norm $\|A(\sigma)\| < K$.Therefore for any $\tau, t \in [a, b]$

$$\|e(t)\| \leq \int_\tau^t K \, \|e(\sigma)\| \, d\sigma, \quad \forall t, \tau \in [a, b]$$

and by the simple variant of Grönwall 's inequality that we proved before, $\|e(t)\| = 0$ for all $t \in [a, b]$ and therefore $y(t) = x(t)$ for all $t \in J_y$.

0.3 Solution space.

We have considered a particular variant of the following theorem in the case of linear systems of ODEs with constant coefficients. The formulation and the proof we suggested are based only on the fact that the set of solutions $\mathbb{S}_h$ is a linear vector space and on the property of the uniquness of solutions. We repeat this argument here again with some corollaries about the structure of the transition matrix $\Phi(t, \tau)$.

Proposition 2.7 (1), p.30, L&R.

Let $b_1, ..., b_N$ be a basis in $\mathbb{R}^N$ (or $\mathbb{C}^N$) and let $\tau \in J$.

Let $\Phi(t, \tau)$ be a transition matrix to the equation

$$x' = A(t)x$$

with $A(t)$ being a matrix valued function $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$), piecewise continuous on the interval J.

Then functions $y_j : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) defined as solutions

$$y_j(t) = \Phi(t, \tau)b_j$$

with $j = 1, ..., N$ to , the equation above form a basis of the solution space $\mathbb{S}_h$ of the equation.

In particular $\mathbb{S}_h$ is N -dimensional and for every solution $x(t) : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) there exist scalars $\gamma_1, ...\gamma_N$ such that

$$x(t) = \sum_{j=1}^{N} \gamma_j y_j(t)$$

for all $t \in J$.

Proof

We can just repeat here the proof that we gave earlier. Point out that it is more general than one given in the book.

Suppose that at some time t solutions $y_j(t)$ are linearly dependent. It means that there are constants $\{a_j\}_{j=1}^N$ not all zero such that

$$\sum_{j=1}^{N} a_j y_j(t) = 0$$

at this time. On the other hand there is a solution that satisfies this condition. It is zero solution $x_*(t) = 0$ for all t.

But then these two solutions must coinside because solutions are unique!!! Namely $\sum_{j=1}^N a_j y_j(t) = 0$ for all times including $t = \tau$.Therefore $\sum_{j=1}^N a_j y_j(\tau) = \sum_{j=1}^N a_j b_j = 0$ because b_j are initial conditions at $t = \tau$ for y_j. It is a contradiction because vectors b_j , $j = 1, ..., N$ are linearly independent. Therefore $y_j(t)$ with $j = 1, ..., N$ are linearly independent for all t in J. ■

Example.

Calculate the transition matrix function $\Phi(t,s)$ for the system of equations

$$\begin{cases} x_1' = t\,x_1 \\ x_2' = x_1 + t\,x_2 \end{cases}$$

$$\begin{aligned} x' &= A(t)x; \quad A(t) = \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix} \\ x(\tau) &= \xi \end{aligned}$$

$$x(t) = \Phi(t,\tau)\xi$$

Here the matrix $A(t)$ is triangular.

The system of ODE above has triangular matrix and can be solved recursively starting from the first equation.

The fundamental matrix $\Phi(t,\tau)$ satifies the same equation, namely

$$\begin{aligned} \frac{d}{dt}\Phi(t,\tau) &= A(t)\Phi(t,\tau) \\ \Phi(\tau,\tau) &= I \end{aligned}$$

$\Phi(t,\tau)$ has columns $\pi_1(t,\tau)$ and $\pi_2(t,\tau)$ that at the time $t=\tau$ have initial values $[1,0]^T$ and $[0,1]^T$, because $\Phi(\tau,\tau) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We will use a general solution to the scalar linear equation $x' = p(t)x + g(t)$ with initial data $x(\tau) = x_0$ calculated using the primitive function $\mathbb{P}(t,\tau)$ of $p(t)$:

$$x(t) = \exp\{\mathbb{P}(t,\tau)\}\,x_0 + \int_\tau^t \exp\{\mathbb{P}(t,s)\}\,g(s)ds$$

A derivation of this formula using the integrating factor idea follows.

$$\begin{aligned}
x' &= p(t)x + g(t), \quad x_0 = x(\tau) \\
\mathbb{P}(t,\tau) &= \int_\tau^t p(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' &= \exp\{-\mathbb{P}(t,\tau)\}\, p(t)x + \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' - p(t)\exp\{-\mathbb{P}(t,\tau)\}\, x &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' + (\exp\{-\mathbb{P}(t,\tau)\})'\, x &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
[\exp\{-\mathbb{P}(t,\tau)\}\, x]' &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\int_\tau^t [\exp\{-\mathbb{P}(s,\tau)\}\, x(s)]'\, ds &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x(t) - \exp\{-\mathbb{P}(\tau,\tau)\}\, x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x(t) - \exp\{0\}\, x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds
\end{aligned}$$

$$\begin{aligned}
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,\tau)\}\exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau)\}\, g(s)ds \\
\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau) &= \int_\tau^t p(z)dz - \int_\tau^s p(z)dz = \int_\tau^t p(z)dz + \int_s^\tau p(z)dz = \\
\int_s^t p(z)dz &= \mathbb{P}(t,s) \\
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,s)\}\, g(s)ds; \\
x(\tau) &= x_0
\end{aligned}$$

In the equation

$$x_1' = t\, x_1$$

the coefficient $p(t) = t$, therefore $\mathbb{P}(t,\tau) = \int_\tau^t s\, ds = \left(\frac{1}{2}s^2\right)\Big|_\tau^t = \frac{1}{2}\left(t^2 - \tau^2\right)$ and the solution

$$x_1(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_1(\tau).$$

The second equation

$$x_2' = t\, x_2 + x_1$$

is similar but inhomogeneous:

$$x_2(t) = \exp(\mathbb{P}(t,\tau))x_2(\tau) + \int_\tau^t \exp(\mathbb{P}(t,s))x_1(s)ds.$$

Substituting $\mathbb{P}(t,\tau)=\frac{1}{2}\left(t^2-\tau^2\right)$ we conclude that$=\exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau)+\int_\tau^t \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)ds$

$$\begin{aligned}
x_2(t) &= \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau)+\int_\tau^t \exp(\frac{1}{2}\left(t^2-s^2\right))\exp(\frac{1}{2}\left(s^2-\tau^2\right))x_1(\tau)ds \\
&= \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau)+\int_\tau^t \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)ds
\end{aligned}$$

And

$$x_2(t)=\exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau)+\exp(\frac{1}{2}\left(t^2-\tau^2\right))(t-\tau)x_1(\tau).$$

The fundamental matrix solution $\Phi(t,\tau)$ has columns that are solutions to $x'=A(t)x$ with initial data - that are columns in the unit matrix: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Taking $x_1(\tau)=1$ and $x_2(\tau)=0$ we get $x_1(t)=\exp(\frac{1}{2}\left(t^2-\tau^2\right))$ with $x_2(t)=\exp(\frac{1}{2}\left(t^2-\tau^2\right))(t-\tau)$

Taking $x_1(\tau)=0$ and $x_2(\tau)=1$ we get $x_1(t)=0$ with $x_2(t)=\exp(\frac{1}{2}\left(t^2-\tau^2\right))$ and the fundamental matrix solution in the form

$$\Phi(t,\tau)=\exp(\frac{1}{2}\left(t^2-\tau^2\right))\begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix}$$

0.4 Group properties of transition matrix. Chapman - Kolmogorov relations.

remeber that in the case with autonomous systems the transition matrix $\Phi(t,\tau)=\exp\left((t-\tau)A\right).$

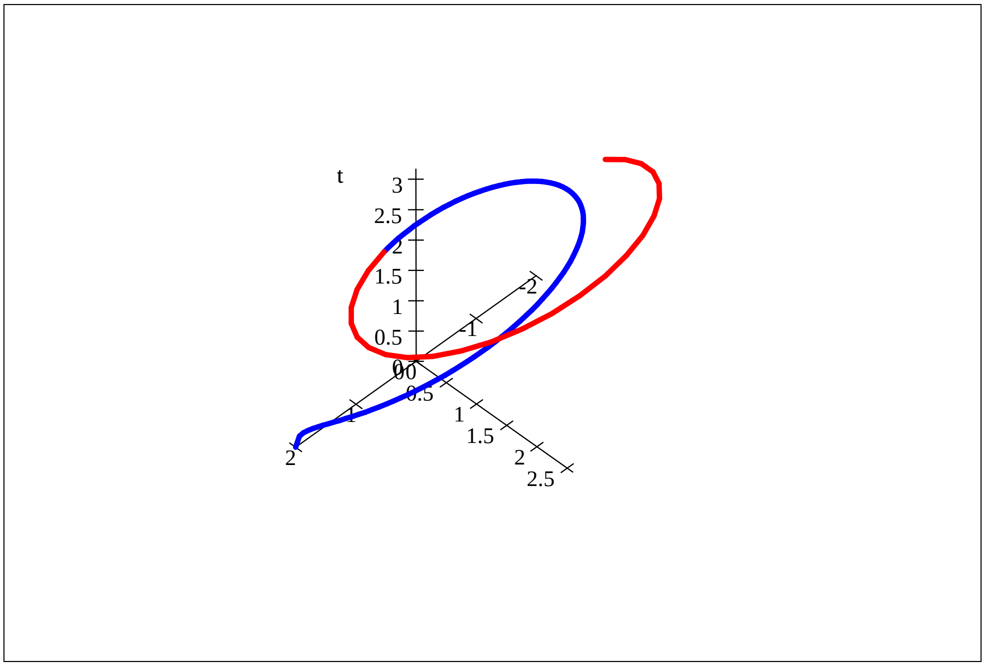
Therefore in this case

$$\begin{aligned}
\Phi(t,\tau) &= \exp\left\{(t-\tau)A\right\}=\exp\left\{(t-\sigma)A\right\}\exp\left\{(\sigma-\tau)A\right\} \\
&= \exp\left\{(t-\sigma)A+(\sigma-\tau)A\right\}=\Phi(t,\sigma)\Phi(\sigma,\tau) \\
\Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau)
\end{aligned}$$

The transition matrix $\Phi(t,\tau)$ defines a **transition mapping** $\varphi(t,\tau,\xi)$, that maps initial data ξ at time τ into the state $\varphi(t,\tau,\xi)=x(t)=\Phi(t,\tau)\xi$ of the system at time t.

Let us consider two consequtive solutions of the equation $x(t)=\Phi(t,\tau)\xi$ and $y(t)=\Phi(t,\sigma)\left(\Phi(\sigma,\tau)\xi\right)$ that continue each other in the time point $t=\sigma$ where the second solution $y(t)$ attains the initial state that is the point where the the first solution $x(t)$ arrives at time $t=\sigma$. Together with the uniqness of solutions, this consideration leads to the group property of the transition mapping and the transition matrix. The group property means that moving the system governed by the equation $x'(t)=A(t)x(t)$ from time τ to time t is the same as to move it first from time τ to time σ (blue curve) and then to move it without break from time σ to time t (red curve)

$$\Phi(t,\tau)\xi=\Phi(t,\sigma)\left[\Phi(\sigma,\tau)\xi\right]$$



Point out that these two "movements" do not need to go both in the positive direction in time as it is in the picture. One of these movements (or both) can go backward in time. Another observation is that the linearity of the system was not essential for this reasoning, only the uniqueness of solutions. We will use a similar argument later for non-linear systems.

We have proven (almost) the following theorem.

Corollary 2.6, p.29 L&R (Chapman - Kolmogorov relations)

For all t, σ, $\tau \in J$

$$\begin{aligned} \Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau), & (5) \\ \Phi(t,t) &= I, \\ \Phi(\tau,t)\Phi(t,\tau) &= \Phi(\tau,\tau) = I \\ \Phi(\tau,t) &= (\Phi(t,\tau))^{-1} & (6) \end{aligned}$$

Proof.

The first statement has been proven already. The second follows from the integral equation for the transfer matrix. The third one follows from the first two. We apply the first statement $\Phi(t,\tau)\ \Phi(\tau,t) = \Phi(t,t) = I$ therefore $\Phi(\tau,t)$ is the right inverse of $\Phi(t,\tau)$. The same argument for this expression with t and τ changed their roles leads to that $\Phi(\tau,t)$ is the left inverse of $\Phi(t,\tau)$.■

0.5 Fundamental matrix solution.

Introducing the transition matrix function $\Phi(t,\tau)$ for non-autonomous system of equations was similar to considering $\exp(A\,(t-\tau))$ for autonomous linear systems. We have got a solution to an arbitrary I.V.P. by multiplying arbitrary initial data $x(\tau) = \xi$ with the the transition matrix function: $x(t) = \Phi(t,\tau)\xi$.

On the other hand we could construct a general solution to an autonomous linear system just by taking a linear combination of N linearly independent solutions to the system, because the dimension of the solution space is equal to N.

The situation is exactly the same for non-autonomous linear systems with the difference that we in general cannot find a basis for the space of solutions analytically. It is possible only in some particular cases, for example for a triangular matrix $A(t)$.

Definition.

The functon $t \longmapsto \Psi(t) \in \mathbb{R}^{n\times n}$is called the **fundamental matrix solution** for the system $x' = A(t)x$, $x \in \mathbb{R}^n$ if it's columns $\Psi_k(t)$, $k = 1, ..., N$ are linearly independent solutions to the system (and therefore build a basis to the solution space): $\Psi_k'(t) = A(t)\Psi_k(t)$

It follows from the definition of the matrix product that

$$\Psi'(t) = A(t)\Psi(t)$$

General solution to the system is a linear combination of these vector valued functions:

$$x(t) = \Psi(t)C$$

with an arbitrary constant vector $C \in \mathbb{R}^N$.

The fundamental matrix solution $\Psi(t)$ is an invertible matrix for all t because it's columns are linearly independent for all t.

There is a simple connection between an arbitrary fundamental matrix solution $\Psi(t)$ and the transition matrix $\Phi(t, \tau)$.

Proposition 2.8 , p. 33

$$\Phi(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$$

Proof.

The product $X(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$ satisfies the equation

$$X'(t, \tau) = A(t)X(t, \tau)$$

in all points $t \in J$ where $A(t)$ is continuous, because each column in $\Psi(t)$ does it. On the other hand $\Psi(\tau)\Psi^{-1}(\tau) = I$. Therefore $X(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$ satisfies the integral equation

$$X(t, \tau) = I + \int_\tau^t A(\sigma)X(\sigma, \tau)d\sigma$$

in all points $t \in J$ because each column in $\Psi(t)$ does it. The same equations are satisfied by $\Phi(t, \tau)$. By the uniqueness of solutions to linear systems $\Phi(t, \tau) = X(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$.

This proposition shows another way to calculate the transition matrix solution, because sometimes it is easier to find some basis for the space of solutions and to put it into a matrix $\Psi(t)$ instead of solving the matrix equation for $\Phi(t, \tau)$.

Point out that it is easy to find a solution to the equation for $\Psi_*(t)$ with initial data $\Psi_*(\tau) = I$. For such a solution the formula connecting $\Phi(t,\tau)$ simplifies to $\Phi(t,\tau) = \Psi_*(t)$ because $\Psi_*^{-1}(\tau) = I$.

0.6 Abel - Liouville's formula.

Lemma about the derivative of a determinant of a matrix valued function.

Let $B : J \to \mathbb{R}^{N\times N}$ be differentiable. Then the derivative of it's determinant satisfies the following formula

$$(\det(B(t))' = \sum_{k=1}^{N} \det\left(U_k(B)\right)$$

where matrices $U_k(B)$ have the same columns $b_k(t)$ as the matrix $B(t) = [b_1(t), ..., b_N(t)]$ except the k -th column exchanged by the column of derivatives of the k-th column in $B(t)$.

$$U_k(B) = \left[b_1(t), ..., \left[\frac{d}{dt}b_k(t)\right], ..., b_N(t)\right]$$

A similar relation can be written for rows instead of columns.

An elementary proof can be carried out using the definition of derivative as a limit of a finite difference and repeated application of the addition formula for determinants. **Prove it as an exercise on properties of determinants!**

Consider a homogeneous linear system of ODEs $x'(t) = A(t)x(t)$ and N solutions $y_1(t), y_2(t),...,y_N(t)$ to it. Consider the matrix $Y(t)$ having these solutions as it's columns:

$$Y(t) = [y_1(t), y_2(t), ..., y_N(t)]$$

Definition.

The determinant

$$w(t) = \det Y(t) = \det\left[y_1(t), y_2(t), ..., y_N(t)\right]$$

is called **Wronskian** associated with solutions $y_1(t), y_2(t), ..., y_N(t)$.

Proposition 2.7 part (2) - Abel - Liouville's formula

Wronskian $w(t)$ associated with solutions $y_1(t), y_2(t),...,y_N(t)$ to the system $x'(t) = A(t)x(t)$ satisfies the following relations:

$$w(t) = w(\tau)\det\Phi(t,\tau)$$

In points t where $A(t)$ is continuous it satisfies the differntial equation

$$w'(t) = \mathrm{tr}(A(t))w(t)$$

and therefore with initial value for $w(\tau)$ at time τ :

$$w(t) = w(\tau)\exp\left(\int_{\tau}^{t} \operatorname{tr}(A(s))ds\right) \tag{7}$$

for all $t \in J.\square$

Proof.

We use here that $y_k(t) = \Phi(t,\tau)y_k(\tau)$ and therefore $Y(t) = \Phi(t,\tau)Y(\tau)$. It implies that

$$w(t) = \det Y(t) = \det Y(\tau)\det\Phi(t,\tau) = w(\tau)\det\Phi(t,\tau)$$

giving the first statement of the Proposition.

We denote by $\varphi_k(t)$ columns in $\Phi(t,\tau)$, so that $\Phi(t,\tau) = [\varphi_1(t), \varphi_2(t), ..., \varphi_N(t)]$.Then we apply the Lemma **about the derivative of a determinant of a matrix valued function** to the case $B(t) = \Phi(t,\tau)$. A direct substitution implies that

$$\frac{\partial}{\partial t}\left(\det\Phi(t,\tau)\right) = \sum_{k=1}^{N}\det\left(U_k(\Phi(t,\tau))\right) = \sum_{k=1}^{N}\det\left(\left[\varphi_1(t), ..., \frac{\partial}{\partial t}\left(\varphi_k(t)\right), ..., \varphi_N(t)\right]\right)$$

where the k-th column in $U_k(\Phi(t,\tau))$ is $\frac{\partial}{\partial t}\left(\varphi_k(t)\right)$ and other columns are columns $\varphi_j(t)$, $j \neq k$, $j = 1, ...N$ from $\Phi(t,\tau)$.

$\frac{\partial}{\partial t}\left(\varphi_k(t)\right) = A(t)\varphi_k(t)$, because $\varphi_k(t)$ are solutions to the system $x'(t) = A(t)x(t)$. We assume here that τ is not a point of discontiuity for $A(t)$. It leads to the more explicit expression:

$$\frac{\partial}{\partial t}\left(\det\Phi(t,\tau)\right) = \sum_{k=1}^{N}\det\left(U_k(\Phi(t,\tau))\right) = \sum_{k=1}^{N}\det\left(\left[\varphi_1(t), ..., A\left(\varphi_k(t)\right), ..., \varphi_N(t)\right]\right)$$

Setting $t = \tau$, into the last formula for we arrive to

$$\left.\frac{\partial}{\partial t}\left(\det\Phi(t,\tau)\right)\right|_{t=\tau} = \sum_{k=1}^{N}\det\left([e_1, ..., A(\tau)e_k, ..., e_N]\right)$$

because $\Phi(\tau,\tau) = I = [e_1, ..., e_k, ..., e_N]$. Observe that $A(\tau)e_k = [A(\tau)]_k$ - is the k-th column in $A(\tau)$.Matrices under the determinant sign in the last formula are diagonal with all elements equal to one except one equal to $[A(\tau)]_k$. Its determinant is the product of diagonal elements $\det\left([e_1, ..., A(\tau)e_k, ..., e_N]\right) = A(\tau)_{kk}$.Therefore

$$\left.\frac{\partial}{\partial t}\left(\det\Phi(t,\tau)\right)\right|_{t=\tau} = \sum_{k=1}^{N}\det\left([e_1, ..., [A(\tau)]_k, ..., e_N]\right) = \sum_{k=1}^{N}A_{kk}(\tau) = \operatorname{tr}A(\tau)$$

$$\det\left[e_1, ..., [A(\tau)]_k, ..., e_N\right] = \det\begin{bmatrix} 1 & 0 & A_{13} & 0 & 0 \\ 0 & 1 & A_{23} & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 \\ 0 & 0 & A_{43} & 1 & 0 \\ 0 & 0 & A_{53} & 0 & 1 \end{bmatrix}, \quad k = 3$$
$$= 1 \times 1 \times A_{33} \times 1 \times 1 = A_{33}$$

Therefore

$$w'(\tau) = w(\tau)\mathrm{tr}A(\tau)$$

The argument given here applies to any $\tau \in J$ that is not a point of discontinuity for $A(t)$. The expression

$$w(t) = w(\tau)\exp\left(\int_\tau^t tr(A(s)ds\right)$$
$$w(t) = \det Y(t)$$

follows by integration of the differential equation for $w(t)$ using method of integrating factor applied to a scalar first order linear equation.■

Interesting observations with application of Abel - Liouville's formula.

The geometric meaning of determinant $\det(C)$ of the matrix $C = [c_1, ...c_N]$ with columns $c_1, ...c_N$ is volume of the parallelepiped V build on vectors $c_1, ...c_N$:

$$|\det(C)| = vol\,(V)$$

One can define V formally as $V = \left\{x \in \mathbb{R}^N : \quad x = \sum_{k=1}^N a_k\, c_k, \quad a_k \in [0,1], k = 1, ..., n\right\}$.

It implies that the Abel - Liouvilles formula gives an exact description of how for example the volume of a unique cube build on standard basis vectors $e_1, ..., e_N$ given at the initial time τ is changing by the "flow" described by the transition matrix function $\Phi(t, \tau)$.

0.7 Non-homogeneous linear systems and Duhamel's formula in general case.

We consider the I.V.P. for non-homogeneous linear system

$$x'(t) = A(t)x(t) + b(t), \quad x(\tau) = \xi, \quad (\tau, \xi) \in J \times \mathbb{R}^N(J \times \mathbb{C}^N)$$

We suppose here that $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) is continuous or piecewise continuous and denote by $\Phi(t, \tau)$ the transition matrix function generated by $A(t)$. We rewrite the I.V.P. for the system also in integral form

$$x(t) = \xi + \int_\tau^t (A(\sigma)x(\sigma) + b(\sigma))\, d\sigma,$$

that allows to consider continuous solutions in the case when A is only peacewie continuous. In this case solutions satisfy the differential form of the problem in time points outside of discontinuties of A.

Theorem 2.15, p. 41 L&R
Let $(\tau,\xi)\in J\times\mathbb{R}^N$.The function

$$x(t)=\Phi(t,\tau)\xi+\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma,$$

is a unique solution to the I.V.P. above.

Proof. A simpler proof can be given for points t outside the discontinuties of A.

Apply the Chapman-Kolmogorov relation to the transition matrix under the integral: $\Phi(t,\sigma)=\Phi(t,0)\Phi(0,\sigma)$ and calculate derivative of the integral in the expression for the solution.

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma\right)\\
=\;&\frac{d}{dt}\left(\int_\tau^t\Phi(t,0)\Phi(0,\sigma)b(\sigma)d\sigma\right)=\frac{d}{dt}\left(\Phi(t,0)\int_\tau^t\Phi(0,\sigma)b(\sigma)d\sigma\right)\\
=\;&\left(\frac{d}{dt}\Phi(t,0)\right)\int_\tau^t\Phi(0,\sigma)b(\sigma)d\sigma+\left(\Phi(t,0)\frac{d}{dt}\left(\int_\tau^t\Phi(0,\sigma)b(\sigma)d\sigma\right)\right)\\
=\;&A\Phi(t,0)\int_\tau^t\Phi(0,\sigma)b(\sigma)d\sigma+\Phi(t,0)\Phi(0,t)b(t)
\end{aligned}$$

Observe that by Chapman -Kolmogorov relation $\Phi(t,0)\Phi(0,t)=\Phi(t,t)=I$, and $\Phi(t,0)\Phi(0,\sigma)=\Phi(t,\sigma)$. It implies simplifications in the last formula and finally

$$\frac{d}{dt}\left(\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma\right)=A\left(\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma\right)+b(t)$$

Therefore $\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma$ is the solution to the inhomogeneous equation with initial condition zero. Together with the solution $\Phi(t,\tau)\xi$ to the homogeneous equation, satisfying the initial condition $\Phi(\tau,\tau)\xi=\xi$ we conclude that $x(t)=\Phi(t,\tau)\xi+\int_\tau^t\Phi(t,\sigma)b(\sigma)d\sigma$, is a solution to the I.V.P. above. The uniqueness follows if we consider difference between two solutions $x(t)$ and $y(t)$ with the same initial condition: $z(t)=x(t)-y(t)$ that evidently satisfies the homogeneous equation $z'(t)=A(t)z(t)$ and the zero initial condition $z(\tau)=0$. The known result for homogeneous linear systems based on Grönwall's inequality implies that $z(t)=0$ on J.

Another proof based on the integral formulation of the problem and on the explicit checking that $x(t)$ expressed as in the formulation of the theorem satisfies it, is given in the book on the page 41.

1 Systems with periodic coefficients: Floquet theory

We consider here linear homogeneous systems of ODE's with $J=R$ and a continuous or piecewise continuous matrix $A:\mathbb{R}\to\mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$), with period $p>0$:

$$x'(t)=A(t)x(t),\quad A(t+p)=A(t),\quad \forall t\in\mathbb{R}$$

Let Φ be a transition function generated by a periodic $A(t)$.

Shifting invariance property.(formula 2.31, p. 45 in L.R.)

We are going to prove an important *shifting invariance property* of this transition matrix function, namely that

$$\Phi(t+p, \tau+p) = \Phi(t, \tau) \tag{8}$$

Structure of the transition matrix for a time interval including a finite number of periods. (formula 2.32, p. 45 in L.R.)

(Motivation to introducing the monodromy matrix)

Another property specifying further how the periodicity of the system influences properties of solutions.

$$\Phi(t+p, \tau) = \Phi(t, 0)\Phi(p, 0)\Phi(0, \tau) \tag{9}$$

$$\Phi(t+n\,p, \tau) = \Phi(t, 0)\left[\Phi(p, 0)\right]^n \Phi(0, \tau) \tag{10}$$

for any $(t, \tau) \in \mathbb{R} \times \mathbb{R}$.

Definition of the Monodromy matrix

The matrix $\Phi(p, 0)$ for a periodic linear system with period p is called the **monodromy matrix** (this standard notion is not used in the book)

Proof of the shifting invariance property.

This first property is untuitively clear.

The matrix $\Phi(t, \tau)$ satisfies the equation

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$$

with initial condition $, \Phi(t, \tau)|_{t=\tau} = I$.

The matrix $\Phi(t+p, \tau+p)$ satisfies the equation

$$\frac{\partial}{\partial t}\Phi(t+p, \tau+p) = A(t+p)\Phi(t+p, \tau+p)$$

with initial condition , $\Phi(t+p, \tau+p)|_{t=\tau} = I$.

Now we observe that $A(t) = A(t+p)$. Substituting it in the second equation we get the equation

$$\frac{\partial}{\partial t}\Phi(t+p, \tau+p) = A(t)\Phi(t+p, \tau+p)$$

with the same initial condition, $\Phi(\tau+p, \tau+p) = I$ on the interval $t \in [\tau, t)$.

It implies that $\Phi(t, \tau)$ and $\Phi(t+p, \tau+p)$ satisfy in fact the same equation with the same initial conditions $\Phi(t+p, \tau+p)|_{t=\tau} = I$. The uniquness of solutions implies that they must be equal: $\Phi(t+p, \tau+p) = \Phi(t, \tau)$.

A prove using the integral form of the equation is presented in the course book.■

Proof of the structure of the transition matrix for periodic system

The proof is based on a combination of the shifting property with the Chapman-Kolmogorov relations.

$$\begin{aligned}
&\Phi(t+p,\tau) \overset{Ch.-Kol.}{=} \Phi(t+p,\tau+p)\Phi(\tau+p,\tau) \overset{Shift}{=} \Phi(t,\tau)\Phi(\tau,\tau-p) \\
&\overset{Ch.-Kol.}{=} \Phi(t,\tau)\Phi(\tau,0)\Phi(0,\tau-p) \overset{Ch.-Kol._and_Shift}{=} \Phi(t,0)\Phi(p,\tau) \\
&\overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(p,0)\Phi(0,\tau)
\end{aligned}$$

The second equality for the shift $n\,p$ in n periods p in time is derived by the repetition of the last argument and induction

$$\begin{aligned}
&\Phi(t+np,\tau) \overset{Ch.-Kol.}{=} \Phi(t+np,\tau+np)\Phi(\tau+np,\tau) \overset{Shift}{=} \Phi(t,\tau)\Phi(\tau,\tau-np) \\
&\overset{Ch.-Kol.}{=} \Phi(t,\tau)\Phi(\tau,0)\Phi(0,\tau-np) \overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(np,\tau) \\
&\overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(np,0)\Phi(0,\tau)
\end{aligned}$$

and from the observation that $\Phi(np,0) = \Phi(np,np-p)...\Phi(kp,kp-p)...\Phi(2p,p)\Phi(p,0) = [\Phi(p,0)]^n$ that follows from the Chapman-Kolmogorov relation and from the fact that $\Phi(t,0)$ satisfies the same equation on each interval $[kp,(k+1)\,p]$, (shift invariance property) because $A(t) = A(t+p)$ is a periodic matrix with period p.

■

Example illustrating ideas of Floquet theory on a scalar linear equation.

Consider the following scalar linear equation with periodic coefficient $A(t) = (\sin(4t) - 0.1)$ with period $p = 0.5\pi$:

$$\frac{dx}{dt} = (\sin(4t) - 0.1)\,x,$$

We will find the monodromy matrix for this simple equation and demonstrate all objects related to the Floquet theorem.

The exact general solution is:

$$x(t) = C\exp\left(-0.25\cos\left(4t\right) - 0.1t\right)$$

with arbitrary constant C, can be found by the method with integrating factor.

To find the solution equal to 1 at $t = 0$ that is the transfer "matrix" in the scalar case, we calculate the expression $\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}\big|_{t=0} = 0.778\,8$ and choose $C = \frac{1}{0.778\,8}$ in the expression for the general solution $x(t)$.

The transfer "matrix" is:

$$\Phi(\,t,0) = \frac{1}{0.778\,8}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}$$

The period of the coefficient in the system is $p = 0.5\pi$ and the **monodromy matrix** is $\Phi(\,p,0) = \Phi(\,0.5\pi,0)$:

$\Phi(p,0) = \frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}\Big|_{t=0.5\pi} = 0.85464$

The eigenvalue μ of the (1x1) "monodromy matrix" $\Phi(p,0)$ coinsides with it's value: $\mu = 0.85464 < 1$ and is strictly less than 1.

Consider the logarithm $G = \ln\left(\Phi(p,0)\right)$ of the **monodromy matrix** $\Phi(p,0)$:

$$G = \ln\left(\Phi(p,0)\right) = \ln\left(\frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}\right)\Big|_{t=0.5\pi} = -0.15708$$

$F = \frac{G}{p} = \frac{-0.15708}{0.5\pi} = -0.1 < 0$

Therefore the eigenvalue $\lambda = -0.1$ of the "matrix" $F = \frac{1}{p}G$ is negative.

The transfer matrix to the system

$$y'(1) = Fy(t)$$

is

$$\exp(Ft) = \exp(t\frac{G}{p}) = \exp(-0.1t).$$

Compare black and green graphs for $\exp(t\frac{G}{p})$ and for $\Phi(t,0) = \frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}$. Observe that $\exp(t\frac{G}{p})$ and $\Phi(t,0)$ coinside in points $t = pn = (0.5\pi)\,n$, $n = 1,2,3...$

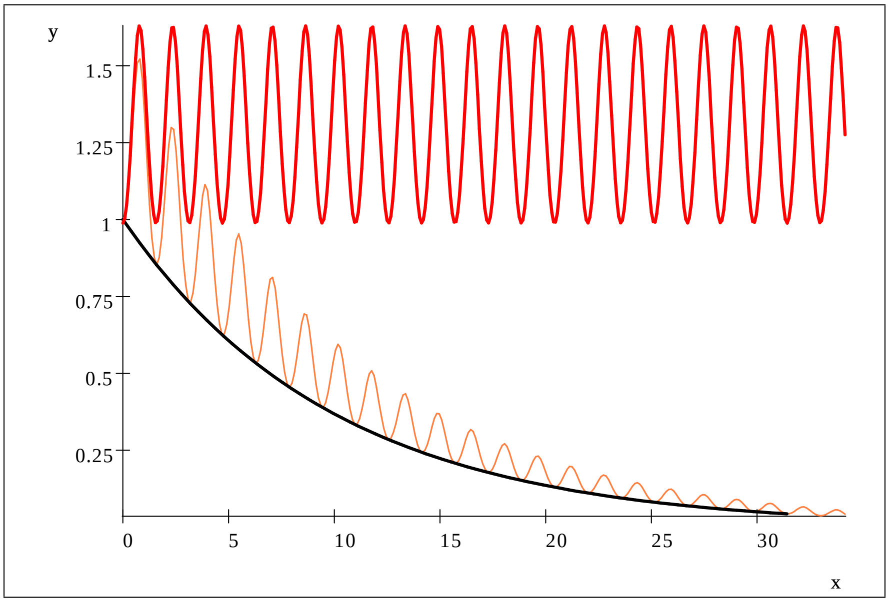
Introduce a "corrector" multiplicator $\Theta(t)$ introduced so that

$$\Phi(t,0) = \Theta(t)\exp(t\frac{G}{p})$$

Observe that

$$\Theta(t) = \frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)$$

is a $p = 0.5\pi$ - periodic function equal to 1 in all points $t = pn = (0.5\pi)\,n$, $n = 1,2,3...$(red graf).



We are going to observe soon that a similar representation of the transfer matrix $\Phi(t,0)$ is possible for an arbitrary periodic linear systems of ODEs and for it's transfer matrix $\Phi(t,0)$.

The main idea of the Floquet theory.

The monodromy matrix $\Phi(p, 0)$ is a particular transition matrix that maps initial data at time $\tau = 0$ to the state of the system after one period p. A particular property of this matrix in the case of periodic systems is that similar the mapping to the state at the time $t = np$ equal to n periods is just

$$\Phi(n \cdot p, 0) = [\Phi(p, 0)]^n$$

This property is similar to properties of autonomous linear systems where $\Phi(t, 0) = \exp(At)$ and therefore

$$\Phi(n \cdot p, 0) = \exp(A(n \cdot p)) = [\exp(A(p))]^n = [\Phi(p, 0)]^n \tag{11}$$

that follows from the factorisation property of the exponent of two commuting matrices:

$$\exp(A + B) = \exp(A)\exp(B)$$

In the case of periodic systems this factorisation applies only for shifts in time that are integer numbers of periods. But it is still a remarkable property. The behaviour of solutions is described by a repeated multiplication by a constant matrix in certain time points: p, $2p$, $3p$, ...:

$$x'(t) = A(t)x(t), \quad x(0) = \xi.$$

$$x(np) = [\Phi(p, 0)]^n \xi, \quad n = 0, 1, 2, ...$$

The first idea of the Floquet theory is to represent $x(np)$ at times $t = np$ similarly as for autonomous systems, namely with the help of an exponent of some constant matrix F times the time argument: $t = np$.

$$x(np) = [\Phi(p, 0)]^n \xi = \exp(np\, F)\xi = [\exp(p\, F)]^n \xi$$

It means that the matrix F in such representation must satisfy the relation

$$\Phi(p, 0) = \exp(pF).$$

Therefore the matrix pF must be something like the logarithm of the monodromy matrix:

$$pF = \log(\Phi(p, 0))$$

Definition. A matrix $G \in \mathbb{C}^{N \times N}$ is called a **loragithm of the matrix** $H \in \mathbb{C}^{N \times N}$ if

$$H = \exp(G)$$

We write in this case $G = \log(H)$.

We are going to prove soon that for any non-singular matrix H there is a logarithml $\log(H)$ in this sense. Point out that the monodromy matrix $\Phi(p, 0)$ is always non-singular, because columns in a transfer matrix $\Phi(\,t,0)$ are always linearly independent.

The logarithm of a matrix is not uniquely defined in the same way as it is not unique for complex and real numbers z:

$$\ln(z) = \ln(|z|) + i\arg(z) \tag{12}$$

because the argument $\arg(z)$ of a complex number is defined only up to $2\pi k$, $k = \pm 1, \pm 2, ...$

One can choose a unique branch for the logarithm function, called the *principle logarithm* or $\mathrm{Log}\,(z)$ by choosing the argument in the last formula (12) only in the interval $[0, 2\pi)$.

We will suspend the discussion of matrix logarithm now and will consider first an application of it to the analysis of solutions to periodic linear systems of ODEs.

The main idea in the Floquet theory is the "approximation" of the transfer matrix $\Phi(\,t,0)$ for a periodic linear system with matrix $A(t) = A(p+t)$ by the transfer matrix $\exp\,(t\,F)$ for an autonomous system

$$y'(t) = [F]\,y(t)$$

with the constant matrix $F = \left[\frac{1}{p}G\right]$ where

$$\begin{aligned} G &= \log(\Phi(\,p,0)) &(13)\\ \exp(G) &= \Phi(\,p,0) &(14)\\ \exp(pF) &= \Phi(\,p,0) &(15)\\ \exp(npF) &= [\Phi(\,p,0)]^n = \Phi(\,np,0) &(16) \end{aligned}$$

These two transfer matrices coinside in points $t = 0, p, 2p, 3p, ...$

$$\Phi(\,np,0) = [\Phi(\,p,0)]^n = \exp\left((np)\,[F]\right) \tag{17}$$

The deviation of $\Phi(\,t,0)$ from $\exp\,(t\,F)$ in intermediate points within one period can be expressed by a factor $\Theta(t)$ so that

$$\Phi(\,t,0) = \Theta(t)\exp\,(tF)$$

The matrix function $\Theta(t)$ must be equal to the unit matrix I in the points $t = 0, p, 2p, ...$ because in these points these two transfer functions coinside by construction, see (17).

The exact formulation of the properties of such factorization is given in the following Theorem by Floquet.

Theorem 2.30 , p. 53. Floquet theorem

Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of the monodromy matrix $\Phi(p,0)$.

$$G = \log(\Phi(p,0))$$

There exists a periodic with period p piecewise continuously differentiable function $\Theta(t) : \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular (invertible, all eigenvalues are non-zero) for all t, such that

$$\Phi(t,0) = \Theta(t)\exp\left(\frac{t}{p}G\right), \qquad \forall t \in \mathbb{R} \tag{18}$$

Proof.

We remind the main property (9) of the monodromy matrix for $\tau = 0$:

$$\Phi(t+p,0) = \Phi(t+p,p)\Phi(p,0) = \Phi(t,0)\Phi(p,0)$$

where we applied first the Chapman Kolmogorov relation (5) and then the shift invariance (8) of the transfer matrix function $\Phi(t,\tau)$ for a periodic linear system

We denote $\frac{1}{p}G$ by F for convenience, so that $G = pF$, and define the function $\Theta(t)$ after the desired relation (18)

$$\Theta(t) = \Phi(t,0)\exp\left(-\frac{t}{p}G\right) = \Phi(t,0)\exp(-tF)$$

The function $\Theta(t)$ is well defined in such a way. The problem is to show that it has desired properties: p - periodicity and satisfies initial conditions.

We remind that $\Theta(0) = I$ and even $\Theta(np) = I$ for all $n = 0, 1, 2, 3, ...$

$\Phi(t,0)$ is piecewise contiuously differentiable or contiuously differentiable depending on if $A(t)$ is piecewise continuous or continuous. Therefore $\Theta(t)$ has the same property because $\exp\left(-\frac{t}{p}G\right)$ is continously differentiable. $\Theta(t)$ is also invertible for all t as a product of two invertible matrices $\Phi(t,0)$ and $\exp(-tF)$.

We check now that $\Theta(t)$ is p - periodic, namely that $\Theta(t+p) = \Theta(t)$ for all $t \in \mathbb{R}$.

$$\begin{aligned}\Theta(t+p) &= \Phi(t+p,0)\exp(-(t+p)F) \\ &= \Phi(t+p,0)\exp(-pF)\exp(-tF) = \Phi(t+p,0)\overbrace{\exp(-G)}^{\Phi(0,p)}\exp(-tF)\end{aligned}$$

We remind that $\exp(G) = \exp(\log(\Phi(p,0)) = \Phi(p,0)$, therefore $\exp(-G) = (\exp(G))^{-1} = \Phi(p,0)^{-1} = \Phi(0,p)$. Therefore, using the main relation for the monodromy matrix (**??**) $\Phi(t+p,0) = \Phi(t,0)\Phi(p,0)$ together with the relation $\exp(-G) = \Phi(0,p)$, we arrive to

$$\Theta(t+p) = \Phi(t,0)\overbrace{\Phi(p,0)\Phi(0,p)}^{\Phi(p,p)=I}\exp(-tF) = \Phi(t,0)\,(I)\exp(-tF) \stackrel{\text{def}}{=} \Theta(t),$$

where we also used that $\Phi(p,0)\Phi(0,p) = I$ in the last step. Therefore $\Theta(t)$ is periodic with period p.■

1.1 Logarithm of a matrix. Existence and calculation.

We will formulate a theorem and give a proof to it (simpler than in the book) about the existence of a matrix logarithm.

Definition

The matrix G is a **logarithm of matrix** H or $G = \log(H)$ if $\exp(G) = \exp(\log(H)) = H$.

Consider a nonsingular matrix H and it's a canonical Jordan form J:

$$H = TJT^{-1}$$

where T is invertible matrix. Then if there is $Q \in \mathbb{C}^{N\times N}$, such that $\exp(Q) = J$ that means

$$Q = \log(J), \qquad J = \exp(Q)$$

then according to the properties of the exponent of similar matrices, and the definition of matrix logarithm

$$\begin{aligned} H &= TJT^{-1} = T\exp(Q)T^{-1} = T\exp(\log(J))T^{-1} = \\ &= \exp\left(T\log(J)T^{-1}\right) \stackrel{\text{def}}{=} \exp(\log(H)) \end{aligned}$$

and

$$\log(H) = T\log(J)T^{-1}$$

where we used that if $A = TBT^{-1}$ then $\exp(A) = T\exp(B)T^{-1}$.

It means that to calculate logarithm of an arbitrary matrix H it is enough to calculate the logarithm of it's Jordan canonical form. For $H = TJT^{-1}$

$$\log(H) = T\log(J)T^{-1}$$

Definition.

We say that G is a principal logarithm $G = \mathrm{Log}(H)$ of the matrix H if G is a matrix logarithm of H and

$$\begin{aligned} \sigma(H) &= \{\exp(\lambda) : \quad \lambda \in \sigma(G)\} \\ \sigma(G) &= \{\mathrm{Log}(\mu) : \quad \mu \in \sigma(H)\} \end{aligned}$$

where $\mathrm{Log}(\mu)$ is the scalar principal logarithm:

$$z = e^{\mathrm{Log}(z)}; \quad \arg(\mathrm{Log}\,(z)) = \mathrm{Im}(\mathrm{Log}\,(z)) \in [0, 2\pi).$$

This definition implies the explicit one to one correspondence between eigenvalues to H and eigenvalues to G. Essentially the second relation is non-trivial.

Theorem.Proposition 2.29, p. 53.

If $H \in \mathbb{C}^{N\times N}$ is invertible, then there exists a principle logarithm $\mathrm{Log}\,(H)$.

Proof.

We have established above that it is enough to investigate existence of logarithm for the similar canonical Jordan form J of the matrix. So without loss of generality we may assume that H is canonical Jordan form J. Exponent of a Jordan matrix consists of exponents of it's blocks. Therefore it is enough to establish the existence of logarithm for each Jordan block J_j in J, $j = 1, ..., s$ where s is the number of distinct eigenvectors to H and J_j has size $n_j \times n_j$

$$J_j = \begin{bmatrix} \lambda_j & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & \lambda_j & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_j \end{bmatrix}$$

$J_j = \lambda_j \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right)$ where

$$\mathcal{N}_j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

From the classical Maclaurin series for $\log(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p$ valid for $|x| < 1$, and for exp we get

$$\exp(\log(1+x)) = 1 + x$$

We formally write the Maclaurin series for $\log(1 + \frac{1}{\lambda_j} \mathcal{N}_j)$:

$$\log \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right) = \sum_{p=1}^{n_j - 1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j} \mathcal{N}_j \right)^p$$

and observe that the Maclaurin series for $\log(1 + \frac{1}{\lambda_j} \mathcal{N}_j)$ is a **finite sum** because all larger powers of $\mathcal{N}_j$ in the series cancel. We have therefore that

$$\exp \left(\log \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right) \right) = I + \frac{1}{\lambda_j} \mathcal{N}_j$$

and

$$\begin{aligned} & \exp\left(\log(\lambda_j) I\right) \exp \left(\log \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right) \right) \\ \exp \left(\log(\lambda_j) I + \log \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right) \right) \quad = \quad & \lambda_j \left(I + \frac{1}{\lambda_j} \mathcal{N}_j \right) \end{aligned}$$

We define

$$G_j \stackrel{\text{def}}{=} \log(\lambda_j)I + \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p$$

Then we check that this expression G_j is actually a matrix logarithm $\log(J_j)$ for the Jordan block J_j by checking that is satisfies the definition of the matrix logarithm. Point out that the diagonal matrix $\log(\lambda_j)I$ commutes with any matrix. Therefore applying formula $\exp(\log(1+x)) = 1+x$ for series for $\exp(x)$ and $\log(1+x)$ to similar converging series of commuting matrices we arrive to the desired relation.

$$\begin{aligned}
\exp(G_j) &= \exp\left(\log(\lambda_j)I + \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\
&= \exp\left(\log(\lambda_j)I\right)\exp\left(\sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\
&= \exp\left(\log(\lambda_j)I\right)\exp\left(\log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = \lambda_j\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right) = J_j
\end{aligned}$$

In the Jordan canonical form J eigenvalues stand on diagonal and are easy to control. All calculations that we have carried out are correct because $\lambda_j \neq 0$. We can choose logarithms $\log(\lambda_j)$ in these calculations as principle values of logarithm $\mathrm{Log}(\lambda_j)$. In this case the logarithm of J_j will be principal logarith, because there will be one to one correspondence between eigenvalues λ_j to J_j and eigenvalues $\mathrm{Log}(\lambda_j)$ to $\mathrm{Log}(J_j)$ that are diagonal elements in corresponding matrices. They will have the same algebraic multiplicity and the same geometric multiplicity 1 (one linearly independent eigenvector for each Jordan block)

Therefore the existense of the principal logarithm is established also for J and for H, that is a matrix similar to J. The same correspondence as above is valid for the eigenvalues to H and to $\mathrm{Log}(H)$ because eigenvalues to similar matrices H and J are the same. The number of linearly independent eigenvectors corresponding to each distinct eigenvalue (geometric multiplicity) will be also the same.■

1.2 Floquet multiplicators and exponents and bounds of solutions to periodic systems. equations.

Definition.

Eigenvalues of the monodromy matrix $\Phi(p, 0)$ are called **Floquet's multipliers** or **characteristic mutipliers.**

A Floquet multiplier is called semisimple if it is semisimple as an eigenvalue to the monodromy matrix $\Phi(p, 0)$.

Definition.

Eigenvalues of the logarithm of the monodromy matrix are called Floquet's exponents or characteristic exponents.

Theorem 2.31, p.54 on boundedness and zero limits of solutions to periodic linear systems.

1) Every solution to a periodic linear system is bounded on $\mathbb{R}_+$ if and only if the abosolute value of each Floquet multiplier is not greater than 1 and any Floquet multiplier with absolute value 1 is semisimple.

2) Every solution to a periodic linear system tends to zero at $t \to \infty$ if and only if the absolute value of each Floquet multiplier is strictly less than 1.

Proof.

By Floquet theorem any solution $x(t)$ to system

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \forall t \in \mathbb{R} \tag{19}$$

satisfying initial conditions

$$x(\tau) = \xi$$

is represented as

$$\begin{aligned} x(t) &= \Phi(t, \tau)\xi = \Theta(t)\exp(tF)\Phi(0, \tau)\xi = \Theta(t)\overbrace{\exp(tF)\zeta}^{y(t)} \\ &= \Theta(t)y(t) \end{aligned}$$

where

$$F = \frac{1}{p}\mathrm{Log}(\Phi(p, 0)), \quad \zeta = \Phi(0, \tau)\xi.$$

$\Theta(t)$ is a p - periodic continuous or piecewise continuous matrix valued function. $\Theta(t)$ is invertible for all t.

We define $y(t) = \exp(tF)\zeta$ as a solution to the equation

$$y'(t) = F\,y, \quad y(0) = \zeta \tag{20}$$

$y(t) = \Theta^{-1}(t)x(t)$, and $x(t) = \Theta(t)y(t)$. The mapping $\Theta(t)$ determines a one to one correspondence between solutions $x(t)$ to the periodic system (19) and solutions $y(t)$ to the autonomous system (20).The periodicity and continuity properties of $\Theta(t)$ and $\Theta^{-1}(t)$ imply that there is a constant $M > 0$ such that $\|\Theta(t)\| \leq M$ and $\|\Theta^{-1}(t)\| \leq M$ for all $t \in \mathbb{R}$. It implies that $\|x(t)\| \leq M\,\|y(t)\|$ and $\|y(t)\| \leq M\,\|x(t)\|$.

Therefore

1)$\|x(t)\|$ is bounded on $\mathbb{R}_+$ if and only if corresponding $\|y(t)\| = \|\exp(tF)\zeta\|$ is bounded on $\mathbb{R}_+$.

2) $\|x(t)\| \to 0$ when $t \to \infty$ if and only if corresponding $\|y(t)\| \to 0$ when $t \to \infty$.

Since $\operatorname{Log}(\Phi(p,0)) = G = pF$, and $\Phi(p,0) = \exp(pF)$ it follows that

$$\begin{aligned}
\sigma\left(\Phi(p,0)\right) &= \{\exp(\lambda p) : \quad \lambda \in \sigma(F)\} \\
\sigma\left(F\right) &= \left\{\frac{1}{p}\operatorname{Log}(\mu) : \mu \in \sigma(\Phi(p,0))\right\}
\end{aligned}$$

and that algebraic and geometric multiplicities of each $\lambda \in \sigma(F)$ coinside with those of $\exp(p\lambda) \in \sigma\left(\Phi(p,0)\right)$.We use now that

$$\begin{aligned}
\operatorname{Log}(z) &= \ln(|z|) + i\arg(z) \\
\exp(z) &= \exp(\operatorname{Re} z)(\cos(\arg z) + i\sin(\arg z))
\end{aligned}$$

The following connections between properties of Floquet multipliers and propertis of corresponding eigenvalues to the matrix $F = \frac{1}{p}\operatorname{Log}(\Phi(p,0))$ are a direct consequence:

a) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has$|\mu| < 1$ if and only if $\operatorname{Re}\operatorname{Log}(\mu) < 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F has $\operatorname{Re}\operatorname{Log}(\mu) < 0$.

b) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has $|\mu| \leq 1$ if and only if $\operatorname{Re}\operatorname{Log}(\mu) \leq 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F has $\operatorname{Re}\operatorname{Log}(\mu) \leq 0$.

c) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$, with $|\mu| = 1$ is semisimple if and only if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F having $\operatorname{Re}\operatorname{Log}(\mu) = 0$ is semisimple.

Known relations between properties of solutions to an autonomous system and the spectrum of corresponding matrix applied to the system $y'(t) = F\,y$ and to the spectrum $\sigma(F)$ of the matrix F together with statements 1),2), a),b),c) in the present proof imply the statement of the theorem.$\blacksquare$

Proposition 2.20. p. 45. On periodic solutions to periodic linear systems

The system $x'(t) = A(t)x(t)$ with p - periodic $A(t) = A(t+p)$ has a non-zero p - periodic solution if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue $\lambda = 1$. A more general statement is also valid.

The system has a non-zero $n\,p$ - periodic solution for $n \in \mathbb{N}$ if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = 1.\square$

Proof. Consider an eigenvector v corresponding to this eigenvalue λ. Then $v \neq 0$, $\Phi(p,0)v = \lambda v$ and

$$[\Phi(p,0)]^n\, v = \lambda^n v = v$$

We will show that the solution to the system, with initial data $x(0) = v$ has period np. This solution is given by the transition matrix: $x(t) = \Phi(t,0)v$. Using this representation and applying the factorisation property of transition matrices for periodic systems we arrive to

$$x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\,[\Phi(p,0)]^n\, v = \Phi(t,0)v = x(t), \quad \forall t \in \mathbb{R}$$

It shows that $x(t)$ is periodic with period $n\,p$.

Supposting that there is a periodic solution $x(t+np) = x(t)$ and repeating the same calculation backwards we arrive that $x(0) = v$ is an eigenvalue corresponding to an eigenvalue λ such that $\lambda^n = 1$.

Carry out this backward argument as an exercise!

■

Corollary 2.33, p. 59

We consider a periodic linear system $x'(t) = A(t)x(t)$, $A(t+p) = A(t)$.

If $\int_0^p \mathrm{tr}(A(s)ds$ has a positive real part, then the equation has at least one solution $x(t)$ that is unbounded, or formulating it more formally, the upper limit of it's norm is infinity: $\limsup_{t\to\infty} \|x(t)\| = \infty\square$

Proof.

We remind that the transfer matrix $\Phi(t,\tau)$ satifies the initial value problem:

$$\begin{aligned} \frac{d}{dt}\Phi(t,\tau) &= A(t)\Phi(t,\tau) \\ \Phi(\tau,\tau) &= I \end{aligned}$$

Arbitrary solution to the initial problem $x'(t) = A(t)x(t)$, $x(\tau) = \xi$ will be expressed as

$$x(t) = \Phi(t,\tau)\xi$$

According to Abel - Liouville's formula and considerations before

$$\begin{aligned} |\det(\Phi(t,0)\,)| &= \left|\det(\Phi(0,0))\exp\left(\int_0^t tr(A(s)ds\right)\right| = \\ \left|\exp\left(\int_0^t tr(A(s)ds\right)\right| &= \left|\exp\left(\mathrm{Re}\left(\int_0^t tr(A(s)ds\right)\right)\right| \end{aligned}$$

Therefore, if $\mathrm{Re}\left(\int_0^p tr(A(s)ds\right) > 0$ then

$$|\det(\Phi(p,0)\,)| = \left|\exp\left(\mathrm{Re}\int_0^p tr(A(s)ds\right)\right| > 1.$$

On the other hand $\det(\Phi(p,0))$ is a product of eigenvalues μ_k to the monodromy matrix $\Phi(p,0)$ with multiplicities m_k (it follows from the structure of similar Jordan matrix)

$$|\det(\Phi(p,0)| = \prod_{k=1}^{s} |\mu_k|^{m_k} > 1$$

To have this product greater than 1 we must have at least one eigenvalue μ_p with $\left|\mu_p\right|$ >1. Therefore, according to one of Floquet theorems, there is a solution $x(t)$ that is not bounded and therefore $\limsup_{t\to\infty} \|x(t)\| = \infty$. ■

For example we can choose the initial condition $x(0) = v_p$ with v_p being the eigenvector to $\Phi(p,0)$ corresponding to the eigenvalue $\left|\mu_p\right| > 1$.Then the solution

$$\begin{aligned} x(t) &= \Phi(t,0)v_p \\ \Phi(np,0)v_p &= [\Phi(p,0)]^n v_p = \left(\mu_p\right)^n v_p \end{aligned}$$

with $\left|\mu_p\right| > 1$. Therefore $x(t)$ is unbounded and $\limsup_{t\to\infty} \|x(t)\| = \infty$.

We give also a geometric interpretation of this result. Consider a unite cube build on standard base vectors $e_1, ..., e_N$ at time $t = 0$. Consider how the volume $\mathrm{Vol}(t)$ of this cube changes under the action of the linear transformation by the transfer matrix $\Phi(t, 0)$ of our periodic system. Point out that $I = [e_1, ..., e_N]$ .It implies that the figure of interest is the parallelepiped build on columns of the transfer matrix $\Phi(t, 0)$.One of the main properties of periodic system is that $\Phi(np, 0) = [\Phi(p, 0)]^n$. Therefore

$$\mathrm{Vol}(np) = |\det([\Phi(p, 0)]^n)| = |\det([\Phi(p, 0)])|^n = \left[\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds)\right)\right)\right]^n$$

If $\mathrm{Re}\left(\int_0^p tr(A(s)ds\right) > 0$ then $\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds)\right)\right) > 1$. It implies that

$$\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$$

Therefore along the sequence of times$\{t = np, \quad n = 1, 2, 3, ...\}$ $\mathrm{Vol}(n\,p)$ is unbounded. It implies also that

$$\limsup_{t\to\infty} \|\mathrm{Vol}(t)\| = \infty$$

The fact that $\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$ implies that the diameter $D(n\,p)$ of the parallelepiped build on columns of $\Phi(n\,p, 0)$ calculated at these discrete time points, is also unbounded $\sup\lim_{n\to\infty} D(n\,p) = \infty$. It in turn means that there should be a solution that has the property $\limsup_{t\to\infty} \|x(t)\| = \infty$.

3.5 Linear Systems with Periodic Coefficients

In this section, we shall study the linear periodic systems

$$x' = A(t)x, \quad A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}, \tag{LP}$$

where $A(t)$ is continuous on $\mathbb{R}$ and is periodic with period T, i.e., $A(t) = A(t+T)$ for all t. We shall analyze the structure of the solutions $x(t)$ of (LP). Before we prove the main results we need the following theorem concerning the logarithm of a nonsingular matrix.

Theorem 3.5.1 *Let $B \in \mathbb{R}^{n\times n}$ be nonsingular. Then there exists $A \in \mathbb{C}^{n\times n}$, called logarithm of B, satisfying $e^A = B$.*

Proof. Let $B = PJP^{-1}$ where J is a Jordan form of B, $J =$ diag

$(J_0, J_1, \cdots, J_s)$ with

$$J_0 = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & 0 \\ 0 & & \ddots & \\ & & & \lambda_k \end{bmatrix} \text{ and } J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i},$$

$i = 1, \cdots, s.$

Since B is nonsingular, $\lambda_i \neq 0$ for all i. If $J = e^{\tilde{A}}$ for some $\tilde{A} \in \mathbb{C}^{n\times n}$ then it follows that $B = Pe^{\tilde{A}}P^{-1} = e^{P\tilde{A}P^{-1}} \stackrel{\text{def}}{=} e^A$. Hence it suffices to show that the theorem is true for Jordan blocks J_i, $i = 1, \cdots, s$. Write

$$J_i = \lambda_i \left(I + \frac{1}{\lambda_i} N_i \right), \ N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Then $N_i^{n_i} = O$. From the identity

$$\log(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p, \ |x| < 1$$

and

$$e^{\log(1+x)} = 1 + x, \tag{3.11}$$

we formally write

$$\begin{aligned} \log J_i &= (\log \lambda_i) I + \log \left(I + \tfrac{1}{\lambda_i} N_i \right) \\ &= (\log \lambda_i) I + \textstyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p. \end{aligned} \tag{3.12}$$

From (3.12) we define

$$A_i = (\log \lambda_i) I + \sum_{p=1}^{n_i - 1} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p.$$

Then from (3.11) we have

$$e^{A_i} = \exp((\log \lambda_i) I) \exp \left(\sum_{p=1}^{n_i-1} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p \right) = \lambda_i \left(I + \frac{N_i}{\lambda_i} \right) = J_i.$$

$\square$

May 11, 2020

Lecture notes on non-linear ODEs: existence, extension, limit sets, periodic solutions.

Plan

1. Peano theorem on existence of solutions (without proof), Theorem. 4.2, p. 102.

2. Existence and uniqness theorem by Picard and Lindelöf . Th. 4.17, p. 118 (for continuous $f(t,x)$, locally Lipschitz in x), (Proof comes in the last week of the course)

Th.4.22, p.122 (for piecewise continuous $f(t,x)$, locally Lipschitz in x).

3. Maximal solutions. Openness of the maximal existence interval. Prop. 4.4., p. 107.

4. Existence of Maximal solutions. Theorem 4.8.

5. Extensibility of bounded solutions to the boundary time point of the interval. Lemma 4.9, p. 110.

6. Corollary 4.10, p. 111, on solutions enclosed in a compact, implying "infinite" maximal interval.

7. Properties of limits of maximal solutions. Theorem 4.11, p. 112 on the property of solutions with "finite" maximal interval $I_{\max}$, to escape any compact subset C in the space domain $C \subset G$.

8. On infinite existence interval for systems with linear growth estimate for the right hand side. Proposition 4.12, p. 114.

9. Transition map. Definition p. 126. Transition property of the transition map. Translation property for autonomous systems.

Theorem 4.26, p. 126. (similar to Chapman - Kolmogorov relations for transition matrix)

10. Openness of the domain and smoothness of transition map.Theorem 4.29, p. 129.

11. Autonomous systems. §4.6.1. Example 4.33., p. 139. of a transition map.

0.1 Non-linear systems. Existence and uniqueness of solutions.

Second half of the course deals with initial value problems for non-linear systems of ODE's, non-autonomous:

$$x'(t) = f(t,x), \quad f : J \times G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{1}$$

with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f- continuous in $J \times G$, and autonomous systems of ODE's:

$$x'(t) = f(x), \quad f : G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{2}$$

that are a particular case of (1) with $G \subset \mathbb{R}^n$, open, $\tau \in J = \mathbb{R}$, $\xi \in G$, f- continuous in G, where the right hand side f in the equation is independent of the time variable t running over the whole $\mathbb{R}$. The practical meaning of this kind of systems is that the "velocity" f of the system depends only on the position x, but not on time t. So independently of the starting time τ the output $x(t)$ of an evolution depends only on the shift in time $t - \tau$. It lets to choose always $\tau = 0$ for autonomous systems.

In many situations the equivalent integral form of I.V.P. is convenient to use:

$$x(t) = \xi + \int_\tau^t f(s, x(s))ds \tag{3}$$

Another option for requirements to f that is considered in the book by Logemann Ryan is that f is supposed to be piecewise continuous in t and locally Lipschitz with respect to x. We will not consider this case systematically in this part of the course.

Fixed point problems

The existence of solutions to abstract non - linear equations in the form of a so called **fixed point problem**

$$z = \mathcal{B}\{z\}$$

for an operator $\mathcal{B} : H \to H$ defined on a complete vector space H (Banach space) is resolved by one of two general methods.

1. **Compactness principles.** One of examples of this scope of ideas is the theorem by Schauder. For a continuous operator $\mathcal{B} : K \to K$ defined on a convex closed subset K of H and mapping it to a compact set $\mathcal{B}\{K\}$, there is at least one fixed point z in K that is a solution to the equation $z = \mathcal{B}\{z\}$.

2. **Banach's contraction principle.** Convergence of successive approximations: $z_{n+1} = \mathcal{B}\{z_n\}$ to a fixed point for a "small" operator $\mathcal{B}$.

The fundamental question of existence of solutions is answered by the following Peano theorem (with possibility of non-uniqueness of solutions)

Theorem 4.2, p. 102. Peano theorem.

For each (τ, ξ) in $J \times G$ there exists at least one solution to (1) defined on a (possibly small) time interval $I \subset J$, $\tau \in I$.

This result implies also the solvability of the problem (2) that is just a particular case.

The proof of this theorem is based on the compactness principle, one of two main approaches in analysis to the existence of solutions to non-linear equations. We do not give a proof, but will sketch main ideas behind it.

i) One of characteristic properties of compact sets in complete normed spaces is, that any sequence of points $\{z_n\}_{n=1}^{\infty}$ from a compact set C always has a converging subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ with a limit $\lim_{k\to\infty} z_{n_k} = z_*$ that belongs to C: $z_* \in C$.

ii) One approximates solutions to (1) by the explicit Euler method;

$$x(t) = x_k + (t - t_k) f(t_k, x(t_k)), \quad t \in (t_k, t_{k+1})$$

and considers a sequence $\{y_n(t)\}_{n=1}^{\infty}$ of such approximations if steps $(t_{k+1} - t_k)$ in finite differences tend uniformly to zero with $n \to \infty$. Such an approximation in one dimensional case has a graph in form of continuous peacewise linear broken line.

iii) Considering these approximations on a time interval I including τ and choosing this interval small enough (depending on the absolute value of f around (τ, ξ)), one can show that the approximathions $\{y_n(t)\}_{n=1}^{\infty}$, are uniformly bounded and uniformly continuous on I.

iv) Then basing on the property i) and on iii), one can choose a subsequence $\{y_{n_k}(t)\}_{k=1}^{\infty}$

converging uniformly on I, in the space of continuous vector valued functions on I , to a continuous function $y(t)$ that is a solution to (3) and therefore to (1).$\square$

Remark. Choosing different converging subsequences in this construction can in general lead to different limits and to non-unique solutions.

Exercise. Show that the I.V.P. $x'=\sqrt[3]{x}$; $x(0) = 0$, has non-unique solutions.

The uniqueness of solutions to I.V.P. needs additional requirements on regularity of $f(t, x)$ with respect to x variable. A standard requirement is that $f(t, x)$ is supposed to be locally Lipschitz with respect to the space x variable.

We repeat here the definition of locally Lipschitz functions.

Definition. Let A be any subset in a metric space X. The the set U_A is called to be relatively open in A if there is an open subset $U \subset X$ such that $U_A = U \cap A$.

Definition.(p. 115) Locally Lipschitz function

Let $D \subset \mathbb{R}^n$ be a non-empty set. A function $g : D \to \mathbb{R}^M$ is said to be locally Lipschitz if for any $z \in D$ there is a set $U \subset D$, realtively open in D, $z \in U$, and a number $L \geq 0$ (which may depend on U) such that

$$\|g(u) - g(w)\| \leq L \|u - w\|, \quad \forall u, w \in U$$

If L is independent of the choice of U, the function is called globally Lipschitz.

Similarly one defines functions locally Lipschitz with respect to a part of variables.

Definition.(p. 118)

Let $G \subset \mathbb{R}^{\acute{n}}$ be a non-empty open set, J be an interval in $\mathbb{R}$. A function $f : J \times G \to \mathbb{R}^n$ is said to be locally Lipschitz with respect to $x \in G$ if for any $(\tau, x) \in J \times G$ there is a set $S \times U \subset J \times G$, realtively open in $J \times G$ and a number $L \geq 0$ (which may depend on $S \times U$) such that

$$\|g(s, x) - g(s, y)\| \leq L \|x - y\|, \quad \forall (s, x), (s, y) \in S \times U$$

A theorem that gives conditions for both existence and uniquness of solutions to (1) is called the Picard-Lindelöf theorem

We will prove it in the last week of the course by applying the Banach contraction principle, that is the second main approach in analysis to existence of solutions to non-linear equations.

Theorem. Picard-Lindelöf. Theorem 4.17, p. 118 (variant with continuous f).

Let with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f be continuous in $J \times G$. If f is locally Lipschitz with respect to its second argument $x \in G$, then there is a unique **maximal solution** $x : I_x \to \mathbb{R}^n$ to the I.V.P. problem (1). Any other maximal solution with the same initial conditions must coinside with $x(t)$.

Definition. By maximal solution we mean here the solution that cannot be extended to a larger time interval.

A simpler version of this theorem states just that a "local" solution to (1) on a possibly small time interval $I \subset J$, $\tau \in I$, exists and is unique in the sence that any two solutions x and y must coinside on the intersection of the time intervals I_x and I_y where they are defined.

Proof of local uniqueness uses the integral form of the problem and the argument with the Grönvall inequality that was in a similar fashion applied two times earlier for lineary systems.

The same argument with the Grönvall inequality is used for proving well posedness of the I.V.P., namely that solutions to initial value problem (1) considered as functions of three variables t, τ, ξ: $x(t) = \varphi(t, \tau, \xi)$ are continuous and in fact even locally Lipschitz with respect to all three variables t, τ, ξ.

The uniqueness proof.

Consider difference of two solutions $x(t)$ and $y(t)$ to I.V.P. defined on a set $S \times U$ including (τ, ξ)such that the local Lipschitz property is valid for $f(t, x)$ on $S \times U$.

$$
\begin{aligned}
x(t) - y(t) &= \int_\tau^t f(s, x(s)) - f(s, y(s)) ds \\
\|x(t) - y(t)\| &= \left\| \int_\tau^t f(s, x(s)) - f(s, y(s)) ds \right\| \leq \\
&\leq 0 + \int_\tau^t \|f(s, x(s)) - f(s, y(s))\| \, ds \leq L \int_\tau^t \|x(s) - y(s)\| \, ds
\end{aligned}
$$

The Grönvall inequality implies that solutions $x(t)$ and $y(t)$ mus coincide:

$$\|x(t) - y(t)\| \leq 0 \cdot \exp(L\,(t-\tau)) = 0$$

■

0.2 Extensions, maximal solutions and their properties.

The condition in the Proposition 4.12 is not necessary, but simple examples show solutions that blow up in finite time in future or in the past if this condition is not satisfied, as for example the equation $x' = x^2$.

We consider in this section the problem (1) with f continuous and satisfying conditions in the Peano theorem implying existence (but not uniqueness) of "local solutions $x : I \to \mathbb{R}^n$ on an interval $I \subset J$.

Definition. p. 106.

An extension (proper extension) of the solution x is a solution $\widetilde{x} : \widetilde{I} \to \mathbb{R}^n$ to (1) such that $\widetilde{x}(t) = x(t)$ $\forall t \in I$, $I \subset \widetilde{I}$, $\widetilde{I} \neq I$.

Definition. p. 106. Maximal solution and maximal interval of existence.

The interval I is a maximal interval of existence and x is called maximal solution if x does not have an extension to a larger interval that is a solution to (1).

We suggest some simple examples of maximal solutions and maximal intervals that can be calculated explicitely.

Exercise 4.6

$J = [-1, 1]$; $G = R$; $\quad f : J \times G \to R.$

$(\tau, \xi) = (0, 1)$

$$f(t, z) = \frac{3z^2\sqrt{1-|t|}}{2}$$

$t \in [0, 1]$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{3z^2\sqrt{1-t}}{2} \\
\frac{dz}{z^2} &= \frac{3\sqrt{1-t}}{2}dt \\
\frac{-1}{z} &= -(1-t)^{3/2} + C \\
-1 &= -1 + C; \quad (\tau, \xi) = (0, 1) \\
C &= 0 \\
z &= \frac{1}{(1-t)^{3/2}}; \quad t \in [0, 1)
\end{aligned}$$

$t \in [-1, 0]$;

$$\frac{dz}{dt} = \frac{3z^2\sqrt{1+t}}{2}$$

$$\begin{aligned}
\frac{dz}{z^2} &= \frac{3\sqrt{1+t}}{2}dt \\
\frac{-1}{z} &= (1+t)^{3/2} + C \\
-1 &= 1 + C; \quad (\tau, \xi) = (0, 1) \\
C &= -2 \\
\frac{-1}{z} &= (1+t)^{3/2} - 2 \\
z &= \frac{1}{2-(1+t)^{3/2}}; \quad t \in [-1, 0];
\end{aligned}$$

The maximal interval $I_{\max} = [-1, 1)$ - is relatively open in $[-1, 1]$

Exercise 4.7

$J = (-\infty, 1)$; $G = (-\infty, 1)$.

$$f(t,z) = \frac{1}{\sqrt{(1-t)(1-z)}}$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{1}{\sqrt{(1-t)(1-z)}} \\
\int \sqrt{1-z}dz &= \int \frac{dt}{\sqrt{(1-t)}} \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+C \\
\frac{2}{3}(-1)(1) &= -2+C; \quad t=0,\ z=0 \\
4/3 &= 2-2/3 = C \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+\frac{4}{3} \\
\frac{2}{3}(1-z)\left(\sqrt{1-z}\right) &= 2\sqrt{1-t}-\frac{4}{3} \\
(1-z)\left(\sqrt{1-z}\right) &= 3\sqrt{1-t}-2 \\
(1-z)^{3/2} &= 3\sqrt{1-t}-2 \\
(1-z) &= \left(3\sqrt{1-t}-2\right)^{3/2} \\
z &= 1-\left(3\sqrt{1-t}-2\right)^{3/2}
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to 5/9} x(t) &= 1 \\
I_{\max} &= (-\infty, 5/9)
\end{aligned}$$

$I_{\max}$ is open.

Proposition 4.4. Openness of maximal intervals.

Let $x : I \to G$ be a maximal solution to I.V.P. (1).The maximal interval I is relatively open in J (just open if $J = \mathbb{R}$).

It means that $I = J \cap O$ for some open set $O \subset \mathbb{R}$.

For example the interval $[-1, 0.5)$ is relatively open in $[-1, 1)$ and in $[-1, 1]$,because

$$(-2, 0.5) \cap [-1, 1) = [-1, 0.5)$$

Proof. Consider the case when J is an open interval $J = (a, b)$. Suppose that the maximal interval of a maximal solution to I.V.P. $I \subset J$ is not open, for example is $(\alpha, \omega]$.

In this case the point $(\omega, x(\omega)) \in J \times G$ and there is a solution to the differential equation with initial conditions $(\omega, x(\omega))$, existing on a small time interval $[\omega, \omega + \varepsilon)$. This solution is an extension of the original solution. It is a contradiction because we supposed that $(\alpha, \omega]$ was a maximal interval for the maximal solution $x(t)$. Other cases are considered similarly.$\blacksquare$

Theorem 4.8. p. 108. Existence of maximal solutions.

Every solution to (1) can be extended to a maximal solution.

Idea of the proof(not required at exam)

In the case when solutions are unique (for example f is locally Lipschitz with respect to x), one can build the maximal interval of existence just as a union of domains for all extensions of a given solution. Because of the uniqueness of solutions, trajectories cannot make branches in this case and this construction leads to a unique maximal solution that at each time point t attains the value of one of the extensions defined at this time point. The uniqueness of solutions makes that this definition is consistent.

In the general case when trajectories can create branches, the union of extensions can have a tree like geometry, or even be an n-dimensional set. In this case the proof uses Zorn lemma (see appendix in the course book) to choose a maximal solution. It has an existence interval including all existence intervals of all extensions, but is possibly not unique.

The following technical lemma is the main tool in several arguments about maximal solutions.

Lemma 4.9. On the extension to the boundary point of the open existence

time interval for a bounded solution having the closure of the orbit in G,

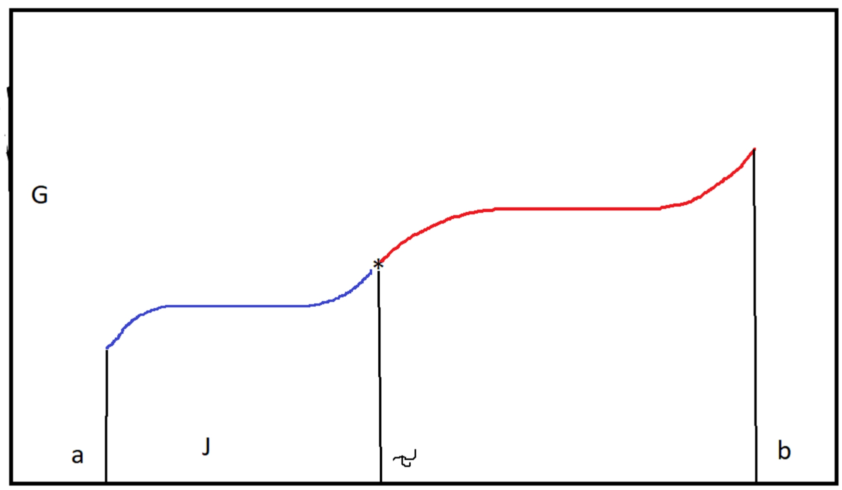
Let $x : I \to G$ be a solution to (1) and denote $a = \inf I$; $b = \sup I$.

(1) If b is in J and not in I (I is open in the right end), and the closure $\overline{O_+}$ of the orbit $O_+ = \{x(t) : t \in [\tau, b)\}$ is a compact subset of G,

then there is a solution $y : I \cup \{b\} \to G$ to (1) that is an extension of x.

(2) a similar statement is valid for the "backward orbit" $O_- = \{x(t) : t \in (a, \tau]\}$ and extension of x to the left end point a.

Comment. Compact sets are sets that are bounded and closed.



Proof. We prove (1). Let C be the closure of $\{x(t), t \in [\tau, b)\}$. Assume that $b \in J \backslash I$ and that C is a compact in G. The continuous function $f(t, x)$ must be bounded on the compact $[\tau, b] \times C$.

$$\|f(t, x)\| < M, \quad (t, x) \in [\tau, b] \times C$$

It implies that the limit

$$\eta = \xi + \lim_{t \to b} \int_{\tau}^{t} f(s, x(s)) ds$$

is well defined for continuous and uniformly bounded function under the integral. We get it because for any sequence $\{t_k\}_{k=1}^{\infty}$, $t_k < b$, $t_k \to b$, with $k \to \infty$, $\int_{\tau}^{t_k} f(s, x(s)) ds$ is a Cauchy sequence:

$$\left\| \int_\tau^{t_p} f(s,x(s))ds - \int_\tau^{t_m} f(s,x(s))ds \right\| = \left\| \int_{t_m}^{t_p} f(s,x(s))ds \right\| \le M\,|t_p - t_m| \to 0, \quad p,m \to \infty$$

that has a limit η independent of the sequence $\{t_k\}_{k=1}^{\infty}$. Then the solution $x(t)$ can be extended to the closed interval $[\tau, b]$ by setting $x(b) = \eta$.■

The following Corollary is a direct consequence of the Lemma 4.9 and Proposition 4.4 and gives a sufficient condition for a maximal solution to have an infinite maximal interval (if J is infinite) or a maximal interval "ifinite with respect to" J , which meaning is specified exactly below.

Corollary 4.10, p. 111. "Eternal life" of solutions enclosed in a compact.

Let $x : I_{\max} \to G$ be a maximal solution to (1).

Suppose that the "future" half - orbit $O_+ = \{x(t) : t \in I_{\max} \cap [\tau, \infty)\}$ of the maximal solution $x(t)$ is contained in a compact subset C of G.

Then the corresponding maximal interval of existence $I_{\max}$ is infinite to the right (future) if $[\tau, \infty) \subset J$), or "infinite to the right with respect to J" meaning that the maximal solution exists on $[\tau, \infty) \cap I = [\tau, \infty) \cap J$ that is the whole part of J to the right of the initial time τ.

Similar statement is valid for the "backward orbit" $O_- = \{x(t) : t \in (a, \tau]\}$. Suppose that the "backward orbit" is contained in a compact subset C of G,

Then the corresponding maximal interval of existence $I_{\max}$ is infinite to the left (past) if $(-\infty, \tau] \subset J$ and is infinite to the left (past) "with respect to" J, that means that the maximal solution exists on $(-\infty, \tau] \cap I = (-\infty, \tau] \cap J$, that is the whole part of J to the left of the initial time τ.

If the whole orbit $O = \{x(t) : t \in I_{\max}\}$ of the maximal solution $x(t)$ is contained in a compact subset of G, then the corresponding maximal interval of existence $I_{\max} = J$ ($I_{\max} = \mathbb{R}$ if $J = \mathbb{R}$). It means that the maximal solution x exists both in the whole past and whole future for the equation.□

Proof. The proof is easy to carry out by a contradiction argument that follows from the Lemma 4.9 and the fact that a maximal interval must be open (relatively to J).■

How to show that a solution has the orbit inside a compact set?

Definition. A set Q is called positively invariant for a system of differential equations if all trajectories of maximal solutions starting inside Q stay inside Q for all future t in it's maximal interval.

We consider here an idea how to show that solutions to a non-linear autonomous system of differential equations belong to a compact set.

A general idea that is used to answer many questions about behaviour of solutions (trajectories) of the equations, is the idea of test functions.

We find a test function $V(x)$ that has some simple level sets $\partial Q = \{x : V(x) = C\}$ that are closed curves (or surfaces in higher dimensions) enclosing a bounded domain Q. Typical examples are $V(x, y) = x^2 + y^2 {=} R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} {=} 1$ - ellipse, etc.

• To show that a particular level set ∂Q bounds a positively - invariant set Q we check the sign of the directional derivative of V along the velocity in the equation: $V_f(x) = (\nabla V \cdot f)(x)$ for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• $\nabla V(x)$ is a normal vector to the level set of V that goes through the point x.Therefore the sign of $V_f(x) = (\nabla V \cdot f)(x)$ shows if trajectories go to the same side of the level set as the gradient ∇V (if $V(x) > 0$) or to the opposite side (if $V_f(x) < 0$).

• If all trajectories go inside a bounded set Q, then all trajectories starting inside Q will stay inside Q forever.

Example.

Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x - (1 - x^2)y \end{cases}$.

Find a compact around the origin that no trajectories escape.

Solution.

We try the test function $V(x, y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative along trajectories:

$$
\begin{aligned}
V_f(x,y) &= \frac{d}{dt}V(x(t),y(t)) = \nabla V(x,y)\cdot\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \\
&= \overrightarrow{N}\cdot\overrightarrow{f} = \\
&= \begin{bmatrix} 2x \\ 4y \end{bmatrix}\begin{bmatrix} 2y \\ -x-(1-x^2)y \end{bmatrix} \\
&= -4y^2(1-x^2) \le 0
\end{aligned}
$$

$\nabla V(x,y)$ is a normal vector to level sets of the form:

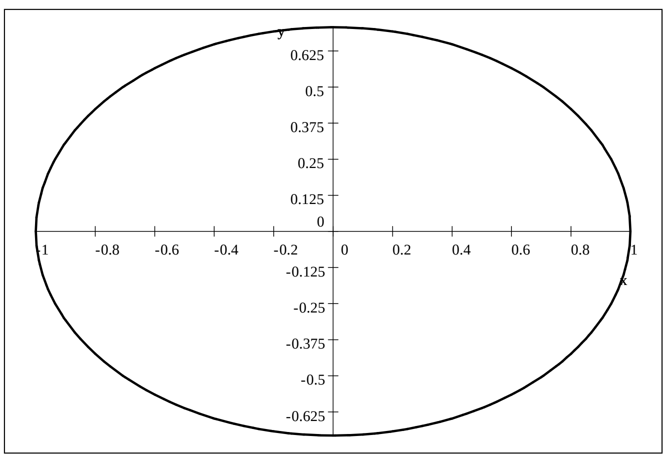
$$x^2 + 2y^2 = C$$

Put $y = 0$, $x = 1$, we get $C = 1$,

$x^2 + 2y^2 = 1$

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2) \le 0$ that is not positive for $|x| \le 1$.

Trajectories starting inside tho compact bounded by this ellips stay inside it forever.



■

The following Theorem describes the situation in a sense opposite to the previous Corollary 4.10. It describes the the behaviour of maximal solutions having bounded maximal interval $I_{\max}$ (if J is $\mathbb{R}$), and in the case when the interval J has bounded endpoints itself, describes maximal solution with maximal interval that is "bounded with respect to J", meaning that $\sup I_{\max} < \sup J$ or $\inf J < \inf I_{\max}$.

Theorem 4.11, p.112. "Short living" maximal solutions escape any compact.

Let $x : I \to G$ be a maximal solution to (1) with maximal interval of existence $I \subset J$ and assume that I is not the whole J: $I \neq J$. Denote endpoints of I as $\alpha = \inf(I)$, $\omega = \sup(I)$.Then one of endpoints does not belong to I: $\omega \in J \backslash I$ or $\alpha \in J \backslash I$.

Statement of the Theorem:

1) In the first case $\omega \in J \backslash I$ for each compact $C \subset G$, there is an "escaping time moment" $\sigma \in I$, $\sigma < \omega$, such that $x(t)$ "escapes" C at time $\sigma : x(t) \notin C$ for all $t \in (\sigma, \omega)$.

This property can be further geometrically specified. If $G \neq \mathbb{R}^n$ the trajectory $x(t)$ tends to the boundary ∂G of G with $t \to \omega$ (if G is bounded). It can also tend to infinity if G has "branches" going to infinity in $\mathbb{R}^n$. If $G = \mathbb{R}^n$, then $\|x(t)\| \to \infty, \quad as\ t \to \omega$. This statement is formulated formally as:

$$\begin{aligned} \lim_{t\to\omega} \min\left\{\operatorname{dist}(x(t), \partial G), 1/\|x(t)\|\right\} &= 0, \quad for\ G \neq \mathbb{R}^n \\ \|x(t)\| &\to \infty, \quad as\ t \to \omega, \quad for\ G = \mathbb{R}^n \end{aligned} \tag{4}$$

2) Similar statements are valid for limits of $x(t)$ as $t \to \alpha$ for the maximal solution having maximal interval with the left end point α "in the past" belonging to J.

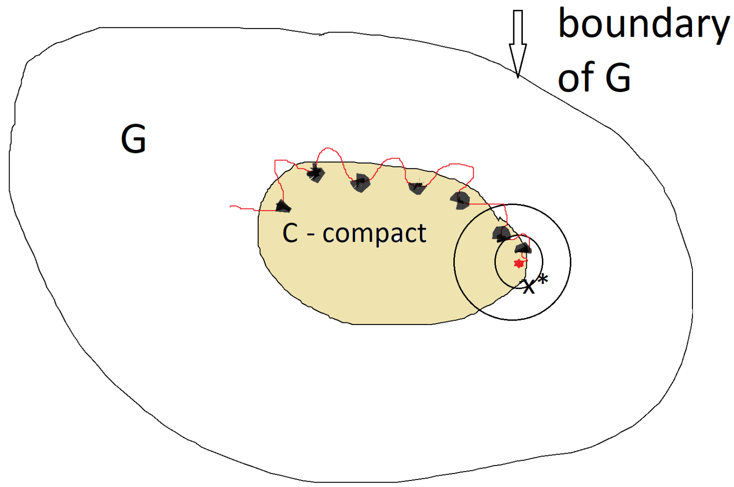
Proof.

We consider the case 1). The fact that the maximal solution must at some time leave any compact C follows from the previous Corollary 4.10 by contradiction, because a solution that stays in a compact must have a maximal interval infinite to the right or $[\tau, \infty) \cap I = [\tau, \infty) \cap J$. It contradicts to the condition that $\omega \in J \backslash I$ that means that the given maximal $x(t)$ solution does not reach the maximal possible time in J.

A more sofisticated argument shows that in our situation the solution $x(t)$ must at some time σ leave any compact C "forever". There is a "last visit" time $\sigma < \omega$, such that $x(t)$

never enters C again after this time.

Suppose the opposite, namely that there is a monotone sequence of times $\{t_m\}_{m=1}^{\infty}$ such that $t_m \nearrow \omega$ with $m \to \infty$ such that $x(t_m) \in C$. C is a compact, therefore there must exist a subsequence (for which we will **keep the same notation** $\{t_m\}_{m=1}^{\infty}$), such that with $m \to \infty$ $t_m \nearrow \omega$ and $x(t_m) \to x_* \in C$.



Choose an r so small that the ball $B((\omega, x_*), r)$ with the center (ω, x_*) would belong to the domain of the equation: $B((\omega, x_*), r) \subset J \times G$. Choose a smaller ball $B \equiv B((\omega, x_*), 2\varepsilon)$ with $\varepsilon = r/3$. Then the closure $\overline{B}$ of B also belongs to the domain of the equation: $\overline{B} \subset J \times G$.

Denote $M = \sup\left\{\|f(t,x)\| : (t,x) \in \overline{B}\right\}$ the supremum of the continuous function $\|f(t,x)\|$ on the compact $\overline{B}$.

Using that $t_m \nearrow \omega$, and the boundedness of $\|f(t,x)\|$ on $\overline{B}$, we will observe that the index m can be chosen so large that the trajectory $\{(t, x(t) : t \in [t_m, \omega)\}$ of the solution $x(t)$ for $t \in [t_m, \omega)$, on the short time interval $[t_m, \omega,)$ belongs to $\overline{B}$.

It can be observed by considering the integral form of the differential equation and using the estimate $M = \sup\left\{\|f(t,x)\| : (t,x) \in \overline{B}\right\}$ for f on $\overline{B}$:

$$
\begin{aligned}
x(t) &= x(t_m) + \int_{t_m}^{t} f(s, x(s))ds \\
x(t) - x_* &= x(t_m) - x_* + \int_{t_m}^{t} f(s, x(s))ds \\
\|x(t) - x_*\| &\leq \|x(t_m) - x_*\| + |t - t_m| \, M \\
&\leq \varepsilon, \quad m > m_*
\end{aligned}
$$

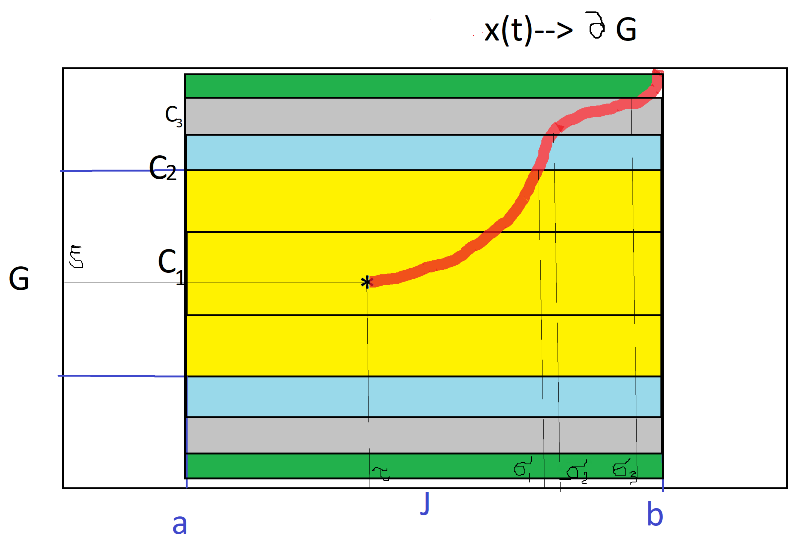
where $\|x(t_m) - x_*\| \to 0$ with $m \to \infty$, and $|t - t_m| \, M \leq |\omega - t_m| \, M \to 0$ with $m \to \infty$. We can choose $m > m_*$ so large that the right hand side in the inequality will be smaller than ε.

Therefore $\|x(t) - x_*\| \leq \varepsilon$, $|t - t_m| \leq \varepsilon$ and the trajectory $\{(t, x(t) : t \in [t_m, \omega)\}$ belongs to $B \equiv B((\omega, x_*), 2\varepsilon)$ and is bounded.

Therefore the closure of this tajectory $\{(t, x(t) : t \in [t_m, \omega)\}$ is compact and belongs to $B((\omega, x_*), r)$ and to $J \times G$.

Therefore according to the Lemma 4.9 the solution $x(t)$ can be extended up to the time ω and also beyond it to an even larger time interval $[t_m, \omega + \delta)$. This fact contradicts the given condition that $x(t)$ is the maximal solution with the maximal interval $I_{\max}$ having $\sup I_{\max} = \omega$.

The property that $x(t)$ tends to the boundary of G can be shown in the following way.



If G is bounded, one can construct a rising sequence of compact sets $\{C_n\}_{n=1}^{\infty}$, $C_n \subset C_{n+1} \subset G$ like "blowing up ballons" tending to the boundary ∂G of G so that $\text{dist}(C_n, \partial G) \to 0$ as $n \to \infty$. For each of these sets there is a time σ_n such that $x(t)$ leaves C_n and therefore has $\text{dist}(x(t), \partial G) < \text{dist}(C_n, \partial G)$ for $t > \sigma_n$. This construction proves the fact that $\text{dist}(x(t), \partial G) \to 0$ as $t \to \omega$.

In the case of $G = \mathbb{R}^n$ one can choose a sequence of test compact sets $\{C_n\}_{n=1}^{\infty}$ as balls with centers in the origin and radii r_n tending to infinity with $n \to \infty$ leading together with the "escaping property" to conclusion that $\|x(t)\| \to \infty, \quad as\ t \to \omega$.

The third case with unbounded G with non-empty boundary ∂G can be proven by a combination of the above arguments.■

Proposition 4.12, p. 114 on "eternal" solutions for equations with linear bound for the right hand side. (proof required at exam)

Consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(\tau) = \xi$$

where $f : J \times \mathbb{R}^N \to \mathbb{R}^N$, continuous and locally Lipschitz in x.

Assume that for any compact interval $K \subset J$ there is $L > 0$ such that for $t \in K$ the following estimate holds for the right hand side:

$$\|f(t, x)\| \leq L(1 + \|x\|). \tag{5}$$

If $x : I \to \mathbb{R}^N$ is a maximal solution to the equation $x'(t) = f(t, x(t))$, then $I = J$. In particular if $J = \mathbb{R}$, the maximal solution is defined for all t.

Proof.

Define $\omega = \sup I$, $\alpha = \inf I$. We use proof by contradiction. Suppose that the statement of the theorem is not true, for example that $\omega \in J$ and $\omega \notin I$and that $\tau < \omega$.

Let choose the konstant L such that the (5) is valid for $t \in [\tau, \omega]$.Then, using the integral form of the I.V.P. and the triangle inequality implies the following estimate

$$
\begin{aligned}
\|x(t)\| &\leq \|x(\tau)\| + \int_\tau^t \|f(s,x(s))\|\,ds \leq \|x(\tau)\| + L\int_\tau^t (1+\|x(s)\|)\,ds \\
& \|x(\tau)\| + L(t-\tau) + L\int_\tau^t \|x(s)\|\,ds
\end{aligned}
$$

for all $t \in t \in [\tau,\omega)$ The Grönvalls inequality implies that $\|x(t)\|$ bounded by a constant C on $[\tau,\omega)$. It makes that the corresponding orbit $\{x(t), t \in [\tau,\omega)\}$ is a bounded and therefore has compact closure in $\mathbb{R}^N$. The Lemma 4.9 implies that the solution can be extended to the closed interval $[\tau,\omega]$ and actually y existence theorem to an even larger interval beyond ω. It contradicts to the supposition that I is a maximal interval for $x(t)$.■

Proof for the case when $\alpha \in J$ and $\alpha \notin I$, $\alpha < \tau$ is treated similarly.

0.3 Transition map

Existence theorems by Picard and Lindelöf (Theorems 4.17 and 4.22) imply that for any point $\tau,\xi \in J \times G$ there is a unique maximal solution that is convenient to consider as a function $\varphi(t,\tau,\xi) : J \times J \times G \to G$ of three variables equal to the maximal solution x of (1). It is a common situation in applications that one is interested not in properties of one solution, but in a description of the family of solutions with all possible initial data as a whole. This type of problems constitute modern theory of differential equations and dynamical systems and motivates introducing the following notion.

Definition. p. 126. Transition map. The mapping $\varphi(t,\tau,\xi)$ defined above is called transition map.

Transition map for autonomous systems. In the case of autonomous systems there is no meaning in considering different initial times τ, because all solutions are functions of the time shift $t-\tau$. In this case we consider transition mappings $\varphi(t,\xi) : J \times G \to G$ with $\varphi(t,\xi) = x(t)$ being the maximal solution of (2) with initial condition $x(0) = \xi$.

Local flow or local dynamical system corresponding to an autonomous system of differential equations.

In the modern theory of ODE and dynamical systems the mapping $\varphi(t,\xi)$ is often called in the local flow or the local dynamical system corresponding to the system of differential equations.

If the maximal interval I_ξ corresponding to the initial point ξ coincides with $\mathbb{R}$ we say that the solution $\varphi(t,\xi)$ is global. If $I_\xi = \mathbb{R}$ for all $\xi \in G$ then $\varphi(t,\xi)$ is said to be a flow or a dynamical system on G.

Example 4.33 of a transition map.

$G = \mathbb{R}$; $f : G \to \mathbb{R}$; $f(x) = x^2$; for $\xi = 0$; $x(t) \equiv 0$.

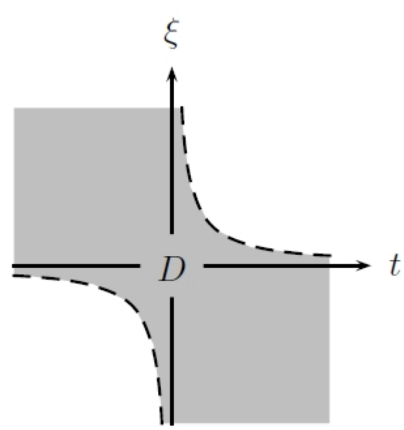
Initial data $x(0) = \xi$

$$\begin{aligned}
\frac{dx}{dt} &= x^2; \quad \int \frac{dx}{x^2} = \int dt; \\
-\frac{1}{x} &= t + C \\
-\frac{1}{x} &= t - \frac{1}{\xi}; \quad -\frac{1}{x} = \frac{t\xi - 1}{\xi} \\
x &= \frac{\xi}{(1 - t\xi)}
\end{aligned}$$

$\xi = 0$; $x(t) \equiv 0$. $\quad \xi > 0$, $I_\xi = (-\infty, 1/\xi)$. $\quad \xi < 0$, $I_\xi = (1/\xi, \infty)$

$$\varphi(t,\xi) = \frac{\xi}{(1 - t\xi)}; \quad D(\varphi) = \{(t,\xi) \in \mathbb{R} \times \mathbb{R}; \quad t\xi < 1\}$$

The domain D of φ is an open set. $\varphi(t,\xi)$ is continuous and even locally Lipschitz.



Proposition. Theorem 4.34, p.139 (consequence of Th. 4.29, p. 129)

The domain $D = \{(t,\xi) \in I_\xi \times G, \quad \xi \in G\}$ of the transition map $\varphi(t,\xi)$ is open and $\varphi(t,\xi)$ is continuous and even locally Lipschitz in D.

Proof of the Lipschitz property with respect to each of the variables follows from the integral form of the I.V.P. and for ξ variable - from an application of Grönwall inequality similar to the proof of uniqueness of solutions to I.V.P.

Proposition. Translation invariance of the transition mapping for autonomous systems

(a non-linear version of the Chapman-Kolmogorov relation) Theorem 4.35, p. 140 -141.

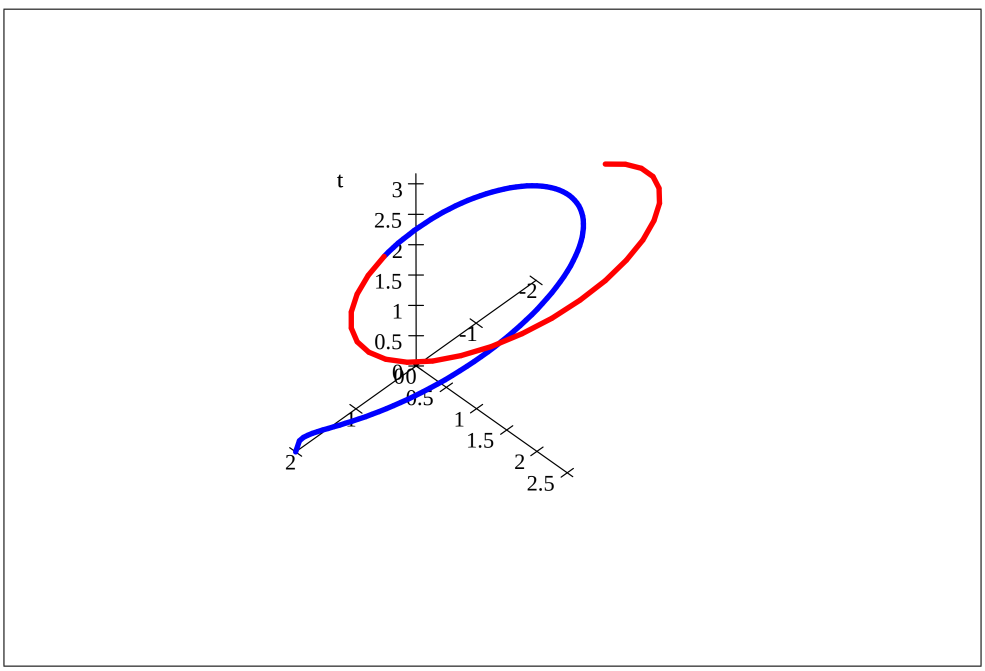
The transition mapping $\varphi(t,\xi)$ has properties

(1) $\varphi(0,\xi) = \xi$ for all $\xi \in G$

(2) if $\xi \in G$ and $\tau \in I_\xi = I_{\max}(\xi)$ - maximal interval for ξ, then

$$\begin{aligned} I_{\varphi(\tau,\xi)} &= I_\xi - \tau \\ \varphi(t+\tau,\xi) &= \varphi(t,\varphi(\tau,\xi)), \quad \forall t \in I_\xi - \tau \end{aligned}$$

Proof of this statement follows is similar to the proof of the Chapman Kolmogorov relations for linear systems.



We consider first a trajectory $\varphi(...,\xi)$ starting at the point $\xi \in G$ and finishing at time τ at the point $\varphi(\tau,\xi)$ (blue curve). Then we continue this movement from the last point $\varphi(\tau,\xi)$ during time t (red curve) coming finally to the point $\varphi(t,\varphi(\tau,\xi))$ in the right hand side of the equation in the conclusion. The fact that the equation is autonomous and independent of time makes that this movement is equivalent to just moving with the flow starting from the point ξ during the total time $t+\tau$, that is the left hand side in the equation. The illustration is borrowed from the proof for the linear systems. The only difference here is that we have a superposition $\varphi(t,\varphi(\tau,\xi))$ of transfer mappings in the non-linear case instead of the product of transfer matrices in the linear case (that is also a superposition for linear mappings).

May 13, 2020

1 Lecture notes on non-linear ODEs: limit sets (attractors), positively invariant sets, periodic solutions, limit cycles.

Plan (continuation after existence and maximal solutions)

• Semi - orbits. Limit sets. p. 142. Positively (negatively) invariant sets p. 142.

• Existence of an equilibrium point in a compact positively invariant set. Theorem 4.45, p. 150.

• Planar systems. Periodic orbits. Poincare-Bendixson theorem. (only idea of the proof is discussed) Theorem 4.46, p. 151.

• Examples on applications of Poincare-Bendixson theorem.

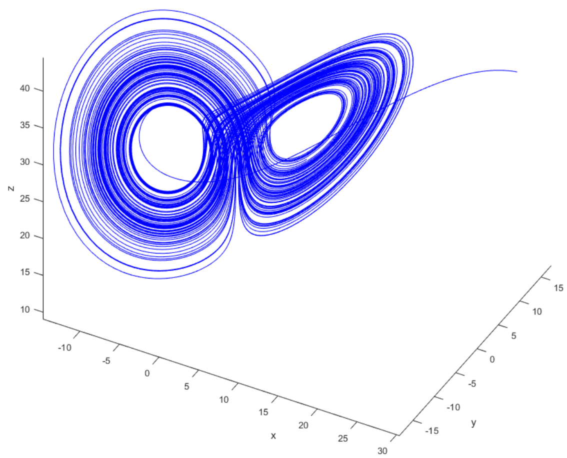
• Generalized Poincare-Bendixson theorem. (missed in the book, only formulation is given)

1.1 Introduction to limit sets and their properties.

We consider flows or dynamical systems corresponding to autonomous differential equations

$$\dot{x} = f(x), \quad f : G \to \mathbb{R}^N, \quad x(0) = \xi \tag{1}$$

with f locally Lipschitz and denote by $\varphi(t, \xi)$ the transition mapping or the local flow generated by f. For $\xi \in G$ let $I_\xi = (\alpha_\xi, \omega_\xi)$ denote the maximal interval - the interval of existence of maximal solution to (1).



Definition. (Positive semi-orbit)

We denote by $O(\xi)$ the orbit of the solution to $(1), O(\xi) = \{x(t) : t \in (\alpha_\xi, \omega_\xi)\}$.

We define the positive semi-orbit $O_+(\xi) = \{x(t) : t \in [0, \omega_\xi)\}$ of ξ - for future, and negative semi-orbit (for the past) $O_-(\xi) = \{x(t) : t \in (\alpha_\xi, 0]\}$ of ξ - for the past.

Definition. (Limit point of ξ)

• A point $z \in R^N$ is called an ω - limit point of ξ (or it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$) if there is a sequence of times $\{t_n\} \in [0, \omega_\xi)$ tending to the "maximal time in the future", $t_n \nearrow \omega_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$

• Similarly a point $z \in R^N$ is called an α - limit point of ξ (or it's negative semi-orbit $O_-(\xi)$ or it's orbit $O(\xi)$) if there is a sequence of times $\{t_n\} \in (\alpha_\xi, 0]$ tending to the "minimal time in the past", $t_n \searrow \alpha_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$.

Definition. (ω **- limit set)**

The ω - limit set $\Omega(\xi)$ of ξ (or it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$) is the set of all it's ω- limit points (in future)

A trajectory approaching the ω - limit set of the Lorentz system

$$\begin{aligned} x' &= -\sigma(x-y) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

for $\sigma = 10$, $r = 28$, $b = 8/7$.

Definition

The α - limit set $\Omega(\xi)$ of ξ (or it's negative semi-orbit $O_-(\xi)$ or it's orbit $O(\xi)$) is the set of all it's α- limit points (in the past).

Definition. (Positively invariant set)

A set $U \subset G$ is said to be positively invariant under the local flow φ generated by f if for each starting point $\xi \in U$ from U the corresponding positive semi - orbit $O_+(\xi)$ is contained in U.

It means that all trajectories $x(t)$ starting in U stay in U as long as they exist in future.

One defines sets negatively invariant similarly, but with respect to the past.

Definition

One also says that the set U is just invariant with respect to the flow $\varphi(t,\xi)$ if $O(\xi) \subset U$ for all $\xi \in U$. It means that all trajectories going through ξ belong to U both in the "whole past" and in the "whole future".

Remark

We know that compact positively invariant sets include trajectories that have "infinite" maximal existence time in the future: $J \cap [0,\infty)$. It makes it meaningfull to investigate limit sets of trajectories that are contained especially in compact positively invariant sets.

The first step in this kind of investigation is to identify possibly small positively invariant sets, that localize solutions. The second step is to classify and to identify ω - limit sets that can be contained there. It particular one is interested in fining ω - limit sets for particular given systems.

1.2 Methods of hunting positively - invariant sets (there is a separate pdf file with this text)

A system of ODEs has naturally many positively - invariant sets, for example the whole domain G is always a positively - invariant set, but it is not very interesting. We like to find possibly narrow positively invariant sets showing more precisely where trajectories or solutions to the equation tend when t tends to the upper bound of the maximal time interval.

How to find a positively - invariant set?

Method 1. A general idea that is used to answer many questions about behaviour of solutions (trajectories) to ODEs, is the idea of test functions. One checks if the velocities $f(x)$ are directed inside or outside with respect to the sets like $Q = \{x \in U : V(x) \leq C\}$ or $Q = \{x \in U : V(x) \geq C\}$ defined by some simple test functions $V : U \to \mathbb{R}$, $U \subset G$. The advantage of the idea with test functions is that one does not need to solve the equation to use it.

• It is convenient to find a test function $V(x)$ that has a level set $\partial Q = \{x : V(x) = C\}$ that is a closed curve (or surface in higher dimensions) enclosing a bounded domain Q.

Typical examples are $V(x, y) = x^2 + y^2 {=} R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} {=} 1$ - ellipse, or more complicated ones as $V(x, y) = x^6 + ay^4$ - smoothed rectangle shape or squeezed ellipse, $V(x, y) = x^2 + xy + y^2 = C$ - ellipse rotated in $\pi/4$ and having axes A and B related as $A/B = \sqrt{3}$etc.

• To show that a particular level set ∂Q bounds an positively - invariant set Q we check the sign of the directional derivative of V along the velocity $f(x)$ in the equation $x' = f(x)$:

$$dV(x(t))/dt = V_f(x) = (\nabla V \cdot f)(x)$$

for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• Point out that the gradient $\nabla V(x)$ is the normal vector to the level set $\{V(x) = C\}$ that goes through the point x. Therefore the sign of $V_f(x)$ shows if the trajectories towards the same side of the level set as the gradient ∇V (if $V_f(x) > 0$) or towards the opposite side (if $V_f(x) < 0$).

• Then if $V(x)$ is rising for x going out of Q, and $V_f(x) < 0$ then the domain Q inside this level set ∂Q (curve in the plane case) will be positively - invariant. Similarly if $V(x)$ is decreasing out of this level set, and $V_f(x) < 0$ on the level set ∂Q then the domain Q inside this level set will be positively - invariant.

In the opposite case the complement to Q that is $\mathbb{R}^N \backslash Q$ will be positively - invariant and trajectories $\varphi(t, \xi)$ starting in this complement: $\xi \in \mathbb{R}^N \backslash Q$ will never enter Q.

First integrals. A very particular case of test functions are functions that are constant on all trajectories $\varphi(t, \xi)$ of the system. It means that $\frac{d}{dt} V(\varphi(t, \xi)) = V_f(x) \equiv 0$. In this case all level sets of the first integral are invariant sets, because velocities $f(x)$ are tangent vectors to the level sets in this case. Such functions are called **first integrals** and represent conservation laws in ODEs. Usually but not always, such test functions have the meaning of the total energy in the system.

Method 2. If it is sometimes difficult to guess a simple test function giving one closed formula for the boundary of an positively - invariant set as in the Method 1, then one can try to identify a boundary for a positively - invariant set as a curve (or a surface in higher dimensions) consisting of a number of simple peaces, for example straight segments.

The simplest positively - invariant set of such kind would be a rectangle (a rectangular box in higher dimensions) with sides parallel to coordinate axes. Then a simple check that this rectangle is a positively - invariant is just to check the sign of x_1 or x_2 - components of $f(x)$ on these segments, showing that trajectories go inside or outside of the rectangle. A bit more complicated analysis is to show that no trajectories can approach these segments in finite time (if one of the segments belongs to the boundary ∂G of G where the equation is not defined). We have such an example in the second home assignment.

Application to Poincare Bendixson theorem

One searches often positively - invariant sets with special properties. For example to apply the Poincare-Bendixson theorem for systems in the plane formulated later in the course, one needs to find a positively - invariant set that does not contain any equilibrium points.

Example of finding positively - invariant sets and ω - limit sets with help of a simple test function.

Consider the system

$$\begin{aligned} x' &= -ay + f(r)x \\ y' &= ax + f(r)y \end{aligned}$$

where $r = \sqrt{x^2+y^2}$. We will try to find an explicit expression for the corresponding flow by introducing polar coordinates $x = \cos(\theta)r$, $y = \sin(\theta)r$. We differentiate $r(t)$ using expressions for r and for x', y' in the equation, and arrive to following formulas:

$$\begin{aligned} \left(r^2\right)' &= 2rr' = (x^2+y^2)' = 2xx' + 2yy' \\ &= 2x(-ay + f(r)x) + 2y\,(ax + f(r)y) = 2f(r)\left(x^2+y^2\right) = 2f(r)r^2 \end{aligned}$$

Therefore:

$$r' = f(r)r$$

The equation for the polar angle θ can be derived by differentiating $\tan(\theta(t))$:

$$\begin{aligned} (\tan(\theta))' &= \theta'\left(\frac{1}{\cos^2(\theta)}\right) = \left(\frac{y}{x}\right)' = \frac{y'x - x'y}{x^2} \\ &= \frac{ax^2 + f(r)xy - (-ay^2 + f(r)xy)}{x^2} = \frac{ax^2 + ay^2}{x^2} = \frac{a}{\cos^2\theta} \end{aligned}$$

Therefore

$$\theta' = a$$

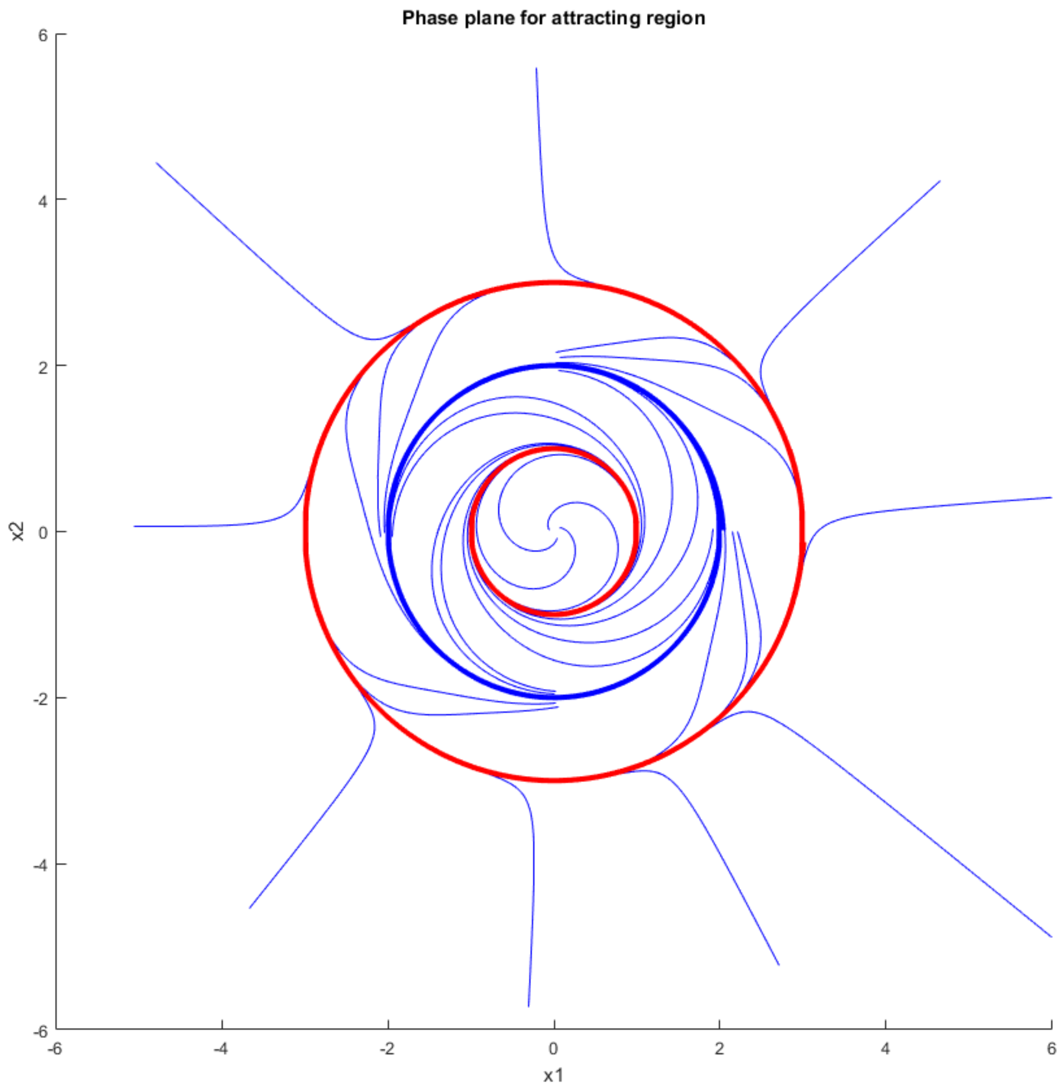
The equation for $r(t)$ can be solved by integration.Each positive root r_* to $f(r)$ corresponds to a periodic trajectoty $r(t) = const = r(0) = r_*,\ \ \theta(t) = \theta(0) + at$

This periodic orbit will attract trajectories, that start nearby if $\frac{df}{dr}(r_*) < 0$ (will be an ω - limit set $\Omega(\xi)$ for points ξ close to the circle $r = r_*$). If r_* is a root of f where the first term in Taylor series is $c(r - r_*)^2$ with $c > 0$, then nearby trajectories will be attracted to the periodic orbit from inside, and will run away from the periodic orbit from the outside of it.

Example with tree periodic solutions, orbits of two of them with $r = 1$ and $r = 3$ (red) are ω - limit sets, the orbit of one with $r = 2$ (blue) is an α - limit set:

$$\begin{aligned} f(r) &= (1 - r^2)(3 - r)(4 - r^2) \\ a &= -10 \end{aligned}$$

We have $dr/dt = f(r)r$



In the following example such kind of system is considered for one more particular function $f(r)$.

Exercise 4.16, p. 140.

Exercise 4.16

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2)\,,\ -z_1 + z_2(1 - \|z\|^2)\big).$$

Show that f generates a local flow $\varphi\colon D \to \mathbb{R}^2$ given by

$$\varphi(t, \xi) = \big(\|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t}\big)^{-1/2} R(t)\xi\,,$$

where the function $R\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ is given by

$$R(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \forall\, t \in \mathbb{R}. \tag{4.27}$$

(and so $R(t)\xi$ is a clockwise rotation of ξ through t radians) and

$$D := \big\{(t, \xi) \in \mathbb{R} \times \mathbb{R}^2 \colon \|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t} > 0\big\}.$$

(*Hint.* Show that, for $\xi = (\xi_1, \xi_2) \neq 0$, the initial-value problem (4.25) may be expressed – in polar coordinates – as

$$\dot r(t) = r(t)\big(1 - r^2(t)\big), \quad \dot\theta(t) = -1, \quad (r(0), \theta(0)) = (r^0, \theta^0),$$

where $r^0 = \|\xi\|$, $r^0 \cos\theta^0 = \xi_1$ and $r^0 \sin\theta^0 = \xi_2$.)

Solution. The equations in polar form follow from the general argument above.

We solve the equation for r :

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2) \\ \frac{dr}{r(1 - r^2)} &= dt \end{aligned}$$

$$\frac{1}{r(1 - r^2)} = \frac{1}{r} - \frac{1}{2\,(r + 1)} - \frac{1}{2\,(r - 1)}$$

$$
\begin{aligned}
\int \frac{dr}{r(1-r^2)} &= \ln r - \frac{1}{2}\ln\left(r^2-1\right)\\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) &= t + C\\
C &= \ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right)\\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \left(\ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right)\right) &= t
\end{aligned}
$$

$$
\begin{aligned}
\exp(t) &= \exp\left(\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \ln|\xi| + \frac{1}{2}\ln\left(|\xi|^2-1\right)\right)\\
\frac{r}{\sqrt{r^2-1}}\frac{\sqrt{|\xi|^2-1}}{|\xi|} &= \exp(t)\\
\frac{(r^2-1)}{r^2}\frac{|\xi|^2}{\left(|\xi|^2-1\right)} &= \exp(-2t)\\
\left(r^2-1\right)|\xi|^2 &= r^2\left(|\xi|^2-1\right)\exp(-2t)\\
r^2(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t)) &= |\xi|^2\\
r^2 &= \frac{|\xi|^2}{(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t))}\\
r &= \frac{|\xi|}{\sqrt{(|\xi|^2 - 1 - |\xi|^2\exp(-2t))}}
\end{aligned}
$$

Example 4.37. p. 142. Do it as exercise.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be as in the Exercise 4.16, the generator of a local flow considered above.

Let Δ be an open unit disc in $\mathbb{R}^2$, namely $\Delta = \{(z_1, z_2) \in \mathbb{R}^2 : z_1^2 + z_2^2 < 1\}$.

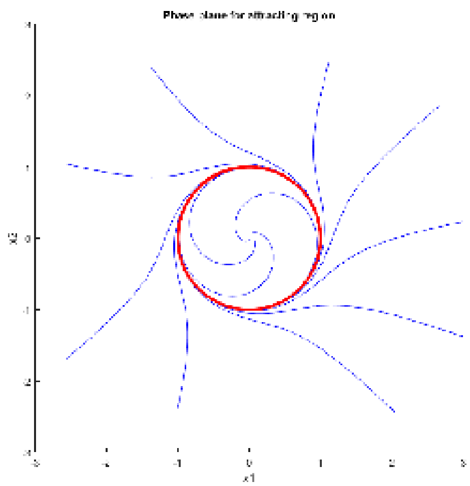
Show that sets Δ, $\partial\Delta, \mathbb{R}^2\backslash\overline{\Delta}$ are invariant and find for every $\xi \in \mathbb{R}^2$ the corresponding ω and α limit set.

Remark. In the case $\|\xi\| > 1$ solutions $\varphi(t,\xi)$ have the maximal interval I_ξ that is not the whole $\mathbb{R}$, but is bounded in the past $I_\xi = (\alpha_\xi, \infty)$.

The calculation of α_ξ using the explicit solution found in the exercise 4.16 is given here:

$$
\begin{aligned}
\|\xi\|^2 + \left(1 - \|\xi\|^2\right) e^{-2t} &= 0 \\
\frac{\|\xi\|^2}{\|\xi\|^2 - 1} &= e^{-2t} \\
\ln\left(\frac{\sqrt{\left(\|\xi\|^2 - 1\right)}}{\|\xi\|}\right) &= t = \alpha_\xi < 0
\end{aligned}
$$

The phase portrait is the following:



1.3 Dynamical systems in plane. Poincare Bendixson theorem, periodic solutions and more positively invariant sets.

Theorem. Poincare-Bendixson theorem.

Suppose that $\xi \in G \subset \mathbb{R}^2$ is such that the closure of the positive orbit $O_+(\xi)$ is compact and is contained in G and the ω - limit set $\Omega(\xi)$ does not contain equilibrium points.

Then the ω - limit set $\Omega(\xi)$ is an orbit of a periodic solution.□

Counterexample: an annulus containitn no periodic orbits, because it is a region of attraction containing an attracting equilibrium.

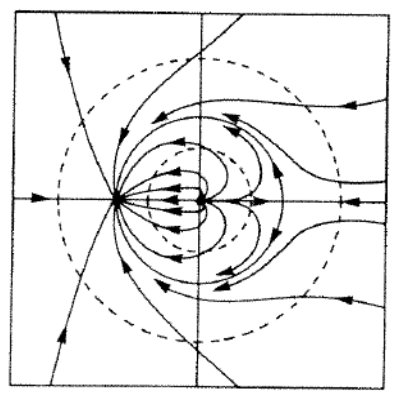


Fig. 3.25. Phase portrait for the system $\dot{r} = r(1-r)$, $\theta = \sin\theta$.

Definition

A periodic orbit γ (an orbit corresponding to a periodic solution) is called an ω - limit cycle (or often just a limit cycle) if $\gamma = \Omega(\xi)$ for some start point $\xi \in G\backslash\gamma$: namely that γ is an ω-limit set for some point ξ outside γ.

This definition excludes the case of phase portraits completely filled periodic orbits, as the system $x' = -y$, $y' = x$, having all orbits being circles arund the origin corresponding to periodic solutions.

Hint to applications. It is difficult to check conditions in the Poincare-Bendixson theorem as they are. It is easier to check that there is a compact positively invariant set $C \subset G \subset \mathbb{R}^2$ such that $\xi \in C$. Then the ω - limit set $\Omega(\xi) \subset C$ is not empty. If C contains no equilibrium points, then the closure of $\Omega(\xi)$ cannot contain equilibrium points either and by the Poincare-Bendixson theorem $\Omega(\xi)$ is an orbit of a periodic solution.

One fundamental fact about positively invariant sets is the following.

Theorem 4.45. p. 150, L&R (slightly generalised, without proof)

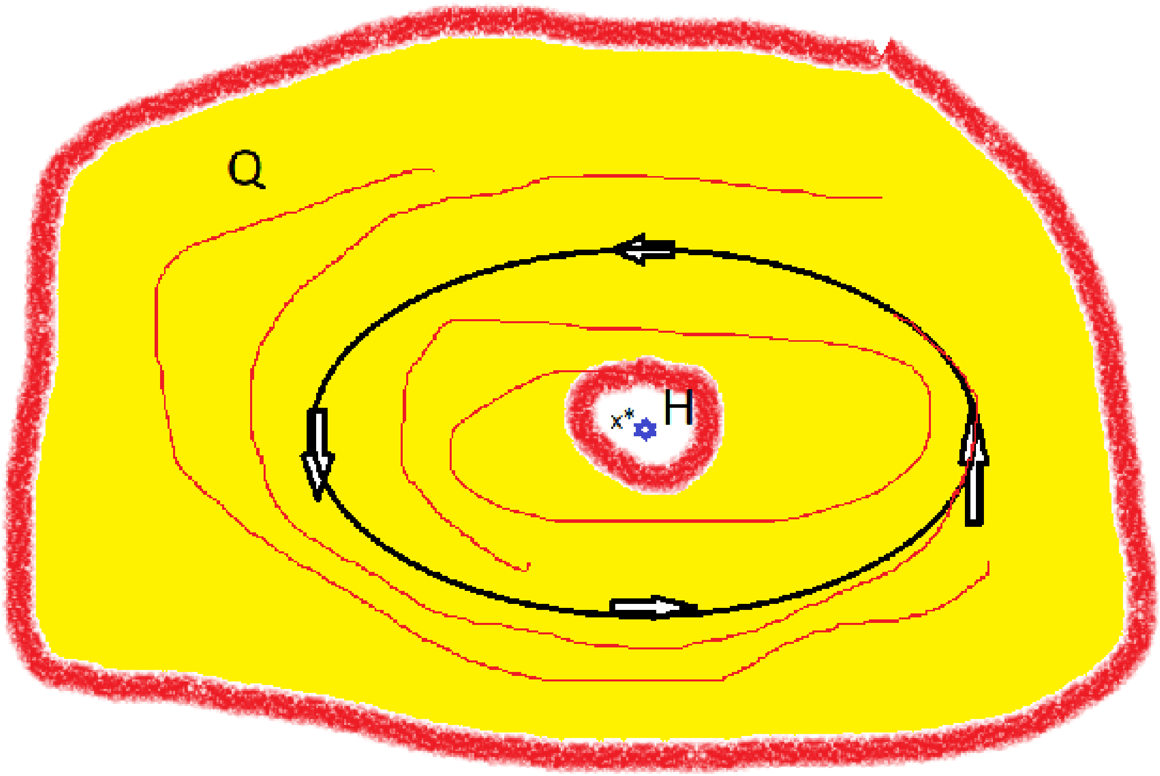
Suppose that $C \subset G \subset \mathbb{R}^2$ is non-empty and compact and is homeomorphic to a circular disc (has no holes). If C is positively invariant under the flow $\varphi(t, \xi)$, then C contains at least one equilibrium point for the corresponding ODE.

Proof of this theorem is based in the Bohl-Brouwer fixed-point theorem about the existence of fixed points $x = F(x)$ of a continuous mapping $F : C \to C$ for a compact $C \subset \mathbb{R}^n$ homeomorphic to a ball. See an Appendix in L.R.

Definition. Two sets A and B in $\mathbb{R}^n$ are homeomorphic if there is a continuous invertible mapping (homeomorphism) $\Theta : A \to B$,and $\Theta^{-1} : B \to A$.

The Theorem 4.45 has an important practical consequence for application of the Poincare Bendixson theorem.

Remark. Considering any periodic orbit in the plane $\mathbb{R}^2$ we see that it encloses a compact positively invariant set Q homeomorfic to a round disc (it follows from Jordan's lemma). Theorem 4.45 suggests that Q includes at least one equilibrium point. It means that any periodic orbit in plane must urround at least one equilibrium point. It makes that typical compact positively - invariant set C considered for applying the Poincare-Bendixson theorem should be a closed ring shaped set with at least one hole in the middle including an equilibrium point.



Check list for application of the Poincare-Bendixson theorem.

• One starts with applying one of the two methods above to find a compact positively - invariant set Q.

• Then we consider if Q has an equilibrium inside. Usually there is one such if our intuition is not wrong. Therefore the set Q does not satisfy conditions in the Poincare-Bendixson theorem yet. It is only the first step to the goal.

• Suppose there is just one equilibrium point x_* inside Q. It might be that this equilibrium is asymptotically stable and attracts all trajectories starting in Q. Then there is no periodic orbit inside Q.

• To have a periodic orbit in Q we need to find a "hole" H around the equilibrium x_* such that no trajectories enter it. Then the closure of the set $Q\backslash H$ without the hole will be a compact ring - shaped set (annulus) that is positively invariant and contains no equilibrium points. Then all trajectories $x(t)$ starting in $\overline{Q\backslash H}$ will have a non-empty ω - limit set that according to the Poincary Bendixson theory is a periodic orbit. There can be several periodic orbits in $\overline{Q\backslash H}$ that are ω - limit sets for different trajectories. There an also be some periodic orbits that are not ω - limit sets!

• The "hole"H that repells trajectories can be found using the method of test functions, sometimes using the same test function V as one used to identify Q, just choosing different level sets for Q and for H.

• Alternatively one can use the linearization to show that this equilibrium is repeller and therefore trajectories cannot enter some small neighbourhood of the equilibrium in the middle of the set Q. This method is convenient in the case when the equilibrium is not the origin

• One must check at the end that the positively invariant annulus (closed ring shaped domain) does not include equilibrium points (no at the boundary either!).

It is often simpler to do it after carrying out estimates for V_f by first checking zeroes of $V_f(x) = 0$ that contain naturally all equilibrium points but is a scalar equation, and then checking zeroes of the system $f(x) = 0$.

Examples on Poincare-Bendixson's theorem

Example. Show that the following system has a periodic solution.

$$\begin{aligned} x' &= y \\ y' &= -x + \left(1 - x^2 - 2y^2\right) y \end{aligned}$$

The test function is chosen as $V(x,y) = \frac{1}{2}(x^2 + y^2)$

Solution. First, we convert Equation (7) to a system of two first-order equations by setting $x = z$ and $y = \dot{z}$. Then,

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + (1 - x^2 - 2y^2)y. \tag{8}$$

Next, we try and find a bounded region R in the $x - y$ plane, containing no equilibrium points of (8), and having the property that every solution $x(t)$, $y(t)$ of (8) which starts in R at time $t = t_0$, remains there for all future time $t \geq t_0$. It can be shown that a simply connected region such as a square or disc will never work. Therefore, we try and take R to be an annulus surrounding the origin. To this end, compute

$$\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right) = x\frac{dx}{dt} + y\frac{dy}{dt} = (1 - x^2 - 2y^2)y^2,$$

and observe that $1 - x^2 - 2y^2$ is positive for $x^2 + y^2 < \frac{1}{2}$ and negative for $x^2 + y^2 > 1$. Hence, $x^2(t) + y^2(t)$ is increasing along any solution $x(t)$, $y(t)$ of (8) when $x^2 + y^2 < \frac{1}{2}$ and decreasing when $x^2 + y^2 > 1$. This implies that any solution $x(t)$, $y(t)$ of (8) which starts in the annulus $\frac{1}{2} < x^2 + y^2 < 1$ at time $t = t_0$ will remain in this annulus for all future time $t \geq t_0$. Now, this annulus contains no equilibrium points of (8). Consequently, by the Poincaré–Bendixson Theorem, there exists at least one periodic solution $x(t)$, $y(t)$ of (8) lying entirely in this annulus, and then $z = x(t)$ is a nontrivial periodic solution of (7).

A more geometric analysis would be to consider the the test function $V(x,y) = (x^2+y^2)/2$

nd its particular level set - the ellipse with the equation

$$x^2 + 2y^2 = 1$$

that separates points (x, y) where $V_f(x, y) > 0$ and $V_f(x, y) < 0$.

$$\begin{aligned} V_f(x,y) &= \frac{d}{dt}V(x(t), y(t)) = \nabla V(x,y) \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \\ \nabla V(x,y) \cdot f(x,y) &= -\left(x^2 + 2y^2 - 1\right) y^2 \leq (\geq) 0 \end{aligned}$$

The negative sign of $V_f(x, y)$ says that trajectories go inside the level set of V (a circle in this case) going through the point (x, y).

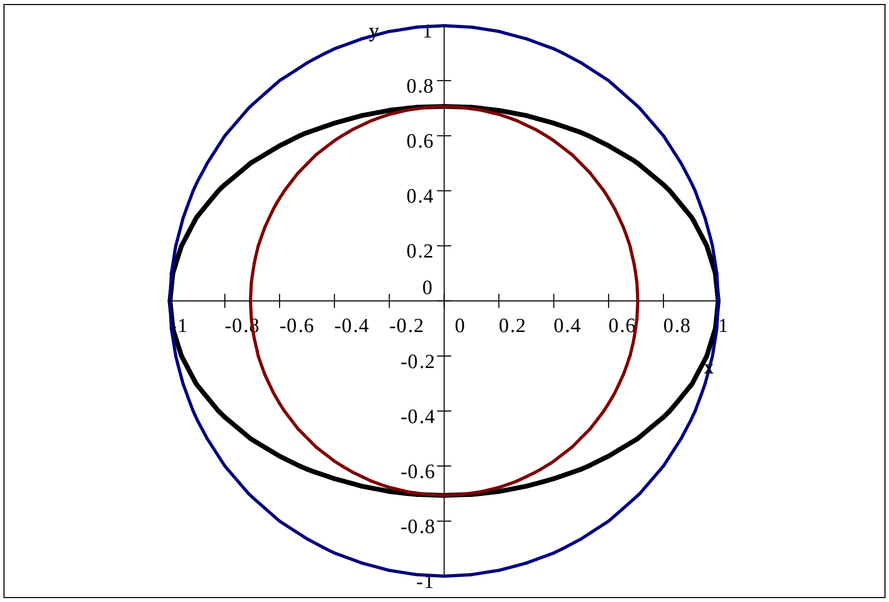
The positive sign of $V_f(x, y)$ says that trajectories go outside the level set of V going through the point (x, y).

The half axes of ellipse $x^2 + 2y^2 = 1$ are expressed from the transformed equation

$$\frac{x^2}{1^2} + \frac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$$

We find the largest level set (circle) of $V(x, y)$ inside this ellipse (red) and the smallest level set of $V(x, y)$ outside this ellipse (blue) to get the smallest positive invariant set that includes a periodic orbit.

By chance they coinside with boundaries of the annulus found earlier by analytical means.



As Theorem 4.45 and examples considered before suggest, the positively invariant set we look for applying the Poincare Bendixson theorem must have a shape of annulus with a hole in the middle that contains at least one equilibrium point. The next Proposition gives a particular hint how to find the "hole" for such an annulus domain with less effort by using the Grobman-Hartman theorem that we studied earlier.

Proposition 4.56. p. 165.

Let $C \subset G$ be a compact set that is positively invariant under the local flow φ fgenerated by the equation $x'(t) = f(x)$. Assume that an interior point x_* is the interior point in C and is the only equilibrium point in C. Assume that f is differentiable in x_*. Let A be the Jacoby matrix of f in x_*: $\frac{Df}{Dx}(x_*) = A$. Let eigenvalues of A have $\operatorname{Re} \lambda_{1,2} > 0$.

Then there exists at least one ω - limit cycle in C.

Proof is an exercise.

Example 4.57

Consider again the system given in Exercise 4.16, with $G = \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2)\,,\, -z_1 + z_2(1 - \|z\|^2)\big)$. Let C be the closed unit disc $\{z \in \mathbb{R}^2 : \|z\| \le 1\}$. Then

$$\langle z, f(z)\rangle = \|z\|^2\big(1 - \|z\|^2\big) = 0 \quad \forall\, z \in \partial C,$$

and so solutions starting in C cannot exit C in forwards time. Thus, the compact set C is positively invariant. Moreover, 0 is the unique equilibrium in C

and

$$A = (Df)(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

with spectrum $\sigma(A) = \{1 + i, 1 - i\}$. Therefore, by Proposition 4.56, we may conclude the existence of a limit cycle in C. This, of course, is entirely consistent with Exercise 4.16 and Example 4.37, the conjunction of which shows (by explicit computation of the local flow) that the unit circle $\gamma = \partial C$ is a periodic orbit and coincides with the ω-limit set $\Omega(\xi)$ of every ξ with $0 < \|\xi\| < 1$. △

Exercise. Rectangular positively invariant set and application of the Poincare Bendixson theorem.

Consider the following system of ODEs :

$$\begin{cases} x' = & 10 - x - \frac{4xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$$

a) show that the point (x_*, y_*) with coordinates $x_* = 2$ and $y_* = 5$ is the only equilibrium point and is a repeller;

b) find a rectangle $[0, a] \times [0, b]$ in the first quadrant $x > 0$, $y > 0$ bounded by coordinate axes and by two lines parallel to them, that is a positively invariant set. Explain why the system must have at least one periodic orbit in this rectangle.

1. **Solution.**

a) $x_* = 2$ and $y_* = 5$ is an equilibrium point: $\left(1 - \frac{5}{1+2^2}\right) = 0$; and $10 - 2 - \frac{4 \cdot 2 \cdot 5}{5} = 10 - 2 - 8 = 0$.

The Jacobi matrix is $A = \begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 & -4\frac{x}{x^2+1} \\ -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 & -\frac{x}{x^2+1} \end{bmatrix}$. It is calculated as:

$$\nabla\left(10 - x - \frac{4xy}{1+x^2}\right) = \begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 \\ -4\frac{x}{x^2+1} \end{bmatrix}\Bigg|_{x=2,\ y=5}$$

$$= \begin{bmatrix} -4\frac{5}{5} + 8\,(4)\,\frac{5}{25} - 1 \\ -4 * \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -4 + \frac{32}{5} - 1 \\ -\frac{8}{5} \end{bmatrix} = \begin{bmatrix} 1.4 \\ -1.6 \end{bmatrix}$$

$$\nabla\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = \begin{bmatrix} -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 \\ -\frac{x}{x^2+1} \end{bmatrix}\Bigg|_{x=2,\ y=5} = \begin{bmatrix} -\frac{5}{5} + 2(4)\frac{5}{25} + 1 \\ -\frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -1 + \frac{8}{5} + 1 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.4 \end{bmatrix}$$

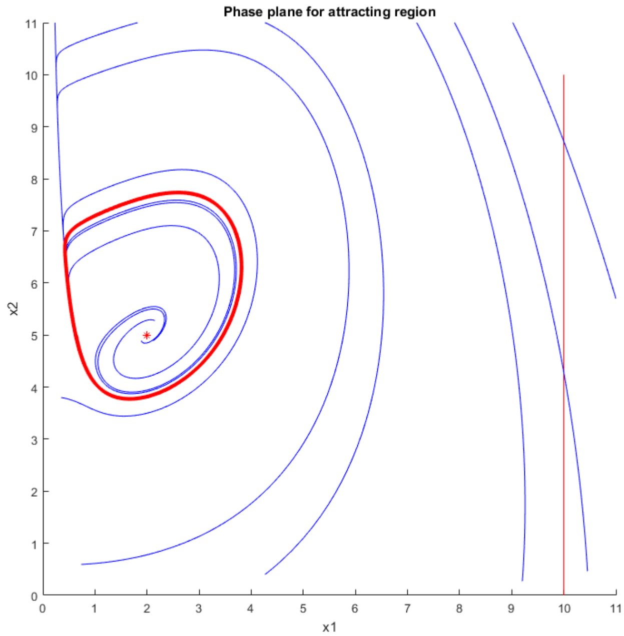
The Jacobi matrix in x_*, y_* is $A = \begin{bmatrix} 1.4 & -1.6 \\ 1.6 & -0.4 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - \lambda + 2 = 0$,

$\text{trace}(A) = 1 > 0$, $\det(A) = 2 > \frac{[\text{trace}(A)]^2}{4} = \frac{1}{4}$ that corresponds to an unstable spiral and it is a repeller, eigenvalues are: $\lambda_1 = 0.5 + \sqrt{0.25 - 2} = 0.5 + i\sqrt{1.75}$, $\lambda_2 = 0.5 - \sqrt{0.25 - 2} = 0.5 - i\sqrt{1.75}$.

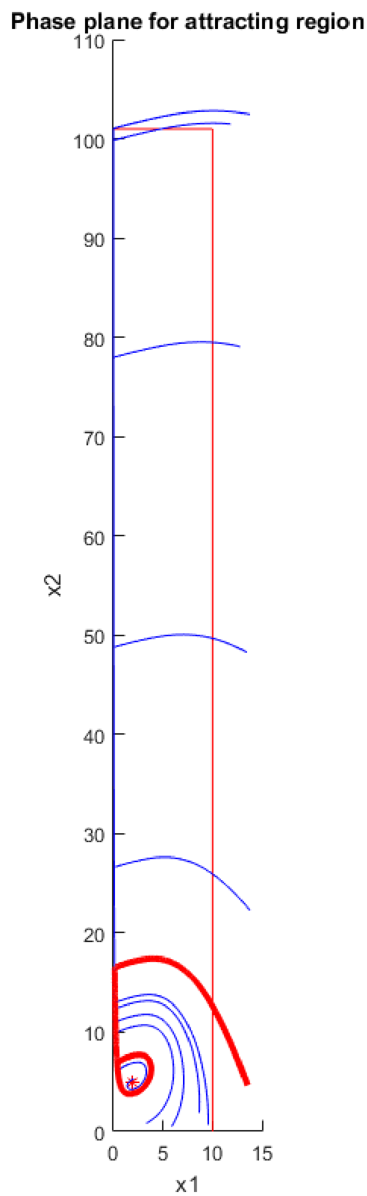
It implies by the Grobman-Hartman theorem, that trajectories cannot enter a small open ball $B((x_*, y_*), \varepsilon)$ with the center in the equilibrium point $(2, 5)$ and some small radius ε. We do not need to specify ε here.

b) Observe that the first quadrant is a positively invariant set. For $y = 0$ we have $\dot{x} = 10 > 0$ and for $y = 0$ and $x > 0$ we have $y' = x > 0$.

Observe also that $\dot{y} < 0$ for $y > 1 + x^2$ and $x > 0$; and that $\dot{x} < 0$ for $x > 10$ and $y > 0$.



It implies that the rectangle $[0,10] \times [0,101]$ is a positively invariant compact set. Excluding a small open set H_ε containing the equilibrium point $(2,5)$ and small diameter ε we get a positively invariant compact set $[0,10] \times [0,101] \setminus H_\varepsilon$ without equilibrium points that according to the Poincare Bendixson theorem must include at least one periodic orbit because each trajectory starting in this set has a non-empty ω - limit set that is a periodic orbit. So in principle there can be several periodic orbits surrounding this equilibrium point.



Phase plane for attracting region
110
100
90
80
70
60
50
40
30
20
10
0
x2
0
5
10
15
x1

Example. 3.9.1 (from A-P) One can instead of the analytical approach shown below, use a more geometric argument, based on considering the curves const $= 3x_1^2 + 2x_2^2$. It is demonstrated later for the **Exercise 4.21.**

Show that the following equation has a limit cycle (a periodic orbit that is an ω - limit set of at least one solution)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + x_2\left(1 - 3x_1^2 - 2x_2^2\right) \end{aligned}$$

write the system in polar coordinates:

$$\begin{aligned} r &= r\sin^2\theta\left(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta\right) \\ \theta' &= -1 + \frac{1}{2}\sin(2\theta)\left(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta\right) \end{aligned}$$

a) Observe that with $r = 1/2$

$$r' = \frac{1}{4}\sin^2\theta\left(1 - \frac{1}{2}\cos^2\theta\right) \geq 0$$

with equality only at $\theta = 0$ and π . Thus $\{x : r > 1/2\}$ is positively invariant (trajectories do not enter the circle $r < 1/2$.

b) The same equation for r' implies that

$$r' \leq r\sin^2\theta\left(1 - 2r^2\right)$$

Thus the annulus $C = \left\{x : 1/2 < r < 1/\sqrt{2}\right\}$ is positively invariant. The only fixed point to the system is outside this annulus. Therefore here is at least one periodic orbit in C that is an ω limit set for all trajectories starting in C (and therefore is a limit cycle).

Exercise 4.21, p. 158

Consider the system $z' = f(z_1, z_2)$:

$$\begin{aligned} z_1' &= z_2 + z_1 g(z_1, z_2) \\ z_2' &= -z_1 + z_2 g(z_1, z_2) \\ g(z_1, z_2) &= 3 + 2z_1 - z_1^2 - z_2^2 \end{aligned}$$

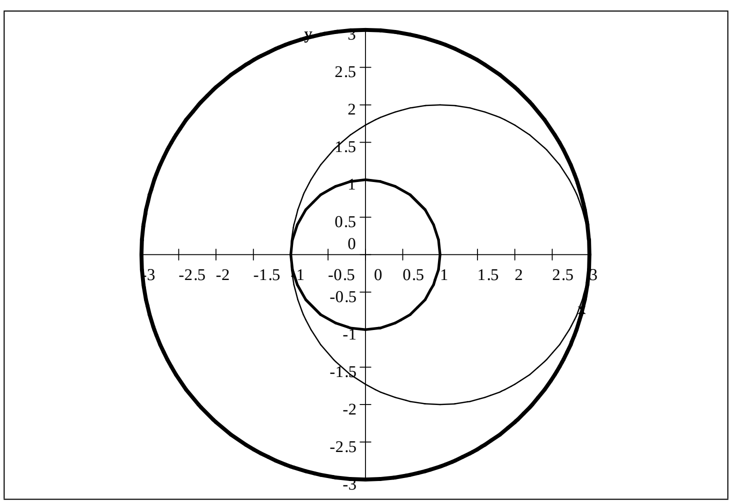
Prove that the system has at least one periodic solution.

Solution.

Consider the test function $V(z_1, z_2) = \left(\frac{z_1^2+z_2^2}{2}\right)$.

$$\begin{aligned} \nabla V \cdot f(z_1, z_2) &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cdot \begin{bmatrix} z_2 + z_1 g(z_1, z_2) \\ -z_1 + z_2 g(z_1, z_2) \end{bmatrix} \\ &= \left(z_1^2 + z_2^2\right) g(z_1, z_2) = \left(z_1^2 + z_2^2\right) \left(3 + 2z_1 - z_1^2 - z_2^2\right) \\ &= r^2(4 - (1 - z_1)^2 - z_2^2) \end{aligned}$$

The circle $4 = (1 - z_1)^2 + z_2^2$ has center in the point $(1, 0)$ and radius 2:



Inside this circle $\nabla V \cdot f(z_1, z_2) > 0$ outside this circle $\nabla V \cdot f(z_1, z_2) < 0$. Therefore as it is easy to see from the picture, $\nabla V \cdot f(z_1, z_2) \geq 0$ on the circle $z_1^2 + z_2^2 = 1$ with center in the origin, and $\nabla V \cdot f(z_1, z_2) \leq 0$ on the circle $z_1^2 + z_2^2 = 9$ with center in the origin. The ring shaped set C: $1 \le r \le 3$ is a positively invariant compact set. The origin is the only equilibrium point fo the system, because from the expression $\nabla V \cdot f(z_1, z_2) = r^2 g(z_1, z_2)$ it follows that other equilibrium points must be on the circle $g(z_1, z_2) = 0 = 4 - (1 - z_1)^2 - z_2^2$. Substitution $g(z_1, z_2) = 0$ into the system leads to the conclusion that there are no equilibrium points on this circle.

Therefore the Poincare Bendixson theorem implies that there exists at least one periodic orbit contained in the ring shaped set C.

Exercise. 3.8.2.

Solve a similar problem for the function $g(z_1, z_2) = 3 + z_1 z_2 - z_1^2 - z_2^2$.

Example 3.8.2. Find the limit cycles in the following systems and give their types:

(a) $\dot{r} = r(r-1)(r-2), \quad \dot{\theta} = 1;$ (3.67)

(b) $\dot{r} = r(r-1)^2, \quad \dot{\theta} = 1.$ (3.68)

Solution

(a) There are closed trajectories given by

$$r(t) \equiv 1, \quad \theta = t \quad \text{and} \quad r(t) \equiv 2, \quad \theta = t. \tag{3.69}$$

Furthermore

$$\dot{r} \begin{cases} > 0, & 0 < r < 1 \\ < 0, & 1 < r < 2 \\ > 0, & r > 2 \end{cases}. \tag{3.70}$$

The system therefore has two circular limit cycles: one stable ($r = 1$) and one unstable ($r = 2$).

(b) System (3.68) has a single circular limit cycle of radius one. However, $\dot{r}$ is positive for $0 < r < 1$ *and* $r > 1$, so the limit cycle is semistable. □

1.

Generalized Poincare-Bendixson's theorem.

The following theorem gives a more complete description of the types of ω - limit sets in the plane $\mathbb{R}^2$.

Theorem (generalized Poincare-Bendixson)

Let M be an open subset of $\mathbb{R}^2$ and $f : M \to \mathbb{R}^2$ and $f \in C^1$. Fix $\xi \in M$ and suppose that $\Omega(\xi) \neq \emptyset$, compact, connected and contains only finitely many equilibrium points.

Then one of the following cases holds:

(i) $\Omega(\xi)$ is an equilibrium point

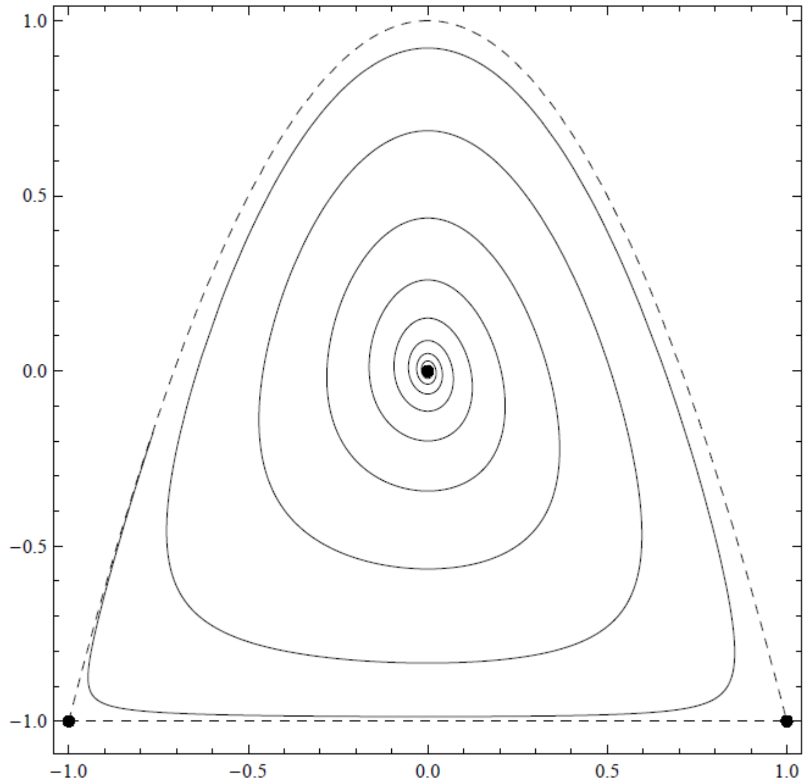
(ii) $\Omega(\xi)$ is a periodic orbit

(iii) $\Omega(\xi)$ consists of finitely many fixed points $\{x_j\}$and non-closed orbits γ such that ω and α - limit points of γ belong to $\{x_j\}$.

Example. While we have already seen examples for case (i) and (ii) in the Poincaré–Bendixson theorem we have not seen an example for case (iii). Hence we consider the vector field

$$f(x,y) = \begin{pmatrix} y + x^2 - \alpha x(y - 1 + 2x^2) \\ -2(1+y)x \end{pmatrix}.$$

First of all it is easy to check that the curves $y = 1 - 2x^2$ and $y = -1$ are invariant. Moreover, there are four fixed points $(0,0)$, $(-1,-1)$, $(1,-1)$, and $(\frac{1}{2\alpha}, -1)$. We will chose $\alpha = \frac{1}{4}$ such that the last one is outside the region bounded by the two invariant curves. Then a typical orbit starting inside this region is depicted in Figure 7.9: It converges to the unstable fixed point $(0,0)$ as $t \to -\infty$ and spirals towards the boundary as $t \to +\infty$. In



particular, its $\omega_+((x_0, y_0))$ limit set consists of three fixed points plus the orbits joining them.

To prove this consider $H(x, y) = x^2(1 + y) + \frac{y^2}{2}$ and observe that its change along trajectories

$$\dot{H} = 2\alpha(1 - y - 2x^2)x^2(1 + y)$$

is nonnegative inside our region (its boundary is given by $H(x, y) = \frac{1}{2}$). Hence it is straightforward to show that every orbit other than the fixed point $(0, 0)$ converges to the boundary. ⋄

1.4 More remarks about hunting ω - limit sets

1.5 How to find an ω - limit set?

We put here this user guide about ω **- limit sets** that refers to some notions that will be discussed later in the course. You can come back to this text when corresponding notions will be introduced.

ω - limit sets live naturally inside ω - invariant sets. In case one can find a very small ω - invarinat set the position and the size of the ω - limit set inside it will be rather well defined.

Description properties of ω - limit sets is the main and the most complicated problem in the theory of dynamical systems. Even numerical investigation of limit sets in dimension higher then 2 is rather complicated and needs advanced mathematical tools.

In autonomous systems the plane $\mathbb{R}^2$ limit sets can be only of three types: a) **equilibrium points, b) periodic orbits, and c) closed curves consisting of finite number of equilibrium points connected by open orbits.** It is an extension of the Poincare-Bendixson theorem.

The analytic identification or at least effective localization of ω - limit sets is possible with help of La Salle's invariance theorem that will be studied later. It states that ω - limit sets are subsets of zero level sets of $V_f(x) = (\nabla V \cdot f)(x)$ for appropriate test function (Lyapunov function) $V(x)$ satisfying $V_f(x) \leq 0$.

This theorem helps in particular to find ω - limit sets that are asymptotically stable equilibrium points, by a rather simple check of the behaviour of the velocity $f(x)$ on the zero level set where $V_f(x) = 0$.

One can also investigate asymptotically stable equilibrium points with help of so called "strong" Lyapunov functions V that satisfy $V_f(x) < 0$ for $x \neq 0$.

It is difficult in practice to find analytically ω - limit sets in plane of two other types. It is possible if one can find analytically a zero level set $V_f^{-1}(0)$ of a test function V that is a closed curve in plane. Then this level set belongs to one of the two other types: periodic orbit or a chain of equilibrium points connected by open orbits.

Such an analytic construction is not known for the equation with periodic orbit in the second home assignment, despite the fact that special techniques were developed to show that the periodic orbit there is unique.

If a system has first integrals: test functions having $V_f(x) = 0$ everywhere, then level sets of first integrals give a good tool to identify ω - limit sets because these level sets consist of orbits and are because of that very narrow invariant sets. The existence of first integrals is usually a sign that the energy of the system is preserved, that is a rather special situation.

Observations above show that in many practical situations we can find ω - limit sets that are asymptotically stable equilibrium points.

For systems in plane we can with help of Poincare Bendixson theorem also show that in certain situations ω - limit sets are orbits of periodic solutions but cannot give a formula for them and cannot state how many they are.

ω - limit sets in the plane that are more complicated than equilibrium points, is possible to describe analytically in the case when for a Lyapunov test function $V(x)$ the zero level set $V_f^{-1}(0)$ is a closed curve in the plane and the corresponding equation can be investigated analytically.

1 User Guide to hunting positively - invariant sets and ω - limit sets.

We consider flows or dynamical systems corresponding to autonomous differential equations

$$\dot{x} = f(x), \quad f : G \to \mathbb{R}^N$$

A system has naturally many positively - invariant sets, for example the whole domain G is always an positively - invariant set, but it is not very interesting. We like to find possibly narrow invariant sets showing more precisely where trajectories or solutions to the equation tend when t tends to the upper bound of the maximal time interval (usually $t \to \infty$ if the trajectory is bounded and has compact closure).

A general idea that is used to answer many questions about behaviour of solutions (trajectories) of the equations, is the idea of test functions. One checks if the velocities $f(x)$ are directed inside or outside with respect to the sets like $Q = \{x \in U : V(x) \leq C\}$ or $Q = \{x \in U : V(x) \geq C\}$ defined by some simple test functions $V : U \to \mathbb{R}$, $U \subset G$. A more refined variant of this idea by Lyapunov is to find test a function that is monotone along the trajectories $\varphi(t, \xi)$ of the equation. The advantage of the idea with test functions is that one does not need to solve the equation to use it.

How to find an positively **- invariant set?**

Method 1. We find a test function $V(x)$ that has some level sets $\partial Q = \{x : V(x) = C\}$ that are closed curves (or surfaces in higher dimensions) enclosing a bounded domain Q. Typical examples are $V(x, y) = x^2 + y^2{=}R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}{=}1$ - ellipse, or more complicated ones as $V(x, y) = x^6 + ay^4$ - smoothed rectangle shape or squeezed ellipse, $V(x, y) = x^2 + xy + y^2 = C$ - ellipse rotated in $\pi/4$ and having axes A and B related as $A/B = \sqrt{3}$etc.

• To show that a particular level set ∂Q bounds an positively - invariant set Q we check the sign of the directional derivative of V along the velocity in the equation: $V_f(x) = (\nabla V \cdot f)(x)$ for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• The sign of $V_f(x)$ shows if the trajectories go to the same side of the level set as the gradient ∇V (if $V(x) > 0$) or to the opposite side (if $V_f(x) < 0$).

• Then if $V(x)$ is rising for x going out of Q, and $V_f(x) < 0$ then the domain Q inside this level set ∂Q (curve in the plane case) will be positively - invariant. Similarly if $V(x)$ is decreasing out of this level set, and $V_f(x) < 0$ on the level set ∂Q then the domain Q inside this level set will be positively - invariant.

In the opposite case the complement to Q that is $\mathbb{R}^N \backslash Q$ will be positively - invariant and trajectories $\varphi(t, \xi)$ starting in this complement: $\xi \in \mathbb{R}^N \backslash Q$ will never enter Q.

First integrals. A very particular case of test functions are functions that are constant on all trajectories $\varphi(t, \xi)$ of the system. It means that $\frac{d}{dt} V(\varphi(t, \xi)) = V_f(x) \equiv 0$. Usually but not always, such test functions have the meaning of the total energy in the system. In this case all level sets of the first integral are invariant sets, because velocities $f(x)$ are tangent vectors to the level sets in this case.

Method 2. If it is difficult to guess a simple test function giving one closed formula for the boundary of an positively - invariant set as in the Method 1, then one can try to identify a boundary for an positively

- invariant set as a curve (or a surface in higher dimensions) consisting of a number of simple peaces, for example straight segments.

The simplest positively - invariant set of such kind would be a rectangle (a rectangular box in higher dimensions) with sides parallel to coordinate axes. Then to check that this rectangle is an positively - invariant one needs just to check the sign of x or y - components of $f(x)$ on these segments, showing that trajectories go inside or outside of the rectangle.

Application to Poincare Bendixson theorem

One searchs often positively - invariant sets with special properties. For example to apply the Poincare-Bendixson theorem one needs to find an positively - invariant set without equilibrium points. On the other hand it is known that any periodic orbit in plane encloses at least one equilibrium point. It means that a typical positively - invariant set for applying the Poincare-Bendixson theorem should be ring shaped with at least one hole in the middle including a repelling non stable equilibrium point.

Check list for application of the Poincare-Bendixson theorem.

- One starts with applying one of the two methods above to find a compact positively - invariant set Q with at least one equilibrium point inside it. Such set Q does not satisfy conditions in the Poincare-Bendixson theorem yet.
- To identify holes around the equilibriums in the middle (one must find all such equilibrium points at the end !), one needs often to find one more test function for each of them, to show that trajectories do not enter a neighbourhood of each of the equilibriums.
- Alternatively one can use the linearization to show that this equilibrium is repeller and therefore trajectories cannot enter some small neighbourhood of the equilibrium in the middle of the set Q.
- One must check at the end that the found positively invariant annulus (closed ring shaped domain) does not include equilibrium points (not at the boundary either!) It is often simpler to do after carrying out estimates for V_f.

How to find an ω - limit set?

ω - limit sets live naturally inside ω - invariant sets. In case one can find a very small ω - invarinat set the position and the size of the ω - limit set inside it will be rather well defined.

Description properties of ω - limit sets is the main and the most complicated problem in the theory of dynamical systems. Even numerical investigation of limit sets in dimension higher then 2 is rather complicated and needs advanced mathematical tools.

In autonomous systems the plane $\mathbb{R}^2$ limit sets can be only of three types: a) **equilibrium points, b) periodic orbits, and c) closed curves consisting of finite number of equilibrium points connected by open orbits.** It is an extension of the Poincare-Bendixson theorem.

The analytic identification or at least effective localization of ω - limit sets is possible with help of La Salle's invariance theorem. It states that ω - limit sets are subsets of zero level sets of $V_f(x) = (\nabla V \cdot f)(x)$ for appropriate Lyapunov functions $V(x)$ satisfying $V_f(x) \leq 0$.

This theorem helps in particular to find ω - limit sets that are asymptotically stable equilibrium points, by a rather simple checking the behaviour of the velocity $f(x)$ on the zero level set where $V_f(x) = 0$.

One can also investigate asymptotically stable equilibrium points with help of so called "strong" Lyapunov functions that satisfy the strict inequality $V_f(x) < 0$ for $x \neq 0$.

It is difficult in practice to find analytically ω - limit sets of two other types. It is possible if one can find analytically a zero level set $V_f^{-1}(0)$ that is a closed curve in plane. Then this level set belongs to one of the two other types: periodic orbit or a chain of equilibrium points connected by open orbits.

Such an analytic construction is not known for the equation with periodic orbit in the second home assignment, despite the fact that special techniques were developed to show that the periodic orbit there is unique.

If a system has first integrals, then level sets of first integrals give a good tool to identify ω - limit sets because these level sets consist of orbits and are very narrow invariant sets themself. The existence of first integrals is usually a sign that energy of the system is preserved, that is a rather special situation.

The observations above show that in many practical situations we can find ω - limit sets that are asymptotically stable equilibrium points.

For systems in plane we can with help of Poincare Bendixson theorem also show that in certain situations ω - limit sets are periodic orbits but cannot give a formula for them and cannot state how many they are.

ω - limit sets in the plane that are more complicated then equilibrium points, is possible to describe analytically in the case when for a Lyapunov function $V(x)$ the zero level set $V_f^{-1}(0)$ is a closed curve in the plane and the corresponding equation can be investigated analytically.

1 Bendixson's citerium for nonexistence of periodic solutions in plane.

Theorem. Let $x' = f(x)$ with $f : G \to \mathbb{R}^2$, $G \subset \mathbb{R}^2$ be open, $f \in C^1(G)$, and let $D \subset G$ be a **simply connected domain** (domain without "holes" even without point holes). It is enough to require that f is locally Lipschitz in G with more knowlege of integration theory.

Suppose that $\mathrm{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly positive (or strictly negative) in D, where $f = [f_1, f_2]^T$.

Then the equation has no periodic solutions with orbits inside D.

Proof 1. Carry out a proof by contradiction. Suppose that there is a periodic trajectory $x(t)$ with period $T > 0$ in D. $x(t+T) = x(t)$ and

$$x_1'(t) = f_1(x(t)), \qquad x_2'(t) = f_2(x(t))$$

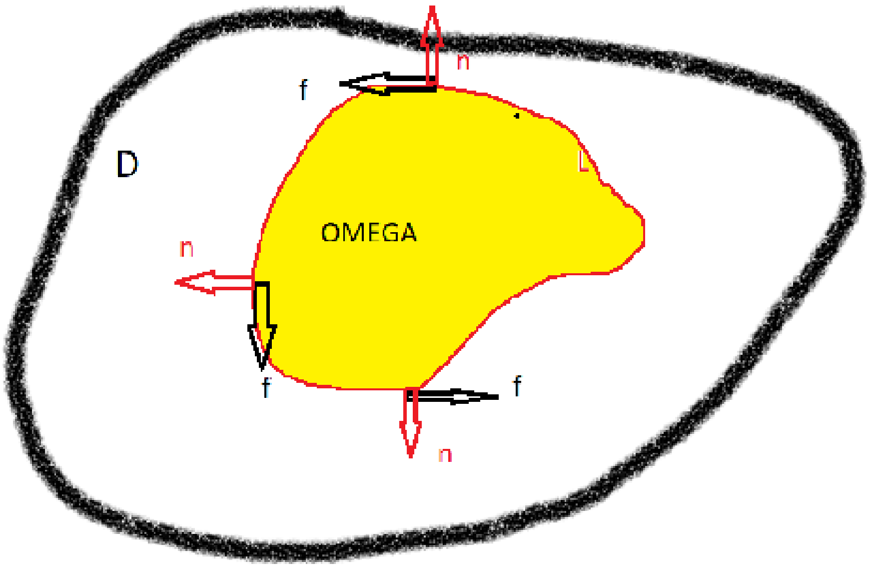
Denote orbit of $x(t)$ by $\mathcal{L} = \{x(t) : \quad t \in [0,T]\}$. It will be a closed curve. Denote the domain inside $\mathcal{L}$ by Ω. Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes. Consider the integral of $\mathrm{div}(f)$ over Ω and apply Gauss theorem:

$$I = \int_\Omega \mathrm{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n \, dl$$

where n is the outward normal to the boundary $\partial\Omega$. Point out that $f(x(t)) = x'(t)$ on $\partial\Omega = \mathcal{L}$ because $\mathcal{L}$ is the orbit of the periodic solution $x(t)$ that we supposed to be existing. Therefore $f(x(t))$ is the tangent vector to $\partial\Omega$ and therefore scalar product of it woth the normal vector is zero $f \cdot n = 0$. Therefore

$$I = \iint_\Omega \mathrm{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n \, dl = 0$$

with the curve integral over $\partial\Omega = \mathcal{L}$ in the right hand side. On the other hand $\mathrm{div}(f) > 0$ (or strictly negative) in the whole $D \supset \Omega$.Therefore the integral $I = \int_\Omega \mathrm{div}(f) dx_1 dx_2$ over a bounded domain Ω must be strictly positive (negative).We arrived to a contradiction: $0 > 0$. Therefore our supposition



was wrong and the system cannot have a periodic orbit in D.■

Proof 2. starts similarly with the supposition that there is a periodic trajectory $x(t)$ with period T in D, $x(t+T) = x(t)$ and

$$x_1'(t) = f_1(x(t)), \qquad x_2'(t) = f_2(x(t))$$

Denote the orbit of $x(t)$: by $\mathcal{L} = \{x(t) : \quad t \in [0, T]\}$. Denote the domain inside $\mathcal{L}$ by Ω. Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes.

We apply the Greens formula:

$$\oint_{\partial\Omega} P dx_1 + Q dx_2 = \iint_{\Omega} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2$$

instead of Gauss theorem.

Choose $P = -f_2$, $Q = f_1$ and express the contour integral in the left side of the Greens formula using the definition of the contour integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 x_2' - f_2 x_1')\, dt$$

Point out that $x_1'(t) = f_1(x(t))$ and $x_2'(t) = f_2(x(t))$ and substitute these expressions into the integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 f_2 - f_2 f_1)\, dt = 0$$

Apply the Greens formula substitute expressions for P and Q, and conclude that in the case $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) = \mathrm{div}(f) > 0$:

$$0 = \oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \iint_{\Omega} \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 > 0$$

that is contradiction: $0 > 0$. In the case if $\mathrm{div}(f) < 0$ in D we arrive to the contradiction < 0. ■

1. **Example.**

Show that the following system of ODE has no periodic solutions.

1. $$\begin{cases} x' = x^3 - y^2x + x \\ y' = -0.5y + y^3 + x^4y \end{cases}$$

   We consider divergence of the right hand side of the system.

   $\mathrm{div}(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes, even point-holes). According to Bendixson's criterium the system cannot have periodic solutions anywhere in the plane.

Example.

Show that the following system of ODE has no periodic solutions.

1. $$\begin{cases} x' = \frac{1}{7} + x^2 - yx + y^2 \\ y' = -\frac{1}{5} - y^2 \end{cases}$$

   Solution

   y' is always strictly negative. It implies that $y(t)$ must be monotone function of time. It contradicts to possibility of having periodic solutions that are always bounded.

6. Formulate the Poincare-Bendixson theorem and use it to show that the following system of ODE has a periodic solution $(x(t), y(t)) \neq (0,0)$.

$$\begin{cases} x' = & y \\ y' = & -f(x,y)\,y - x \end{cases}$$

where f, f'_x , f'_y are continuous, $f(0,0) < 0$ and $f(x,y) > 0$ for $x^2 + y^2 > b^2$.

(4p)

One of formulations of the Poincare Bendixson theorem is: let $\varphi(t)$ be a solution to an autonomous equation $x' = f(x)$ in plane, bounded for all $t > 0$. We suppose that f is Lipschitz to guarantee uniquness of solutions. Then the limit set $\omega(\varphi)$ of $\varphi(t)$ has the following property: it either

i) contains an equilibrium

or

ii) $\varphi(t)$ is periodic itself

or $\omega(\varphi)$ is a periodic orbit.

The theorem cas an important corollary, that is also often called Poincare Bendixson theorem:

If the equation $x' = f(x)$ in plane has a compact positively invariant set (a set that no trajectories can leave) that does not include any equilibrium points, then this set must include at least one periodic orbit.

We apply the corollary to the example above and try to find a set in plane satisfying requirements i the corollary.

We observe first that solutions exist for initial points everywhere in the plane because of the smoothness of f.

Consider a test function $V(x,y) = \left(x^2 + y^2\right)/2$ and its evolution along solutions to the given system.

$V' = xy - y^2 f(x,y) - xy = -y^2 f(x,y)$. $f(x,y) > 0$ for $x^2 + y^2 > b^2$. It implies that $V' \leq 0$ for $x^2 + y^2 = b^2$ and that trajectories of the system cannot leave the disc $x^2 + y^2 \leq b^2$.

$f(0,0) < 0$ and is continuous. It implies that there is a small circle around the origin $x^2 + y^2 \leq \delta^2$

such that $f(x,y) < 0$ and correspondingly $V' > 0$ for $x^2 + y^2 = \delta^2$. It implies that the system cannot leave the ring $\delta^2 \leq x^2 + y^2 \leq b^2$. It is easy to observe that the only equilibrium point of the system is the origin: $y = 0$ for the first equation and therefore $x = 0$ from the second equation.

Alltogether implies that the ring $\delta^2 \leq x^2 + y^2 \leq b^2$ is a positively invariant set without equilibrium points and must include at least one periodic orbit.

iii) Formulate the Poincare- Bendixon theorem and use it to show that the following system of ODE has an ω - limit set that is a periodic orbit.

$$\begin{cases} x' = & -x\left(x^2+y^2-3x-1\right)+y \\ y' = & -y\left(x^2+y^2-3x-1\right)-x \end{cases} \qquad \textbf{(4p)}$$

Solution.

i) $\omega(p)$ is an ω -limit set of a point p for a continuous dynamical system $\pi(t,x)$ if for any point $z \in \omega(p)$ there is a sequence of times $\{t_n\}_{n=1}^{\infty}$ depending on z, such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and $\pi(t_n,p) \underset{n\to\infty}{\longrightarrow} z$.

ii) If the trajectory $\pi([0,\infty],x)$ of the dynamical system is bounded, then the ω -limit set $\omega(p)$ is not empty, closed, connected and positively (or ω) - invariant set.

iii) Poincare Bendixson theorem states that if an ODE in plane has a positively invariant set U without equilibrium points then it must include a periodic orbit that is an ω - limit set for all points in thes set U.

We try to use a simple test function $V(x,y)=\frac{1}{2}\left(x^2+y^2\right)$ to localize a positively invariant set without equilibrium points.

$V'(x(t),y(t)) = -x^2\left(x^2+y^2-3x-1\right)+xy+-y^2\left(x^2+y^2-3x-1\right)-xy = -\left(x^2+y^2\right)\left(x^2+y^2-3x-1\right)$

It implies that for large enough x^2+y^2 we have $V'(x(t),y(t)) \leq 0$ and for small enough $x^2+y^2 \neq 0$ we have $V'(x(t),y(t)) > 0$.

Therefore there are δ and R, $0<\delta<R$ such that the ring $\delta \leq \sqrt{x^2+y^2} \leq R$ is a positively invariant set and therefore must include at least one periodic orbit that is an ω - limit set for all points in this ring.

6. Formulate Poincare-Bendixson theorem. Find a positively invariant set for the following system of ODE. Show that the system has at least one periodic solution.

$$\begin{cases} x' = & -y/3+x(1-3x^2-y^2) \\ y' = & x+y(1-3x^2-y^2) \end{cases} \qquad \textbf{(4p)}$$

Poincare-Bendixson theorem. Consider a system $r' = f(r)$ in the plane R^2. If a limit set $\omega_\sigma(r)$ of a point r is not empty, compact and contains no fixed points, it is a regular periodic orbit.□

A corollary of the theorem is that if C is a compact positively invariant set to a system of ODE in the plane and C does not contain any fixed points, it must contain at least one regular periodic orbit.□

Multiply the first equation by $3x$ and the second equation by y and add:

$\frac{1}{2}\left(3x^2+y^2\right)' = \left(3x^2+y^2\right)\left(1-\left(3x^2+y^2\right)\right)$

The function $V(x,y)=3x^2+y^2$ satisfies the equation: $V'(t)=2V(1-V)$.

We observe that $V(t)$ increases along trajectories of the system for $V<1$ and V decreases for $V>1$. It implies that the set $G=\left\{(x,y): 0.5 \leq 3x^2+y^2 \leq 2\right\}$ (an elliptic ring round the origin) is a positively invariant set.

The same calculation shows that the origin is the only stationary point, because stationary point

must satisfy the equation $V(1-V)=0$. By inserting $1-3x^2-y^2=0$ into the equations one can see that points on the ellips $3x^2-y^2=1$ are not stationary, because thay must at the same time be in the origin. It leaves the only fixed point in the origin.

The corollary to Poincare Bendixson theorem states that in a compact positively invariant set without fixed points there must be at least one periodic solution.

4. **Periodic solutions to ODE.**

Show that the following system of ODE has a periodic solution.

$$\begin{cases} x' = y \\ y' = -x + y(1 - 3x^2 - 2y^2) \end{cases}$$

Hint: transform the system to polar coordinates and consider the equation for polar

Expressing the system in polar coordinates r, θ we get:

$r' = r\sin^2(\theta)(1 - 3r^2\cos^2(\theta) - 2r\sin^2(\theta))$

We observe that for small enough $r \quad r' \geq 0$,

for example for $r = 0.5$: $r' = 0.25\sin^2(\theta)(1 - 0.5\cos^2(\theta)) \geq 0$

One observes also from the equation for r' that

$r' \leq r\sin^2(\theta)(1 - 2r^2)$ that makes $r' \leq 0$ for $r \leq 1/\sqrt{2}$. Equality is attained only for $\theta = 0, \theta = \pi$.

It makes the ring $0.5 < r < 1/\sqrt{2}$ a positively invariant set for the system.

The only fixpoint of the sistem is the origin, therefore by the Poincare-Bendixson theorem it must have a periodic solution in this ring.

Nonexistence of periodic solutions

6. Show that the following system of ODE has no periodic solutions.

$$\begin{cases} x' = \frac{1}{7} + x^2 - yx + y^2 \\ y' = -\frac{1}{5} - y^2 \end{cases} \qquad \textbf{(4p)}$$

Solution

y' is always strictly negative. It implies that $y(t)$ must be monotone function of time. It contradicts to possibility of having periodic solutions that are always bounded.

6. Show that the following system of ODE has no periodic solutions.

$$\begin{cases} x' = x^3 - y^2x + x \\ y' = -0.5y + y^3 + x^4y \end{cases} \qquad \textbf{(4p)}$$

We consider divergence of the right hand side of the system.

$div(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes). According to Bendixson's criterion the system cannot have periodic solutions anywhere in the plane.

1 Lyapunov stability theory (§5.1 in L.R.)

The pioneering work by Lyapunov on stability theory where both the idea of linearization and the idea of test functions were introduced and developed, was his Ph.D thesis published in 1892 and translated to French in 1907.

Consider an autonomous system $x' = f(x)$ with $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ open. We suppose that f is a locally Lipschitz continuous function, so the existence and uniqueness of maximal solutions to I.V.P. are valid.

We repeat for convenience definitions of stable and unstable equilibrium points

(Equilibrium points are considered here at the origin to make it simpler to apply the construction with Lyapunov functions)

Definition

An equilibrium point $0 \in G$ of the system $x' = f(x)$ is said to be stable if for each $\varepsilon > 0$, there is $\delta > 0$ such that for any ξ taken in the ball $B(\delta, 0)=\left\{\xi \in \mathbb{R}^N, \quad |\xi| < \delta\right\}$ the maximal solution $x(t) = \varphi(t, \xi) : I_\xi \to G$ on the maximal interval I_ξ with initial data $x(0) = \xi$ and $0 \in I_\xi$ will stay in the ball $B(\varepsilon, 0)$: $\|\varphi(t, \xi)\| < \varepsilon$ for all $t \in I_\xi \cap \mathbb{R}^+$. In fact $\mathbb{R}^+ \subset I_\xi$ in this case.

Definition

The function $V : U \to \mathbb{R}$, U - open, containing the origin $0 \in U$, is said to be positive definite in U, if $V(0) = 0$ and $V(z) > 0$ for $\forall z \in U$, $z \neq 0$.

Lyapunov's theorem on stability

Theorem. Th.5.2, p.170

Let 0 be an equilibrium point for the system above and there is a positive definite continuously differentiable, $C^1(U)$ function $V : U \to \mathbb{R}$, such that $U \subset G$, $0 \in U$ and $V_f(z) = \nabla V \cdot f(z) \leq 0 \; \forall z \in U$,

then 0 is a stable equilibrium point.

Remark.

A function V with these propertie is usually called the **Lyapunov function of the system.**

Proof.

Take an arbitrary $\varepsilon > 0$ such that $B(\varepsilon, 0) \subset U$. Let $\alpha = \min_{z \in S(\varepsilon,0)} V(z)$ be a minimum of the continuous function V on the boundary of $B(\varepsilon, 0)$, that is the sphere $S(\varepsilon, 0) = \{z : |z| = \varepsilon\}$ and is a compact set (closed and bounded). Then $\alpha > 0$ because $V(z) > 0$ outside the equilibrium point 0.

By continuity of the function V and the fact that$V(0) = 0$ one can find a $0 < \delta < \varepsilon$ such that $\forall z \in B(\delta, 0)$ we have $V(z) < \alpha/2$.

On the other hand for any part of the trajectory $x(t) = \varphi(t, \xi)$, inside U the function $V(\varphi(t, \xi))$ is *non-increasing* because $\frac{d}{dt} V(\varphi(t, \xi)) = (\nabla V \cdot f)(x(t)) \leq 0$. Therefore all trajectories $\varphi(t, \xi)$ with initial points $\xi \in B(\delta, 0)$ satisfy $V(\xi) < \alpha/2$. Therefore $V(\varphi(t, \xi)) < \alpha/2$ and $\varphi(t, \xi)$ cannot reach the sphere $S(\varepsilon, 0)$ where $V(z) \geq \alpha = \min_{z \in S(\varepsilon,0)} V(z)$. Therefore any such trajectory stays within the ball $B(\varepsilon, 0)$ and by the definition, the origin 0 is stable. It implies also that $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ, because the trajectory stays inside a compact set. ■

Remark. The definition of stability and proofs of the theorems are exactly the same if we take an arbitrary equilibrium point x_0 instead of the origin and use balls $B(\varepsilon, x_0)$ around x_0.

Example.

Investigate stability of the equilibrium point in the origin for the following system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

that follows from the second order equation $x'' + (x')^3 + x = 0$.

Try the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$. It is positive definite.

We check the sign of the derivative of V along trajectories of solutions: $V_f(x_1, x_2) = (\nabla V \cdot f)(x_1, x_2) = 2x_1x_2 + 2x_2(-x_1 - x_2^3) = -4x_2^4 \leq 0$.

Point out that $V_f(x_1, x_2) = 0$ along the x_1 axis where $x_2 = 0$, not only in the origin!!!

Example. One dimensional Newton equation. First integrals

Consider a similar example

$$mx'' + g(x) = 0,$$

$xg(x) > 0$, $x \neq 0$, $g(0) = 0$. Suppose that $\int_0^x g(s)ds \to \infty$ as $x \to \infty$.

It describes a spring with non-linear force $-g(x)$. It can be rewritten as a system of ODE's of the first order.

$$\begin{aligned} x_1 &= x \\ x' &= x_1' = x_2, \\ mx_2' &= -g(x_1) \end{aligned}$$

Consider the test function $V(x_1, x_2)$:

$$V(x_1, x_2) = \frac{m}{2}(x_2)^2 + \int_0^{x_1} g(s)ds$$

representing the energy of the system, consisting of two terms: the kinetic energy $\frac{m}{2}(x')^2$ and the potential energy $G(x) = \int_0^x g(s)ds$.

Point out that V is positive definite because of the limitation $xg(x) > 0$, $x \neq 0$.

Consider the derivative V_f of V along trajectories

$$\begin{aligned} (\nabla V \cdot f)(x_1, x_2) &= \left(\frac{\partial}{\partial x_1} V\right) f_1 + \left(\frac{\partial}{\partial x_2} V\right) f_2 \\ &= g(x_1)x_2 + mx_2\left(-\left(\frac{1}{m}\right) g(x_1)\right) = 0 \quad !!!! \end{aligned}$$

The Lyapunov stability theorem implies that the origin is a stable equilibrium point.

We point out also that $(\nabla V \cdot f)(x_1, x_2) = 0$ is zero everywhere and therefore $V(x)$ is constant along trajectories of the system. Such function is called **first integral** of the system.

Definition

Functions that satisfy the relation $(\nabla V \cdot f)(x_1, x_2) = 0$ and are therefore constant on trajectories of the system $x' = f(x)$ are called *first integrals* of the system.

Property of level sets of first integrals.

Level sets of a first integral V have the property that velocities $f(x)$ of the system are tangent or zero on all level sets of V. It implies that these level sets are unions of orbits of the system.

We can express level sets $V(x_1, x_2) = h$ of the first integral V in the example above as

$$x_2 = \pm\sqrt{\frac{2}{m}\left(h - G(x_1)\right)}$$

that is valid in points where the expression under the root is non-negative.

Proposition. 4.54, p. 161

If the first integral V has level sets that are closed curves that do not contain equilibrium points, these curves are orbits of periodic solutiuons.

This idea is almost the only constructive method to calculate periodic orbits for non-linear systems in plane.

Pointing out that $G(x_1) = \int_0^{x_1} g(s)ds$ in the example above is monotone with respect to $|x_1|$, we conclude that those level sets of $V(x_1, x_2)$ that are closed curves and contain no equilibrium points must be orbits of periodic solutions, according to Poincare-Bendixson theorem. It implies in particular that the origin is not asymptotically stable equilibrium point in this example.

Example. Non-linear pendulum without friction.

A particularly interesting example in the form similar to the last one is the equation for pendulum that we considered earlier by using method with linearization.

$$\theta'' = -\frac{g}{l}\sin\theta$$

Let $k^2 = \frac{g}{l}$

$$\begin{aligned}\theta' &= \psi \\ \psi' &= -k^2 \sin\theta\end{aligned}$$

The function $V(\theta, \psi)$

$$\begin{aligned}V(\theta,\psi) &= \frac{\psi^2}{2} + G(\theta) \\ V(\theta,\psi) &= \frac{\psi^2}{2} + k^2(1-\cos\theta)\end{aligned}$$

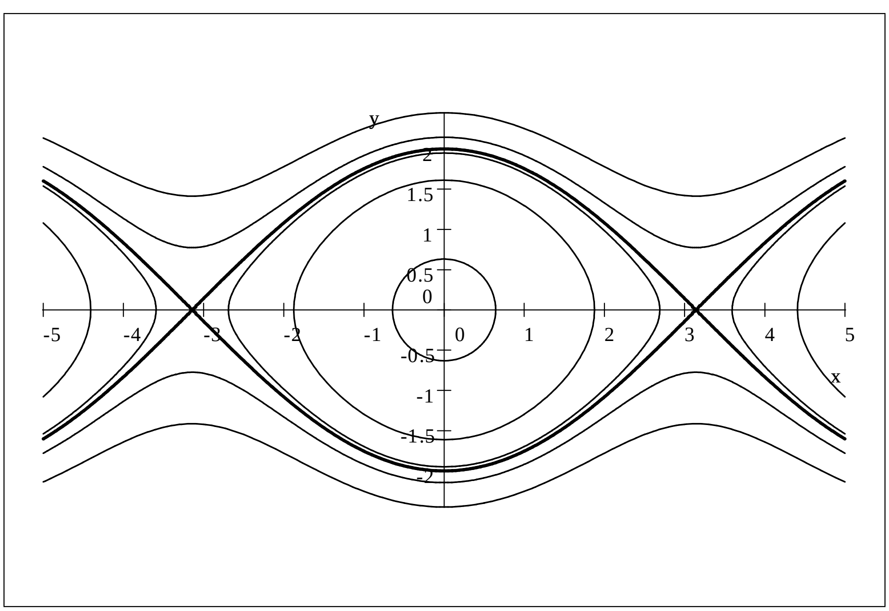
with $G(\theta) = k^2(1-\cos\theta)$ is the first integral of the system describing the pendulum.

Level sets of the function $V(\theta,\psi) = h$ consist of the orbits of the system

$$\psi = \pm\sqrt{2\,(h - G(\theta))}$$

. For $0 < h < 2k^2$ level sets are periodic orbits. For $h > 2k^2$ level sets are wave-looking orbits of trajectories corresponding to the pendulum rotating around the pivot. There are also level sets corresponding to $h = 2k^2$ and consisting of unstable equilibrium points and orbits connecting them and corresponding to trajectories that tend to the upper non-stable equilibrium and not rotating further.

We draw several level sets for the function $\frac{y^2}{2} + 1 - \cos(x) = h$:



Theorem. Asymptotic stability by Lyapunovs functions. Cor. 5.17, p.185,

Let 0 be an equilibrium point for the system above and let V be a positive definite, continuously differentiable function $V : U \to \mathbb{R}$, such that $U \subset G$, U - open, $0 \in U$, and $V_f(z) = \nabla V \cdot f(z) < 0$ (strict inequality outside the origin!) $\forall z \in U$, $z \neq 0$,

then 0 is an asymptotically stable equilibrium point.

Definition. Lyapunov functions satisfying conditions in this theorem are often called *strong Lyapunov functions.*

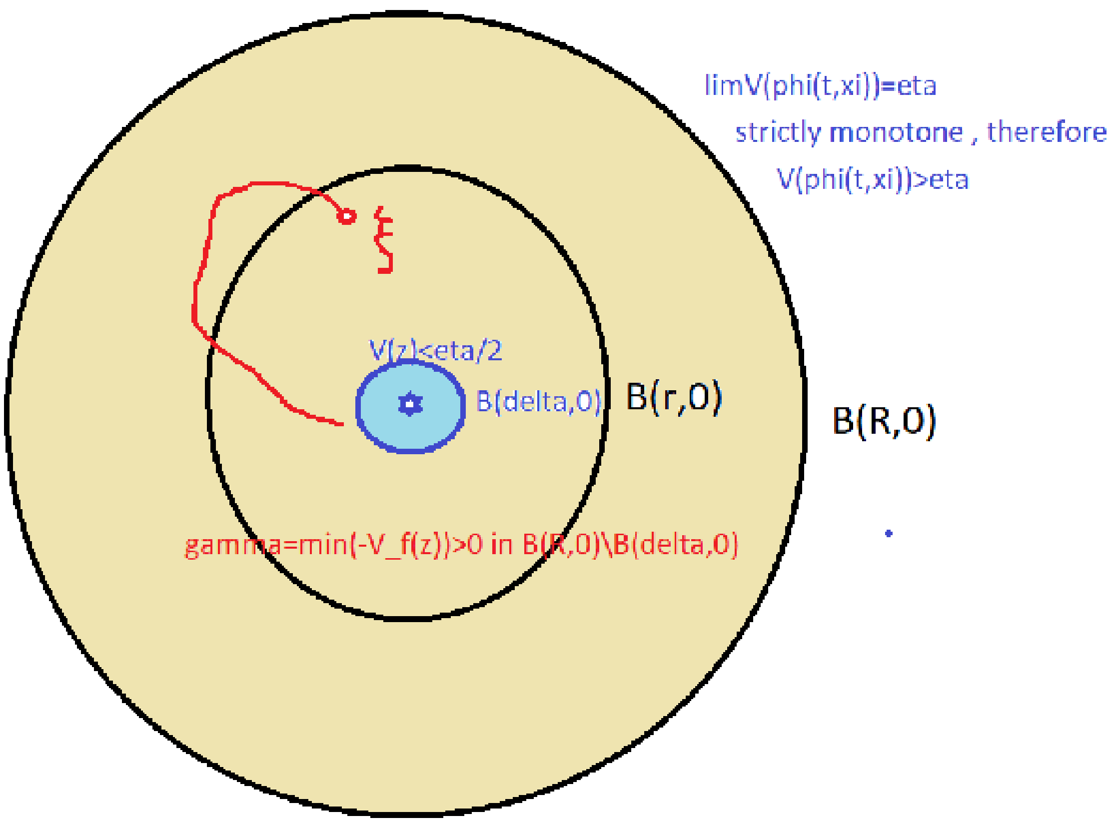
Proof.

In the course book this theorem is considered as a corollary to a more general LaSalle's invariance principle. We give here an independent proof to asymptotic stability. By the Lyapunov's stability theorem the origin is a stable equilibrium and therefore for any ball $B(R,0)$ there is a ball $B(r,0) \subset U$ such that for any $\xi \in B(r,0)$, $\varphi(t,\xi) \in B(R,0)$ for any time $t \in I_\xi$, and $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ.

Therefore we need only to show that the origin is an attractor. Namely we need to show that there is a ball $B(r,0) \subset U$, such that for any $\xi \in B(r,0)$ it follows that $\varphi(t,\xi) \to 0$ as $t \to \infty$.

It suffices to show that $\lim_{t\to\infty} V(\varphi(t,\xi)) = 0$ because V is continuous and is positive outside the origin, where $V(0) = 0$. It will imply that $\varphi(t,\xi) \to 0$ as $t \to \infty$.

It is easy to proof by the following contradiction argument. If $\varphi(t,\xi)$ does not tend to the origin, then there is a sequence of times $t_k \to \infty$ as $k \to \infty$ such that $\|\varphi(t_k,\xi)\| > \varepsilon > 0$. It implies that $V(\varphi(t_k,\xi)) > q > 0$ for some positive q. But it is not compatible with supposition that $\lim_{t\to\infty} V(\varphi(t,\xi) = 0.\square$



Now we continue proving $\lim_{t\to\infty} V(\varphi(t,\xi) = 0$. By conditions of the theorem $\frac{d}{dt}V(\varphi(t,\xi) < 0$, therefore

$0 \leq V(\varphi(t, \xi)$ is a monotone strictly decreasing function of t and must have a limit

$$\lim_{t \to \infty} V(\varphi(t, \xi) = \eta, \quad t \to \infty.$$

Suppose that this limit is not zero: $\eta > 0$. Then $V(\varphi(t, \xi) > \eta > 0$ for all $t \geq 0$ because $V(\varphi(t, \xi)$ is stricly monotone decreasing.

Now we like to find a ball $B(\delta, 0)$, $\delta < r$ around the origin so small that the trajectory $\varphi(t, \xi)$ cannot reach it. The idea is that outside this ball (where our trajectory $\varphi(t, \xi)$ is situated) the decreasing rate for $V(\varphi(t, \xi)$ along the trajectory is never close to zero. This fact would lead us to a contradiction with out supposition.

Continuity of V and the fact that $V(0) = 0$ imply that there is a ball $B(\delta, 0)$, $\delta < r$ such that $0 \leq V(z) < \eta/2$ for all $z \in B(\delta, 0)$. Hence $\varphi(t, \xi)$ cannot reach it: $\|\varphi(t, \xi)\| \geq \delta$ for all $t \geq 0$, because $V(\varphi(t, \xi) > \eta > 0$ for all $t \geq 0$ by our supposition that $V(\varphi(t, \xi)) \searrow \eta$ as $t \to \infty$.

Now we will estimate the smalles rate of decrease for $V(\varphi(t, \xi)$ that follows from our conclusions. Consider the closed spherical slice $S = \{z : \delta \leq \|z\| \leq R\}$ where the trajectory $\varphi(t, \xi)$ is situated, and point out that $\gamma = \min_{z \in S} (-V_f(z)) > 0$ exists because S is compact and V_f is continuous.

$\gamma > 0$ by the condition of the theorem that $V_f < 0$ outside the origin. Therefore

$$-\frac{d}{dt} V(\varphi(t, \xi)) \geq \gamma = \min_{z \in S} (-V_f(z))$$

and

$$\frac{d}{dt} V(\varphi(t, \xi)) \leq -\gamma$$

By integration from 0 to t we arrive to

$$V(\varphi(t, \xi)) - V(\xi) \leq -\gamma t \to -\infty$$

as $t \to \infty$ that contradicts to the supposition that $V(z) \geq 0$.

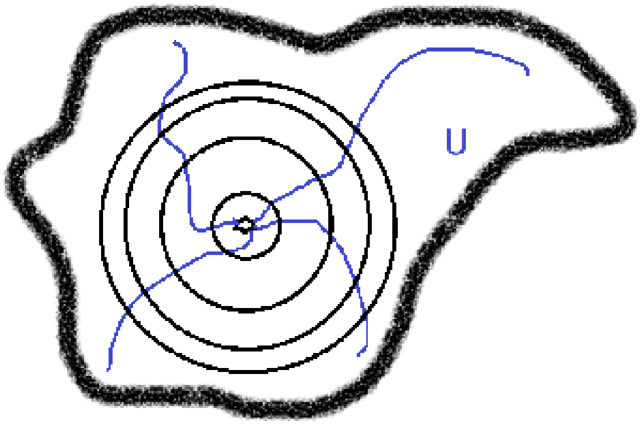
It implies that our supposition that $\lim_{t \to \infty} V(\varphi(t, \xi) = \eta > 0$ was wrong and that $\lim_{t \to \infty} V(\varphi(t, \xi) = 0$.As we pointed out at the beginning of the proof, the last fact implies, that $\lim_{t \to \infty} \varphi(t, \xi) = 0$ and therefore the origin is an attractor and is an asymptotically stable equilibrium point.■

Remark.

This theorem on asymptotic stability has a (very difficult!) inversion (proven in 1949,1956) by José Luis Massera, Uruguay, stating that for any system with an asymptotically stable equilibrium point, there is a "strong" Lyapunov function V such that $V_f(z) < 0$ in a neighborhood of this equilibrium point (outside the point $z = 0$ itself).

Definition. Region of attraction for an asymptotically stable equilibrium point.

A domain $U \subset G$ is called the region of attraction for an asymptotically stable equilibrium point $x_* \in U$ if for any $\xi \in U$, the maximal existence interval I_ξ of the the solution $x(t) = \varphi(t, \xi)$ contains $\mathbb{R}^+ \subset I_\xi$ and $\varphi(t, \xi) \to x_*$ as $t \to \infty$.



Example. Consider the system of equations

$$\begin{aligned} x' &= -x + 2xy^2 \\ y' &= -(1-x^2)y^3 \end{aligned}$$

Investigate stability of the equilibrium in the origin and find possible region of attraction.

Point out that for the right hand side in the equation the Jacoby matrix J in the origin is degenerate $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ and the linearization of the system does not give any information about stability of the equilibrium in the origin.

Consider the simplest test function $V(x, y) = x^2 + y^2$.

$$\begin{aligned} V_f(x, y) &= (\nabla V \cdot f)(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x + 2xy^2 \\ -(1-x^2)y^3 \end{bmatrix} \\ &= 2x\left(-x + 2xy^2\right) + 2y\left(-(1-x^2)y^3\right) = 4x^2y^2 - 2y^4 - 2x^2 + 2x^2y^4 \\ &= -2x^2(1-2y^2) - 2y^4(1-x^2) \end{aligned}$$

$V_f(x, y) < 0$ in the rectangle $(-1, 1) \times \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$, $(x, y) \neq 0$. Therefore the origin is the asymptotically stable equilibrium with the region of attraction - the largest circle around the origin that fits into this rectangle: $x^2 + y^2 < 1/2$.

This region of attraction is just one we could find using this particular Lyapunov function, it can exist a larger region of attraction.

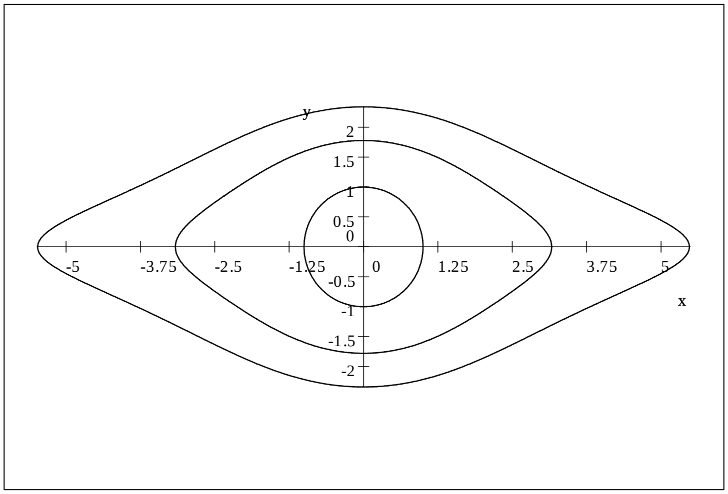
Example. Consider the system of equations

$$\begin{aligned} y_1' &= -y_1^3 - 2y_1y_2^2 \\ y_2' &= y_1^2y_2 - y_2^3 \end{aligned}$$

Investigate stability of the equilibrium point in the origin finding a suitable Lyapunov function. Consider the following test function:

$$V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4$$

The test function V is positive definite. We draw several level sets for $V(x, y) = x^2 + x^2y^2 + y^4 = h$, for $h = 1, 20, 30$.



We choose the form of the test function in such a way that on the level set curves of this function velocities $f(y_1, y_2)$ point inward: $(\nabla V \cdot f)(y_1, y_2) < 0$. We have chosen the term y_2^4 having $\frac{\partial}{\partial y_2}(y_2^4) = 4y_2^3$ that after multiplication with the term $-y_2^3$ from f_2 gives a "good" negative term $-4y_2^6$. Similarly $\frac{\partial}{\partial y_1}(y_1^2) = 2y_1$ multyplied by the term $-y_1^3$ from f_1 gives a good negative term $-2y_1^4$. The tricky step is to play with "bad" indefinite mixed products in such a way that they (in the best case!) give no terms in $(\nabla V \cdot f)(y_1, y_2)$ with indefinite sign.

$$\begin{aligned}
V_f(y_1, y_2) &= (\nabla V \cdot f)(y_1, y_2) = \\
&\quad \left(2y_1 + 2y_1y_2^2\right)\left(-y_1^3 - 2y_1y_2^2\right) + \\
&\quad + \left(2y_1^2y_2 + 4y_2^3\right)\left(y_1^2y_2 - y_2^3\right) \\
&= -2y_1^4 - 4y_1^2y_2^2 - 2y_1^4y_2^2 - 4y_1^2y_2^4 + 2y_1^2y_2^4 - 4y_2^6 + 2y_1^4y_2^2 \\
&= -2y_1^4 - 4y_2^6 - 4y_1^2y_2^2 - 2y_1^2y_2^4 \\
&= \left(-y_1^4 - 2y_2^6 - 2y_1^2y_2^2 - y_1^2y_2^4\right)2 \\
&\leq \left(-y_1^4 - 2y_2^6\right)2 < 0, \quad (y_1, y_2) \neq (0, 0)
\end{aligned}$$

Therefore according to the last theorem, the origin $(0, 0)$ is an asymptotically stable equilibrium point. The test function tends to inifinity with $\|(y_1, y_2)\| \to \infty$. It implies that the equilibriuh has the whole plane $\mathbb{R}^2$ as the region of attraction. All trajectories $\varphi(t, \xi)$ tend to the origin with $t \to \infty$: $\varphi(t, \xi) \underset{t\to\infty}{\longrightarrow} (0, 0)$.

Remark

One can arrive to indefinite terms after calculation of $V_{f.}$ .It is still not the end of hope. One can check that these indefinite terms are not large and might be compensated by negative definite terms in the expression for $V_{f.}$. For example the expression $-x^2 - y^2 + xy < 0$ for $(x, y) \neq (0, 0)$ because $2|xy| \leq x^2 + y^2$.

One can also use known criteria for positive and negative definite quadratic forms in such problems.

Lyapunov's theorem on instability

We give here a slightly weaker variant of the instability theorem comparing with one in the book. An advantage of the variant here is that it suggests a more constructive proof.

Students are free to choose any of these two variants at the examination.

Definition.

An equilibrium point $0 \in G$ of the system is said to be unstable if it is not stable.

Explicit version of the same definition.

There is a ball $B(R, 0) \subset G$ such that for any $\delta > 0$ there is a point $\xi \in B(\delta, 0)$ such that for the trajectory $\varphi(t, \xi)$ starting in ξ there is time $t_* \in I_\xi$ such that $\varphi(t, \xi) \notin B(R, 0)$.

Another reformulation of this definition is possible.

Another explicit version of the same definition

There is a ball $B(R, 0) \subset G$ and a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = 0$ such that for each maximal solution $\varphi(t, \xi)$ with initial data $\xi = x_n$ there is time $t_* \in I_\xi$ such that $\varphi(t, \xi) \notin B(R, 0)$.

Theorem. On a criterium of instability of an equilibrium using test functions.

Let the origin 0 - be the equilibrium point of the system $x' = f(x)$. Assume that there is a neighbourhood $U \subset G$, $0 \in U$ and a continuously differentible $C^1(U)$ function $V : U \to \mathbb{R}$ satisfying the following hypotheses.

1) $V_f(z) = \nabla V \cdot f(z) > 0$ for every $z \in U$, $z \neq 0$

2) For every $\delta > 0$ there exists $z \in U$ with $\|z\| < \delta$ and $V(z) > 0$

3) $V(0) = 0$.

Then the origin 0 is an unstable equilibrium.

Remark. The Theorem Th. 5.7, p. 174 formulted in the book is stronger. It has the same conclusion with the condition 1) changed to a weaker one:

1) $V_f(z) = \nabla V \cdot f(z) > 0$ for every point $z \in U$, where $V(z) > 0$, and 3) is not required.

Proof of the weaker variant of the Theorem

The idea of the proof is to show that any trajectory starting from a point ξ arbitrarily close to 0 where $V(\xi) > 0$ will leave a fixed ball $B(R/2, 0)$ such that a larger ball $\overline{B(R, 0)} \subset U$.

We point out that for any part of the trajectory $\varphi(t, \xi)$ of the maximal solution in U the function $V(\varphi(t, \xi))$ is monotone increasing because $\frac{d}{dt}V(\varphi(t, \xi)) = V_f(\varphi(t, \xi)) > 0$.

It means that $\varphi(t, \xi)$ stays outside the origin because $V_f(z)$ is continuous and $V_f(0) = 0$. It in turn means that $(\nabla V \cdot f)(\varphi(t, \xi)) = \frac{d}{dt}V(\varphi(t, \xi) \geq K > 0$ for all $t \in I_\xi \cap \mathbb{R}^+$.

To prove this inequality one can carry out a more formal argument that follows.

Let $\xi \in B(R, 0)$ be an arbitrary point where $V(\xi) > 0$. V is a continuous function and $V(0) = 0$. It implies that there is $0 < \varepsilon < R/2$ such that $V(z) < V(\xi)/2$ for $\|z\| < \varepsilon$.

Therefore the trajectory $\varphi(t, \xi)$ must stay outside the ball $B(\varepsilon, 0)$ for all $t \in I_\xi \cap R^+$.

The function $(\nabla V \cdot f)(z)$ is continuous in U and must attaint its minumum $K = \min_{z\in\overline{B(R,0)}\setminus B(\varepsilon,0)} (\nabla V \cdot f)(z)$ on the compact set $\overline{B(R, 0)} \setminus B(\varepsilon, 0)$ that is a slab between two spheres. The number K is positive $K > 0$ because $(\nabla V \cdot f)(z) > 0$ for $z \in U$ outside the origin.

Therefore

$$(\nabla V \cdot f)(\varphi(t,\xi)) = \frac{d}{dt} V(\varphi(t,\xi)) \geq K > 0, \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

and by the integration of the left and right hand side over $[0,t]$ we get

$$V(\varphi(t,\xi) \geq Kt + V(\xi), \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

There are two possibilities depending on if $I_\xi \cap \mathbb{R}^+$ is a bounded interval or $\mathbb{R}^+ \subset I_\xi$. In the first case the trajectory $\varphi(t,\xi)$ must leave any compact in G in particlular the ball $\overline{B(R,0)}$. In the second case having possibility to take t arbitrary large in the inequality $V(\varphi(t,\xi) \geq Kt + V(\xi)$ leads to conclusion, that for some time $t_* > 0$ large enough $V(\varphi(t_*,\xi))$ will become larger than $\max_{z\in\overline{B(R/2,0)}} V(z)$ - the maximum of $V(z)$ over the half ball $\overline{B(R/2,0)}$. It means that the point $\varphi(t_*,\xi)$ of the trajectory must be outside the ball $\overline{B(R/2,0)}$ at such time t_*.

Therefore according to the definition, the origin 0 is an unstable equilibrium, because there are trajectories starting arbitrarily close to the equilibrium 0, such that they move outside the ball $\overline{B(R/2,0)} \subset G$ at some time t_*.■

Remark. If we suppose in the formulation of the theorem above that $V_f(z) > 0$ for all $z \in U$, $z \neq 0$, then the origin is a repeller, meaning that for some ball $B(R,0)$ around the origin, any solution $x(t) = \varphi(t,\xi)$ with $\xi \in B(R,0)$ will leave this ball in finite time.

Example.

Consider the system

$$\begin{aligned} x' &= x^3 + yx^2 \\ y' &= -y + x^3 \end{aligned}$$

Show that the origin is unstable equilibrium by using the test function $V(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$.

Point out that the linearization has matrix $J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ that is degenerate. Therefore the Grobman - Hartman theorem cannot be applied.

$$\begin{aligned} V_f(x,y) &= \begin{bmatrix} x \\ -y \end{bmatrix} \cdot \begin{bmatrix} x^3 + yx^2 \\ -y + x^3 \end{bmatrix} = \\ y^2 + x^4 - yx^3 + x^3 y &= y^2 + x^4 > 0 \end{aligned}$$

$V(x,y) > 0$ on the x-axis, arbitrary close to the origin. There is a ball $B(0,R)$ around the origin such that trajectories starting on the x-axis arbitrary close to the origin will leave it in finite time by the Lyapunov instability theorem.

1 General properties of ω -limit sets and La Salle's invariance principle and it's applications to asymptotic stability §5.2

Example. An elementary introduction to LaSalle's invariance principle.

We like to investigate stability of equilibrium poin tin the origin for the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

Using the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$ we observe that it is a Lyapunov function for the system:

$$V_f(x_1, x_2) = \nabla V \cdot f(x_1, x_2) = 2x_1x_2 - 2x_1x_2 - 2x_2^4 = -2x_2^4 \leq 0$$

and the origin is a stable equilibrium point. On the other hand V is not a strong Lyapunov function, because $V_f(x_1, x_2) = 0$ not only in the origin, but on the whole x_1 - axis where x_2 is zero.

On the other hand considering the vector field of velocities of this system on the x_1 - axis, we observe that they are crossing the x_1 - axis (even are orthogonal to it in this particular example) in all points except the origin. It means that all trajectories of the system cross and immediately leave the x_1 - axis that is the line where $V_f(x_1, x_2) = 0$ (the Lyapunov function is not strong). This observation shows that in fact the Lyapunov function $V(\varphi(t, \xi))$ is strictly monotone along trajectories $\varphi(t, \xi)$ everywhere except discret time moments, when $\varphi(t, \xi)$ crosses the x_1 - axis. More explicitely in polar coordinates r and θ:

$$\left(r^2\right)' = -2r^4 \sin^4 \theta$$

We can therefore conclude that $V(\varphi(t, \xi)) \searrow 0$ as $t \to \infty$ and therefore, the origin is asymptotically stable equilibrium of this system of equations.

One can also get a more explicit picture of this dynamics by looking on the equation for the polar angle θ:

$$\begin{aligned}
\left(\frac{x_2}{x_1}\right)' &= (\tan(\theta))' = \frac{\theta'}{\cos^2(\theta)} \\
\frac{x_2' x_1 - x_1' x_2}{x_1^2} &= \frac{\left(-x_1 - x_2^3\right) x_1 - (x_2)\, x_2}{x_1^2} \\
&= \frac{\left(-x_1^2 - x_2^2 - x_1 x_2^3\right)}{x_1^2} = \frac{-r^2 - \cos\theta \sin^3\theta\, r^4}{r^2 \cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
\theta' &= -1 - \cos\theta \sin^3\theta\, r^2 = -1 - \frac{\left(\sin 2\theta \sin^2\theta\right)\, r^2}{2} \\
&= -1 - \frac{\sin 2\theta (1 - \cos 2\theta) r^2}{4} < 0, \quad r < 2
\end{aligned}$$

We see that for $r < 2$ the trajectories tend to the origin going (non-uniformly) as spirals clockwise around the origin.

This example demonstrates the main idea with applications of the LaSalles invariance principle to asymptotic stability of equilibrium points.

Proposition. Simple version of applying LaSalle's invariance principle for asymptotic stability of equilibrium points by using "weak" Lyapunof functions.

(The complete version of LaSalle's invariance principle is Theorem 5.15. p. 183 that is considered a bit later)

We find a simple "weak" Lyapunov function $V_f(z) \leq 0$ for $z \in U$ in the domain $U \subset G$, $0 \in U$. This fact implies stability of the equilibrium. Then we check what happens on the set $V_f^{-1}(0)$ where $V_f(z) = 0$. If the set $V_f^{-1}(0)$ contains no other orbits except the equilibrium point, this equilibrium point in the origin must be asymptotically stable.

Any trajectory starting in W will have a positive orbit with compact closure. We need this property for applying LaSalle's invariant principle describing ω- limit sets for positive orbits of solutions to ODEs.

Exercise.

Show that all trajectories of the system

$$\begin{aligned} x' &= y \\ y' &= -x - (1 - x^2)y \end{aligned}$$

that go through points in the domain $\left\| [x, y]^T \right\| < 1$, tend to the origin. Or by other words, show that the origin is an asymptotically stable equilibrium and that the circle $\left\| [x, y]^T \right\| < 1$ is it's domain of attraction.

Consider $V(x, y) = x^2 + y^2$.

$$\begin{aligned} V_f(x, y) &= 2xy - 2xy - (1 - x^2)y^2 = -(1 - x^2)y^2 \leq 0 \\ V_f^{-1}(0) &= \{(x, y) : y = 0\} \end{aligned}$$

The only invariant set is $\{0\}$, therefore for trajectories starting in $\left\| [x, y]^T \right\| < 1$ the origin is and attractor and it is asymptotically stable with $\left\| [x, y]^T \right\| < 1$ being the domain of attraction.

More general formulation and a proof of the LaSalle's invariance principle use some general properties of transition mappings, and ω - limit sets. We collect them here and give some comments about their proofs.

We consider I.V.P. and corresponding transition map $\varphi(t,\xi)$ for the system

$$\begin{aligned} x' &= f(x) \\ x(0) &= \xi \end{aligned}$$

with $f : G \to \mathbb{R}^n$, G - open, $G \subset \mathbb{R}^n$, f is locally Lipschitz, $\xi \in G$.

1.1 Main theorem on the properties of limit sets.

The next theorem on the properties of ω - limit sets collects properties of ω - limit sets valid for systems of any dimension, in contrast with the Poincare - Bendixson theorem and it's generalization, that gives a

description of ω - limit sets only for systems in plane, or on 2-dimensional manifolds.

Main theorem about properties of limit sets. Theorem 4.38, p.143

We keep the same limitations and notations for the autonomous system as above.

Let $\xi \in G$. Let the closure of the postive semi-orbit $O^+(\xi)$ be compact and contained in G,

Then $\mathbb{R}_+ \subset I_\xi$ and the ω - limit set $\Omega(\xi) \subset G$ is

1) non-empty

2) compact (bounded and closed)

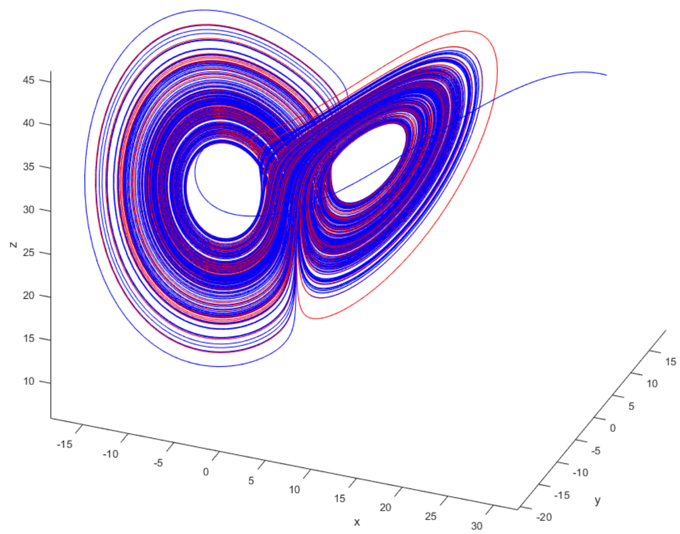
3) connected

4) invariant (both positively and negatively) under the local flow $\varphi(t,\xi)$ generated by the ODE: namely for any ω - limit point $\eta \in \Omega(\xi)$, the maximal interval $I_\eta = \mathbb{R}$ for initial data in η, and $\varphi(t,\eta) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.

5) $\varphi(t,\xi)$ approaches $\Omega(\xi)$ as $t \to \infty$:

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\xi),\, \Omega(\xi)) = 0$$

Example. The Lorentz equation. Trajectory - blue, limit set $\Omega(\xi)$ - red



$$
\begin{aligned}
x' &= -\sigma(x-y) \\
y' &= rx - y - xz \\
z' &= xy - bz
\end{aligned}
$$

A trajectory for $\sigma = 10$, $r = 28$, $b = 8/7$.

Remark

The most interesting statement in the theorem is statement 4). It means that ω - limit sets consist of orbits of solutions to the system. Taking a starting point η on the limit set $\Omega(\xi)$ we get a trajectory $\varphi(t,\eta)$ that stays within this set $\Omega(\xi)$ infinitely long both in the future and in the past.

A simple tool to satisfy conditions in this theorem is to find a compact positively invariant set for the system that contains the point ξ. It can be done using one of two methods discussed earlier.

Proofs of statements in the Theorem **4.38,** are based on: general properties of compact sets for 1) ,2), simple contradiction arguments and the definition of limit sets for 3) and the translation property of the transition mapping $\varphi(t,\xi)$, together with continuity of $\varphi(t,\xi)$ for 4), and a contradiction argument togehter with the definition of ω - limit sets for 5).

We will give a **proof to 4).**

Let η be a limit point for $\varphi(t,\xi)$: $\eta \in \Omega(\xi)$. By definition there is a sequence of times $\{t_n\}$, $t \to \infty$ such that $\varphi(t_n,\xi) \to \eta$.

Consider the trajectory $\varphi(t,\eta)$ starting at η. Denote by I_η corresponding maximal interval and consider an arbitrary $t \in I_\eta$, belonging o the maximal interval I_η. We like to show that $\varphi(t,\eta) \in \Omega(\xi)$ that a trajectory starting in a limit point stays in the limit point forever in future and in the past.

For n large enough $t + t_n \overset{def}{=} s_n \in I_\xi$ - belongs to the maximal interval I_ξ of the solution $\varphi(t,\xi)$ for n large enough.

We apply the group relation for φ (similar to Chapmen-Kolmogorov relation for linear systems)

$$\varphi\left(s_n,\xi\right) = \varphi\left(t+t_n,\xi\right) = \varphi\left(t, \varphi(t_n,\xi)\right)$$

It is possible since the domain D of $\varphi(.,.)$ is open, $(t,\eta) \in D$ therefore there is a ball B around (t,η) such that $(t,\varphi(t_n,\xi)) \in B \subset D$ for n large enough because $\varphi(t_n,\xi) \to \eta$. Therefore $t \in I_{\varphi(t_n,\xi)}$.

By continuity of φ it follows:

$$\varphi\left(s_n,\xi\right) = \varphi\left(t+t_n,\xi\right) = \varphi\left(t, \overset{\lim = \eta}{\varphi(t_n,\xi)}\right) \to \varphi(t,\eta), \quad n \to \infty$$

It means that $\varphi(t,\eta)$ is a limit point for $\varphi\left(t,\xi\right)$ for any $t \in I_\eta$.■

LaSalle's invariance principle

We formulate now LaSalle's invariance principle that generalizes ideas that we discussed in the introductory example and gives a handy instrument for localizing ω - limit sets of non-linear systems in arbitrary dimension.

Theorem 5.12, p.180

Assume that f is locally Lipschitz $f : G \to \mathbb{R}^n$ as before and let $\varphi(t, \xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0$. for all $z \in U$. If $\xi \in U$ is such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U,

i) then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and

ii) as $t \to \infty$, $\varphi(t, \xi)$ approaches the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

Proof given in the solution of Exercise 5.9, on p. 312.

Set $x(t) = \varphi(t, \xi)$. By continuity of V and compactness of the closure $cl(O^+(\xi))$, V is bounded on $O^+(\xi)$ and therefore the function $V(\varphi(t, \xi))$ is bounded.

• Since

$$\frac{d}{dt}\left(V\left(x(t)\right)\right) = V_f(x(t)) \leq 0$$

for all $t \in \mathbb{R}_+$, $V(x(t))$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t))$ of the non-increasing function $V(x(t))$ must exist and is finite. We denote it by λ:

$$\lim_{t\to\infty} V(x(t)) = \lambda$$

• Take an arbitrary an arbitrary point $z \in \Omega(\xi)$ in the ω - limit set $\Omega(\xi)$. Then by the definition of ω - limit sets, there is a sequence $\{t_n\}$ in $\mathbb{R}_+$

such that $\lim_{n\to\infty} t_n = \infty$ and

$$x(t_n) = \varphi(t_n, \xi) = x(t_n) \longrightarrow z, \quad n \to \infty$$

We apply V to the left and right hand side in this limit calulation.

For any continuous function F and any convergent sequence $\{g_n\}$it is valid that

$$F(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} (F(g_n)$$

• By the continuity of V it follows that $V(z) = \lim_{n\to\infty} V(x(t_n))$ and $\lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t))$. Therefore

$$V(z) = \lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t)) = \lambda.$$

This key point in the proof $\Big(!!!\Big)$ implies that for all z in the omega limit set $\Omega(\xi)$ the test function V has the same value:

$$V(z) = \lambda, \quad \forall z \in \Omega(\xi) \tag{1}$$

• By the invariance of $\Omega(\xi)$ with respect to $\varphi(t, .)$, if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.**(it is why the theorem is called the invariance ˜principle!!!)**

Therefore $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$ is a constant function of time t. A constant function must have zero derivative:

$$\frac{d}{dt} V(\varphi(t, z)) = V_f(\varphi(t, z)) = 0$$

for all $t \in \mathbb{R}$. Since $\varphi(0, z) = z$ and z is an arbitrary point in $\Omega(\xi)$ it follows that

$$V_f(z) = \left. \frac{d}{dt} V(\varphi(t, z)) \right|_{t=0} = 0, \quad \forall z \in \Omega(\xi) \tag{2}$$

and therefore $\Omega(\xi) \subset V_f^{-1}(0)$.

• The statement of the theorem follows now from the Main theorem about limit sets (Theorem 4.38), that states: $\Omega(\xi)$ is an invariant set under the action of $\varphi(t,.)$, and $\varphi(t,\xi)$ apporachs $\Omega(\xi)$ as $t \to \infty$.$\blacksquare$

Comment. It can be tempting to simplify the proof by concluding (1) from the fact that $(\nabla V)(z) = 0$ from all $z \in \Omega(\xi)$ which would imply (2).

However this conclusion is not valid, because the set $\Omega(\xi)$ is not open and therefore $V(z) = \lambda, \quad \forall z \in \Omega(\xi)$ does not imply $V_f(z) = 0, \quad \forall \in \Omega(\xi)$.

The invalidity of this conclusion is illustrated by the following simple example: $V(z) = \|z\|$, $\Omega(\xi) = \left\{z \in \mathbb{R}^N : \|z\| = 1\right\}$, then $v(z) = 1$ for all $z \in \Omega(\xi)$, but$(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.

The following theorem follows rather directly from LaSalle's invariance principle and gives a practical criterium for asympototically stable equilibrium points using "weak" Lyapunov's functions.

Theorem 5.15. p. 183.

Let U be an open domain $U \subset G$, such that $0 \in U$ and a continuously differentiable function $V : U \to \mathbb{R}^n$ such that

$$V(0) = 0, \quad V(z) > 0, \forall z \in U \setminus \{0\}, \quad V_f(z) \leq 0, \forall z \in U \setminus \{0\}$$

and $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$, then 0 is an asymptotically stable equilibrium.$\square$

Proof follows from LaSalle's invariance principle and is a good exercise.

Theorem 5.22, p. 188. On global asymtotic stability

Assume that $G = \mathbb{R}^n$. Let the hypothesis of the Theorem 5.15 hold with $U = G = \mathbb{R}^n$.

Namely for a continuously differential function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for all $z \in U\setminus\{0\}$, $V_f(z) \leq 0$ for all $z \in U\setminus\{0\}$, the origin $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$.

If in addition the Lyapunov function V is radially unbounded:

$$V(z) \to \infty, \quad \|z\| \to \infty$$

then the origin 0 is a globally stable equilibrium that means that all solutions $\|\varphi(t,\xi)\| \to 0$, as $t \to \infty$.

Exercise 5.17

The aim of this exercise is to show that the condition of radial unboundedness in Theorem 5.22 is essential.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) = \begin{cases} (-z_1, z_2) & \text{if } z_1^2 z_2^2 \geq 1 \\ (-z_1, 2z_1^2 z_2^3 - z_2) & \text{if } z_1^2 z_2^2 < 1\,. \end{cases}$$

Define $V : \mathbb{R}^2 \to \mathbb{R}$ by

$$V(z) = V(z_1, z_2) = z_1^2 + \frac{z_2^2}{1 + z_2^2}\,.$$

(a) Show that the equilibrium 0 of (5.1) is asymptotically stable.

(b) Show that the equilibrium 0 is *not* globally asymptotically stable.

(c) Show that V is not radially unbounded.

Examples of using La Salle's principle. Investigate stability of equilibrium points in the origin.

Example.

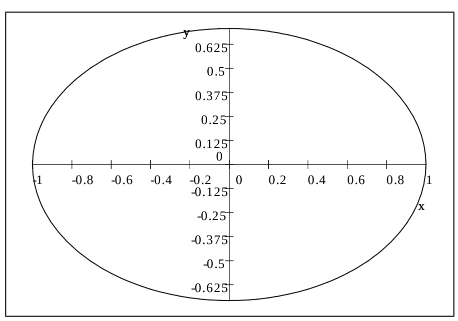
Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x - (1-x^2)y \end{cases}$.

Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction. **(4p)**

Solution.

We try the test function $V(x,y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories:

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x,y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x,y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. LaSalle's invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x,y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$:



The next theorem gives a simple criterion for having the whole space as the domain of attraction for an asymptotically stable equilibrium point.

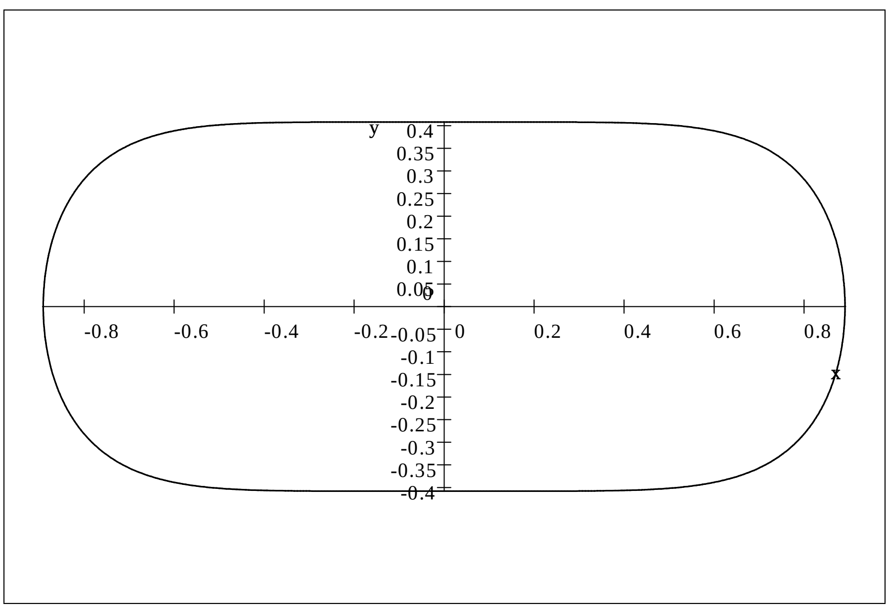
Example. Investigate stability of the equilibrium point in the origin.

$$
\begin{aligned}
x' &= -y - x^3 \\
y' &= x^5
\end{aligned}
$$

We try our simplest choice of the Lyapunov function: $V(x, y) = x^2 + y^2$ and arrive to

$$V_f(x, y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x, y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x, y) = x^6 + \alpha y^2$ where $\partial V/\partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$.The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x, y) = 6x^5(-y - x^3) + 2\alpha y x^5 = -6x^5 y + 2\alpha x^5 y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice $\alpha = 3$ cancels them:

$$V_f(x, y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x, y) = 0$ on the whole $y-$axis that in our "general" theory is denoted by $V_f^{-1}(0)$.We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is $y-$axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \to \infty$ as $\|z\| \to \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.$\blacksquare$

1. **Lyapunov function and stability of stationary points.**

   Consider the system of ODE: $$\begin{aligned} x' &= y \\ y' &= -y + y^3 - x^5 \end{aligned}$$

   Write the definition of asymptotically stable stationary point.

   Find Lyapunov function $V(x, y)$ for the given equation and show that stationary point in the origin is asymptotically stable. **(2p)**

   Hint. Use $V(x, y)$ in the form form $V(x, y) = ax^6 + by^2$ and choose the parameters a, b so that $V(x, y)$ would be a Lyapunov function.

   Solution A stationary point x_* is asymptotically stable if there is a neighborhood N of this point such that for any initial point $x(o) \in N$ the corresponding trajectory $x(t) \to x_*$ for $t \to +\infty$.

   $V'(x, y) = \frac{\partial V}{\partial x} y + \frac{\partial V}{\partial x}(-y + y^3 - x^5) = 6ax^5y - 2by^2 + 2by^4 - 2bx^5y$

   Taking $6a = 2b$ and $a = 1$ makes $V'(x, y) = 6y^2(1 - y^2) < 0$ for $|y| < 1$ and $y \neq 0$.

   For $y = 0$ $y' = -x^5$ and therefore $y' = 0$ only for $x = 0$. It implies that the origin iz asymptotically stable.

1. **Lyapunovs funktioner och stabilitet hos stationära punkter.**

   a) Formulera ett criterium for asymptotiskt stabil stationär punkt till en ODE med hjälp av en Lyapunovs funktion som inte är stark Lyapunovs funktion. **(2p)**

   b) Betrakta ekvationssystemet: $$\begin{aligned} x' &= -y/3 - x(3x^2 + y^2) \\ y' &= x - y(3x^2 + y^2) \end{aligned}$$

   Hitta en stark Ljapunovs funktion $V(x, y)$ för att visa att stationära punkten i origo är asymptotiskt stabil. **(2p)**

   Tips. Använd $V(x, y)$ på formen $V(x, y) = ax^2 + y^2$ och välj parametern a så att $V(x, y)$ blir en stark Ljapunovs funktion.

   $V(x, y)$ kan väljas som $V(x, y) = 3x^2 + y^2$.

2. **Lyapunovs functions and stability of stationary points.** Formulate the criterion for asymptotic stability of a stationary point of an ODE using only a weak Lyapunov function.

   Consider the system of equations: $\begin{cases} x' = -x + y^2 \\ y' = -xy - x^2 \end{cases}$

   Show that $V(x, y) = x^2 + y^2$ is a weak Lyapunov function and decide if the stationary point at the origin is asymptotically stable. **(4p)**

$\frac{d}{dt}(V) = \nabla V \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = 2x\left(-x + y^2\right) + 2y\left(-xy - x^2\right) = -2x^2 - 2x^2 y = -2x^2(1 + y)$;

For $|y| < 1$ $\frac{d}{dt}(V) \le 0$. It implies that the origin is a stable stationary point. On the line $x = 0$ $\frac{d}{dt}(V) = 0$ so V is only a weak Lyapunovs funktion. But we observe that on the line $x = 0$ the velocity is not zero $-x + y^2 \neq 0$ exept the stationary point itself. It implies that the origin is an asymptoticaly stable stationary point.

2. **Ljapunovs functions and stability of stationary points.**

   Consider the system of equations: $\begin{cases} x' = -x + 2xy^2 \\ y' = -(1 - x^2)y^3 \end{cases}$

   Show that the origin is an stable fixed point. **(4p)**

   Solution

   $V(x, y) = x^2 + y^2$

   $V' = 2x\left(-x + 2xy^2\right) + 2y\left(-(1 - x^2)y^3\right) = -2x^2 + 4x^2y^2 - 2y^4\left(1 - x^2\right) =$
   $-2x^2\left(1 - 2y^2\right) - 2y^4\left(1 - x^2\right)$

   We see that $V' < 0$ for $|x| < 1$ and $|y| < \sqrt{1/2}$

Exercise.
Consider the system of equations

$$\begin{aligned} x' &= -x + 2xy\sin(y) \\ y' &= -\cos(x)y \end{aligned}$$

Investigate stability of the fixed point in the origin.
Linearization gives Jacoby matrix:
$A(x,y) = \begin{bmatrix} -1+2y\sin(y) & 2x(\sin(y)+y\cos(y)) \\ \sin(x)y & -\cos(x) \end{bmatrix}$
$A(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. It implies that the equilibrium is asymptotically stable.

We try also using a Lyapunov function. But it feels an overkill comparing with linearization!

But with hard additional work estimating constants in errors of linearizations we could specify a region of attraction. We will not do it here.

We try $V(x,y) = x^2 + y^2$ and use the Cauchy inequality $2ab \le a^2 + b^2$ and linearization for $\sin(y)$ and $\cos(x)$ when $y \to 0$ and $x \to 0$: $\sin(y) = y + O(y^2)$ and $(1-\cos(x)) = x^2 + O(x^3)$. The notation $O(x)$ means that $O(x)/x$ is bounded when $x \to 0$.

$V_f(x,y) = 2x(-x + 2xy\sin(y)) + 2y(-\cos(x)y) = 4x^2y\sin y - 2y^2\cos x - 2x^2 =$

$4x^2y\sin y + 2y^2(1-\cos x) - 2y^2 - 2x^2 = 4x^2y(y + O(y^2)) + 2y^2(O(x^2)) - 2y^2 - 2x^2 =$

$-2y^2 - 2x^2 + 4x^2y^2 + x^2O(y^3) + 2y^2O(x^2) \le -2(x^2+y^2) + 2(x^4+y^4) + x^2O(y^3) + 2y^2O(x^2)$

For small neighbourhood N of the origin the first term $-2(x^2+y^2)$ dominates all other terms of higher order.

It implies that in $V_f(x,y) < 0$ for $(x,y) \neq (0,0)$ in N and the origin is asymptotically stable.

Example 1. Simple strong Lyapunov function.

> Exercise 15 Show that $(x(t), y(t)) = (0, 0)$ is an asymptotically stable solution of
>
> $$\begin{cases} \dot{x} = -x^3 + 2y^3 \\ \dot{y} = -2xy^2. \end{cases}$$

5

Example 2. Stability by Linearization

For the following system of equations find all equilibrium points and investigate their stability and their type by linearization.

$$\begin{cases} x' = & \ln(2 - y^2) \\ y' = & \exp(x) - \exp(y) \end{cases}$$

1. **Solution.** There are two equilibrium points: $x_1 = (1, 1)$and $x_2 = (-1, -1)$.

The Jacobian of the right hand side is: $\begin{bmatrix} 0 & -2\frac{y}{-y^2+2} \\ e^x & -e^y \end{bmatrix}$. Its values in x_1 and x_2 are $A_1 = \begin{bmatrix} 0 & -2 \\ e & -e \end{bmatrix}$, and $A_2 = \begin{bmatrix} 0 & 2 \\ 1/e & -1/e \end{bmatrix}$. The eigenvalues to A_1 are $-\frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 8e}$,and $\frac{1}{2}\sqrt{e^2 - 8e} - \frac{1}{2}e$ that are conjugate complex numbers with negative real parts. Therefore we observe stable spiral around the equilibrium point x_1. The eigenvalues to A_2 are , eigenvalues: $\frac{1}{e}\left(-\frac{1}{2}\sqrt{8e+1} - \frac{1}{2}\right), \frac{1}{e}\left(\frac{1}{2}\sqrt{8e+1} - \frac{1}{2}\right)$, one postive and one negative. Therefore we x_2 is a saddle point and is unstable.

Example 3.

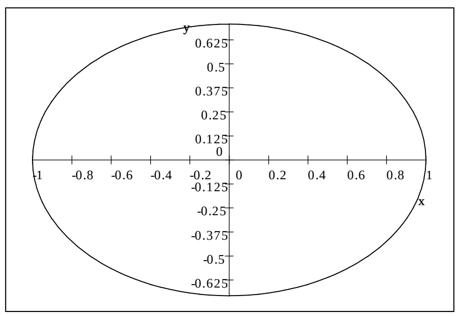
Consider the following system of ODEs:$\begin{cases} x' = & 2y \\ y' = & -x - (1 - x^2)y \end{cases}$.

Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction.

Solution.

We try the test function $V(x, y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories:

$V_f(x, y) = 4xy - 4xy - 4y^2(1 - x^2) = -4y^2(1 - x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x, y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x, y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalles invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x, y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$:



How to find a Lyapunov function?

If the right hand side of the equation is a higher degree polynomial, then it is often convenient to find to find Lyapunov's function in a systematical way in the form of polynomial with unknown coefficients and unknown even degrees like $2m$.

Consider the system

$$\begin{aligned} x' &= -3x^3 - y \\ y' &= x^5 - 2y^3 \end{aligned}$$

Try a test function $V(x,y) = ax^{2m} + by^{2n}$, $a, b > 0$.

$$\begin{aligned} V_f(x,y) &= \nabla V \cdot f(x,y) = \\ &= a2m(x)^{2m-1} \cdot \left(-3x^3 - y\right) + b2n(y)^{2n-1} \left(x^5 - 2y^3\right) \\ &= \underbrace{-6amx^{2m+2}}_{good<0} \underbrace{-2ma(x)^{2m-1}y}_{bad-indefinite} + \underbrace{2nby^{2n-1}x^5}_{bad-indefinite} \underbrace{-4nby^{2n+1}}_{good<0} \end{aligned}$$

We choose first powers m and n so that indefinit terms would have same powers of x and y.

$$\begin{aligned} 2m - 1 &= 5; \Longrightarrow m = 3 \\ 2n - 1 &= 1; \Longrightarrow n = 1 \end{aligned}$$

Then $V_f(x,y) = -18ax^8 - 6x^5y + 2bx^5y - 4nby^4$.We choose $a = 1$ and $b = 3$ to cancel indefinite terms. Then

$$\begin{aligned} V(x,y) &= x^6 + 3y^2 \\ V_f(x,y) &= -18x^8 - 12y^4 < 0, \quad (x,y) \neq (0,0) \end{aligned}$$

Therefore V is a strong Lyapunov's function in the whole plane and the equilibrium is a globally asymptotically stable equilibrium point, because $V(x,y) = x^6 + 3y^2 \to \infty$ as $\|(x,y)\| \to \infty$.

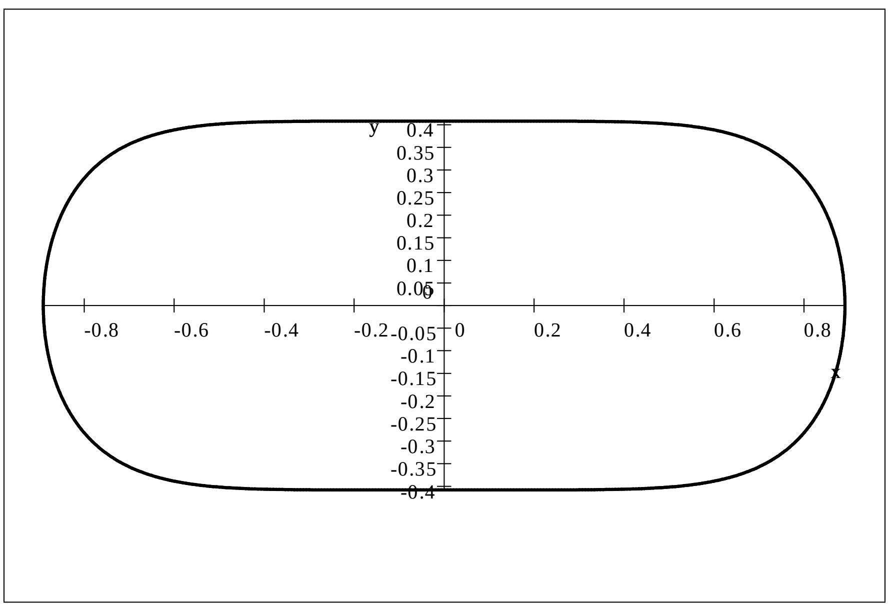
Example 4. Investigate stability of the equilibrium point in the origin.

$$x' = -y - x^3$$
$$y' = x^5$$

We try our simplest choice of the Lyapunov function: $V(x,y) = x^2 + y^2$ and arrive to

$$V_f(x,y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x,y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x,y) = x^6 + \alpha y^2$ where $\partial V/\partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$.The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x,y) = 6x^5(-y - x^3) + 2\alpha yx^5 = -6x^5y + 2\alpha x^5y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice

$\alpha = 3$ cancels them:

$$V_f(x,y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x,y) = 0$ on the whole $y-$axis that in our "general" theory is denoted by $V_f^{-1}(0)$.We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is $y-$axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \to \infty$ as $\|z\| \to \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.

How to find a strong Lyapunov's function?

Example 4.

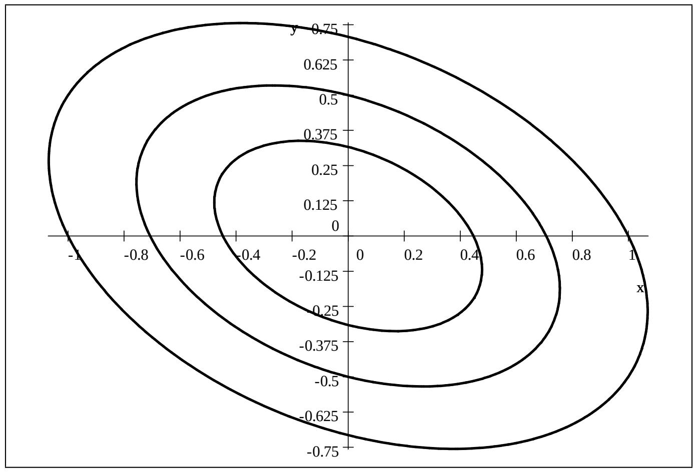
It is theoretically possible to find a strong Lyapunov function for the same system as in the Example 3.

Looking on the previous week Lyapunovs function $x^6 + 3y^2$ we see that it's "weekness" followed from the fact that both level sets of V and velocities of the system were orthogonal to the y - axis. It implied that $V_f(z) = 0$ on the y - axis. To go around this problem a strong Lyapunov function must have level sets that deviate slightly from the normal to the y - axis. Adding a relatively small indefinite term xy^3 to the function $x^6 + 3y^2$ we get this effect. A level set corresponding $x^6 + xy^3 + 3y^2 = 0.7$ of this new Lyapunovs function looks as a slightly rotated version of level sets for the previous (weak) Lyapunovs function.

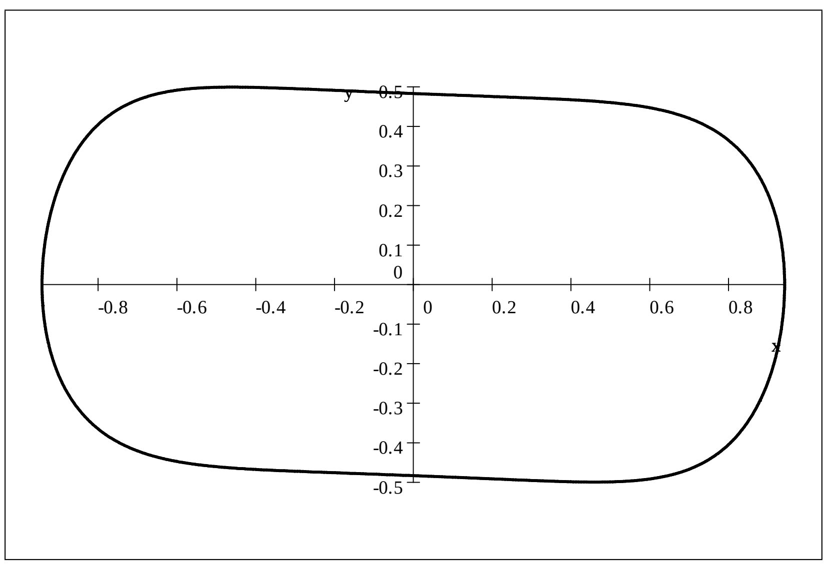
Why like that ? Take a simpler example with an ellipse curve $x^2 + 2y^2 = 1$ and another that is $x^2 + xy + 2y^2 = 1$

This quadratic form is positive definite: the matrix is $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$.A quadratic form $x^T A x = Q(x)$ is positive definite if and only if $\det A > 0$ and all submatrices A_i from the upper left corner have positive determinants: $\det A_i > 0$.

Level sets of the positive definite quadratic form with mixed tems like $x^2 + xy + 2y^2$ are ellipses with symmetry axes (that are orthogonal eigenvectors to A) and are rotated with respect to coordinate axes:



We try to introduce the test function $V(x, y) = x^6 + xy^3 + 3y^2$ with an indefinite mixed term xy^3 added, that would similarly with the ellipses, give slightly rotated level sets so that trajectories would cross them strictly inside on the y - axis:



We claim that the test function $V(x, y) = x^6 + xy^3 + 3y^2$ is positive definite and is a strong Lyapunovs function namely that $V_f(x, y) < 0$ for $(x, y) \neq (0, 0)$.

Because of the geometry of the vector field f of our equation $z' = f(z)$ velocities on the y axis cross such level sets strictly towards inside, implying the desired strict inequality $V_f(z) < 0$, $z \neq 0$ on the y axis. We need to check that

$V(x, y) = x^6 + xy^3 + 3y^2$ is positive definite (it is not trivial) and to show that $V_f(z) < 0$, $z \neq 0$ for all $z \in \mathbb{R}^2$ (it requires some non-trivial analysis).

A very useful inequality in analysis is **Young's inequality**

Lemma. If $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying the conjugacy relation.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The simplest example of Young's inequality:

$$ab \leq \frac{1}{2}\left(a^2 + y^2\right)$$

We show that the test function $V(x, y) = x^6 + xy^3 + 3y^2$ is positive definite in a domain around the origin.

Now, let $V = x^6 + xy^3 + 3y^2$. Applying Young's inequality with $a = |x|$, $b = |y|^3$, $p = 6$, and $q = 6/5$, we see that

$$|xy^3| = |x||y|^3 \leq \frac{|x|^6}{6} + \frac{5|y|^{18/5}}{6} \leq \frac{1}{6}x^6 + \frac{5}{6}y^2$$

if $|y| \leq 1$, so

$$V \geq \frac{5}{6}x^6 + \frac{13}{6}y^2$$

if $|y| \leq 1$. Also,

We calculate $V_f = \dot{V}$ for the system from the Example 3:

$$\begin{aligned} x' &= -y - x^3 \\ y' &= x^5 \end{aligned}$$

$$\begin{aligned}\dot{V} &= -6x^8 + y^3\dot{x} + 3xy^2\dot{y} = -6x^8 - y^3(y + x^3) + 3x^6y^2 \\ &= -6x^8 - x^3y^3 + 3x^6y^2 - y^4.\end{aligned}$$

Applying Young's inequality to the two mixed terms in this orbital derivative, we have

$$|-x^3y^3| = |x|^3|y|^3 \leq \frac{3|x|^8}{8} + \frac{5|y|^{24/5}}{8} \leq \frac{3}{8}x^8 + \frac{5}{8}y^4$$

if $|y| \leq 1$, and

$$|3x^6y^2| = 3|x|^6|y|^2 \leq 3\left[\frac{3|x|^8}{4} + \frac{|y|^8}{4}\right] = \frac{9}{4}x^8 + \frac{3}{4}y^8 \leq \frac{9}{4}x^8 + \frac{3}{64}y^4$$

if $|y| \leq 1/2$. Thus,

$$\dot{V} \leq -\frac{27}{8}x^8 - \frac{21}{64}y^4$$

if $|y| \leq 1/2$, so, in a neighborhood of 0, V is positive definite and $\dot{V}$ is negative definite, which implies that 0 is asymptotically stable.

Example 5.

Consider the Lienard equation: $x'' + x' + g(x) = 0$, and investigate stability of the equilibrium in the origin. The second order equation can be rewritten as a system $z' = f(z)$:

$$\begin{aligned} x' &= y \\ y' &= -g(x) - y \end{aligned}$$

where g satisfies the following hypothesis: g is continuously differentialble for $|x| < k$ for some $k > 0$, $xg(x) > 0$, $x \neq 0$.

Physically this equation is a Newton equation for a non-linear spring. For example if $g(x) = \sin(x)$ it describes a pendulum with friction where air resistance is proportional to velocity.

A Lyapunov function is naturally to choose as a total energy of the system:

$$V(x,y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$$

Indeed it is positive definite in the region $\Omega = \{(x,y) : |x| < k \quad \}$ because $g(s)s > 0$ in Ω according to given conditions. The directional derivative of V along f is

$$V_f(x,y) = y\left(-g(x) - y\right) + g(x)y = -\left(y\right)^2$$

V is a Lyapunov's function, but not strong because $V_f(x,y)$ is negative definite in Ω. $V_f^{-1}(0)$ is the whole x - axis. Checking values of f on $V_f^{-1}(0)$ we observe that trajectories of the system are orthogonal to $V_f^{-1}(0)$ in all points on $V_f^{-1}(0)$ except the origin. It implies that $\{0\}$ is the only invariant set on $V_f^{-1}(0)$ that attracts all trajectorie starting in a small neighborhood of the origin. Therefore the origin is asymptotically stable.

Our next problem is to find a possibly large domain or region of attraction for the equilibrium point.If we find a closed level set for V in Ω, it will be a boundary for a domain of attraction. It will might not be the largest possible and depends on a clever choice of Lyapunov's function V.

We cannot solve this problem for a general expression $V(x,y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

Example 6.

Choose a particular $g(x) = x + x^2$ in the previous example.

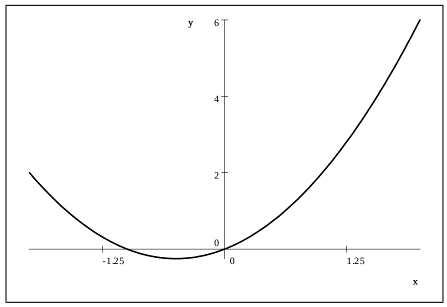
$$\begin{aligned} x' &= y \\ y' &= -(x + x^2) - y \end{aligned}$$

Observe that the system has two equilibrium points: $(-1,0)$ and $(0,0)$

Linearization gives Jacoby matrix $A(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2x & -1 \end{bmatrix}$; $A(-1,0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ Observe that $\det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 0 - 1 = -1 < 0$ it implies by Grobman - Hartman that $(-1,0)$ is a saddle point.

$A(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $\det \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = 1 > 0$, $trace \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -1 < 0$,

$(trace A(0,0))^2/4 = 1/4 < 1 = \det A(0,0)$. It imples that the origin is stable spirals.



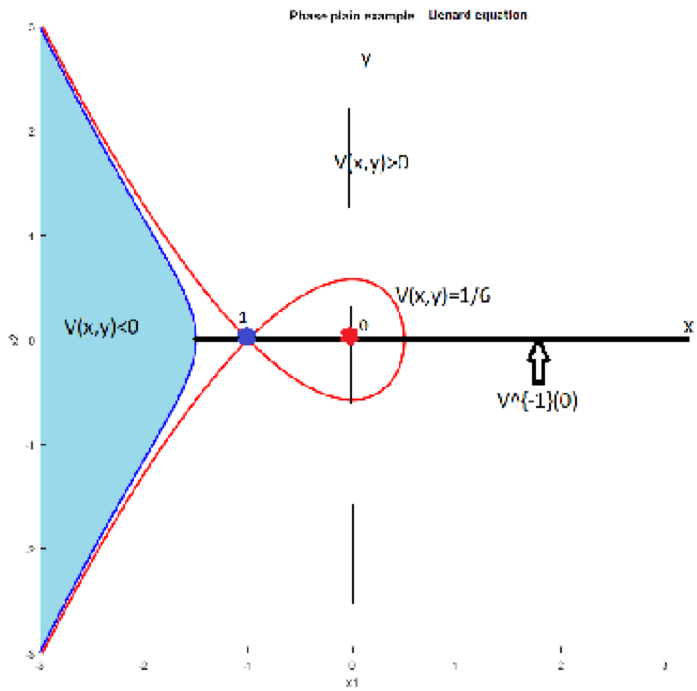
$g(x) = x + x^2$

We can find an explicit expression for the Lyapunov's function $V(x,y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

$$V(x,y) = \frac{(x)^2}{2} + \frac{(x)^3}{3} + \frac{(y)^2}{2}$$

This function is positive definite on the set $\Omega = \left\{ (y)^2 > -(x)^2 - \frac{2}{3}(x)^3 \right\}$

The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{1}{6}$ is depicted by the red line.The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = 0$ is depicted by the blue line. We will investigate them analytically a bit later.



$V_f(x,y) = \nabla V(x,y) \cdot f = xy + (x)^2 y - (y)^2 - xy - (x)^2 y = -(y)^2 \leq 0$ valif in the whole plane $\mathbb{R}^2$.

We check which invariant sets are contained in $V_f^{-1}(0)$ on Ω that is a part of x - axis $\{(x,0) : x > -3/2\}$ that is a thick black line on the picture above.

Notice that $V_f^{-1}(0)$ on Ω contains two equilibrium points $(-1,0)$ and $(0,0)$ and they both are invariant sets. We like to find a largest domain $\Omega_{\ 1} \subset \Omega$ bounded by a part of a level set of V such that $\Omega_{\ 1}$ does not include the point $(-1,0)$. Then $\Omega_{\ 1}$ contains only one invariant set that is the origin $(0,0)$. This set $\Omega_{\ 1}$ is the domain of attraction for the asymptotically stable equilibrium in $(0,0)$.

Such largest level set of V must go through the second equilibrium point $(-1,0)$ and it's value there is $V(x,y) = V(-1,0) = 1/6$. The domain of attraction Ω^* is the egg - shaped domain bounded by the closed curve $(y)^2 = 1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)$ or as a union of explicit two branches:

$$y = \pm\sqrt{1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)}$$

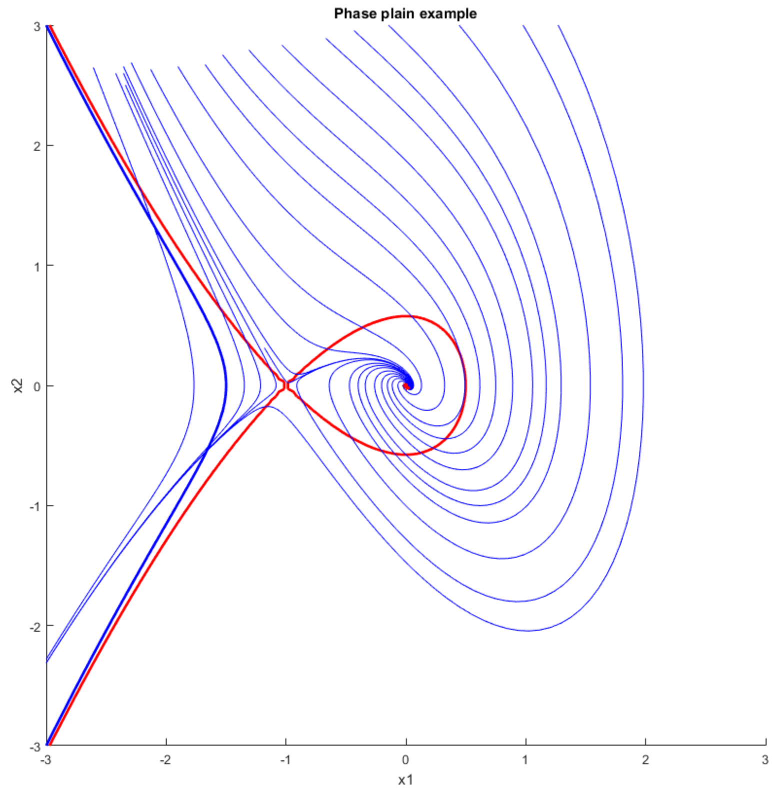
It is a part of the red level set on the picture. To see that this curve is closed e consider derivative of the function

$\frac{d}{dx}\left(1/3-\left((x)^2+\frac{2}{3}(x)^3\right)\right) = -2x-2x^2 = (-2)\,x\,(x+1)$. It implies that the functions has a maximum in $x=0$, and minimum at $x=-1$. $V(x)$ has zero in $x=-1$ and another zero in $x=1/2$:

$1/3-\left((x)^2+\frac{2}{3}(x)^3\right)\Big|_{x=1/2} = 1/3-\left((1/2)^2+\frac{2}{3}(1/2)^3\right) = 1/3-\left((1/4)+\frac{1}{3}(1/4)\right) = 1/3-1/3=0,$

■

One can try to find an even larger region of attraction Ω^{**} for the equilibrium point in the origin. It cannot include the equilibrium in $(-1,0)$ because it is unstable (a saddle point). We can extend Ω_1 to a rectangle $[1,0]\times\left[0,\sqrt{3}/3\right]$in the second quadrant by checking signs of x' and y' on it's left and upper sides. Actual region of attraction is even a bit larger as one can see on the phase portrait



Example 7. Exercise 5.13 from L.R.

Investigate stability of the equilibrium point in the origin and find a possible domain of attraction for the following system.

$$\begin{aligned} x_1' &= -x_2(1+x_1x_2) \\ x_2' &= 2x_1 \end{aligned}$$

We try choose the Lyapunov function V as

$$V(x_1, z_2) = 2x_1^2 + x_2^2$$

We could try first a function $V(x_1, x_2) = ax_1^2 + x_2^2$, check V_f and then decide which value a suites best.

$$\begin{aligned} V_f(x_1,x_2) &= -2ax_1x_2(1+x_1x_2)+2x_2 2x_1 \\ &= 4x_1x_2-2ax_1x_2-2ax_1^2x_2^2=-2ax_1^2x_2^2\le 0 \\ for \quad a &= 2 \end{aligned}$$

We conclude that the equilibriom 0 is stable. $V_f\ (x_1,x_2)=-2ax_1^2x_2^2=0$ on both coordinate axes. We chech which invariant sets are contained in $V_f^{-1}\ (0)$.

If $x_1=0$, then $x_1'=-x_2$, $x_2'=0$. Therefore only $\{0\}$ is an invariant set on the x_2 axis.

If $x_2=0$, then $x_1'=0$, $x_2'=2x_1$. Therefore only $\{0\}$ is an invariant set on the x_1 axis.

Trajectories $\varphi(t,\xi)$ starting inside ellipses $V(x_1,z_2)=2x_1^2+x_2^2=C>0$ are contained inside these ellipses. It implies that their positive orbits $O_+(\xi)$ are bounded and have compact closure in $\mathbb{R}^2$.

It implies according to the LaSalle's theorem that all these solutions $\varphi(t,\xi)$ approach the maximal invariant set in $V_f^{-1}\ (0)$ that in our particular case consists of one point $(0,0)$. Therefore the equilibrium point in the origin is asymptotically stable. It is also globally stable because the Lyapunov function $V(x)$ tends to infinity as $\|x\|\to\infty$, making that arbitrary large elliptic discs from the family $2x_1^2+x_2^2<C$ are regions of attraction.

■

Example 8.

Consider the following system of ODE: $\begin{cases} x' = & -x-2y+xy^2 \\ y' = & 3x-3y+y^3 \end{cases}$.

1. Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

   Hint: applying Lyapunovs theorem, use the elementary Young's inequality $2xy\le\left(x^2+y^2\right)$ to estimate indefinite terms with xy. **(4p)**

   Solution. Choose a test function $V(x,y)=\frac{1}{2}\left(x^2+y^2\right)$

   $V_f=x(-x-2y+xy^2)+y\left(3x-3y+y^3\right)=xy-x^2-3y^2+y^4+x^2y^2$

   $=-x^2\left(1-y^2\right)-y^2\left(3-y^2\right)+xy\le 0 \quad ?????$

We apply the inequality $2xy \le \left(x^2+y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$V_f \le -x^2\left(0.5-y^2\right)-y^2\left(2.5-y^2\right)$

It implies that $V_f < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the origin is asymptotically stable.

The attracting region is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely $\left(x^2+y^2\right) < 1/2$.

Another more clever choice of a test function is $V(x,y) = 3x^2+2y^2$.

$V_f = 6x(-x-2y+xy^2)+4y(3x-3y+y^3) = 4y^4-12y^2-6x^2+6x^2y^2 = -4y^2\left(3-y^2\right)-6x^2\left(1-y^2\right) < 0$

for $|y| < 1$, therefore the ellipse $3x^2+2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin by linearization with variational matrix

$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2+4\lambda+9=0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the set of attraction.■

Example 9 on instability

Consider the following system of ODEs. Prove the instability of the equilibrium point in the origin, of the following system

$$\begin{cases} x' = & x^5+y^3 \\ y' = & x^3-y^5 \end{cases} \qquad \textbf{(4p)}$$

using the test function $V(x,y) = x^4-y^4$ and Lyapunov's instability theorem.

Solution.

Denoting $f(x,y) = \begin{bmatrix} x^5+y^3 \\ x^3-y^5 \end{bmatrix}$, consider how $V(x,y) = x^4-y^4$changes along trajectories of the system. $f(x,y)\cdot\nabla V(x,y) = \begin{bmatrix} x^5+y^3 \\ x^3-y^5 \end{bmatrix} \cdot \begin{bmatrix} 4x^3 \\ -4y^3 \end{bmatrix} =$

$x^5 4x^3 + y^3 4x^3 - x^3 4y^3 + y^5 4y^3 = x^5 4x^3 + y^5 4y^3 = 4(x^8+y^8) > 0.$

Point out that the function $V(x,y) = x^4-y^4$ is positive along the line $y = x/2$, $x > 0$ arbitrarily close to the origin. It implies according to the instability theorem, that the origin is an unstable equilibrium.■

May 27, 2020

1 Banach's contraction principle. Picard-Lindelöf theorem.

We consider in this chapter the theorem by Picard and Lindelöf about existence and uniqueness of solutions to the initial value problem to the system of differential equations in the form

$$x'(t) = f\,(t, x(t)) \tag{1}$$

$$x(\tau) = \xi \tag{2}$$

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous with respect to time variable t and space variable x. J is an interval, G is an open subset of $\mathbb{R}^n$.

One can reformulate the I.V.P. (1),(2) in the form of the integral equation

$$x(t) = \xi + \int_\tau^t f\,(s, x(s))\,ds \tag{3}$$

If f is continuous, then these two formulations are equivalent by the Newton-Leibnitz theorem.

Fixed points of operators.

Consider a vector space X with a subset $C \subset X$ and an operator $K : C \to C$.

Definition

A point $\overline{x} \in C$ is called the **fixed point** of the operator K if

$$K(\overline{x}) = \overline{x} \tag{4}$$

A general idea behind the analysis of many types of equations is to reformulate them as a fixed point problem.

Consider the right hand side of the integral equation (3) as an operator

$$K(x)(t) \stackrel{\text{def}}{=} \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

acting from the vector space of continuous functions $C(I)$, where $I \subset J$ is a closed interval including τ . Point out that t can be smaller than τ $(t < \tau)$.

The expression $\|x\|_{C(I)} = \sup_{t \in I} \|x(t)\|$ defines a norm on the space $C(I)$ because it satisfies the triangle inequality and we know that uniformly convergent sequences of continuous functions on the compact set (I in this case) converge to continuous functions.

This space is even complete in the sense that Cauchy sequences of functions in $C(I)$ converge uniformly to continuous functions. It means more explicitely that if the sequence $\{x_n\} \in C(I)$ has the Cauchy property:

$$\|x_m - x_n\|_{C(I)} = \sup_{t \in I} \|x_m(t) - x_n(t)\|_{C(I)} \underset{m,n \to \infty}{\to} 0$$

then there is a continuous function $\overline{x} \in C(I)$ such that $x_n \underset{n \to \infty}{\to} \overline{x}$ uniformly on I, or that is the same, $\|x_n - \overline{x}\|_{C(I)} \underset{n \to \infty}{\to} 0.$

Definition.

We call a normed vector space a Banach space if it is complete with respect to it's norm.

This notion was introduced by Polish mathematician Stefan Banach who lead the greatest school in functional analysis at the university of Lwow in the beginning of 20th century.

Examples.

The space $C(I)$ is a Banach space.

Elementary examples of Banach spaces are given by $\mathbb{R}^n$ supplied with norms $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ with $p \geq 1$.

A slight extension of this example is a set l_p, $p \geq 1$ of real sequences $\{x_i\}_{i=1}^{\infty}$ with finite norms in the form $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

One of the most popular classes of Banach spaces is the space of "integrable functions" $f : G \to \mathbb{R}$ where $G \subset \mathbb{R}^n$, with norms $\|f\|_{Lp} = \left(\int_G |f(z)|^p \, dz\right)^{1/p}$

"Integrable functions" and the integral here are in the sense of Lebesque, that is a contemporary notion of integral, studied in the course "Integration theory" given for master and PhD students.

Remark.

We point out for convenience that different norms are used through out the text. Notation $\|\|$ means usual euclidean norm in $\mathbb{R}^n$. For a Banach space X the notation $\|x\|_X$ means the norm in the space X.

The operator K defined above, acts from $C(I)$ to itself. It makes that the I.V.P. above can be considered as a fixed value problem (**4**) on $C(I)$ or on some subset of it.

A classical theorem that guarantees the existence and uniqueness of fixed points to operators in Banach and more generally in metric spaces, is Banach's contraction principle.

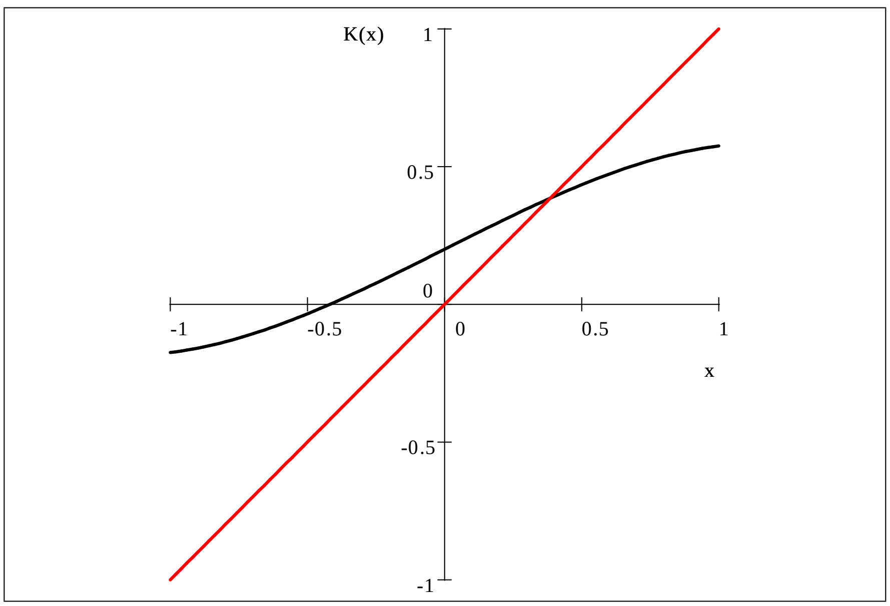
Definition. Operator $K : A \to A$, where $A \subset X$, and X is a Banach space, is called contraction on A if there is a constant $0 < \theta < 1$ such that for any $x, y \in A$

$$\|K(x) - K(y)\|_X \leq \theta \|x - y\|_X$$

Example. An elementary example is a smooth function K acting from an interval $[a,b]$ to itself and having absolute value of derivative $\left|\frac{d}{dt}K(t)\right| < \theta < 1$ for all $t \in [a,b]$. By intermediate value theorem for any $x, y \in [a,b]$ there is a point $c \in (x,y)$ such that $K(x) - K(y) = (x-y)K'(c)$.Therefore

$$|K(x) - K(y)| = |(x-y)|\,|K'(c)| \leq \theta\,|(x-y)|$$

It implies that K is a contraction in on the interval $[a,b]$. Example: $K(x) = 0.5\,(x - 0.25x^3) + 0.2$



Another example would be a Lipschitz function with Lipschitz constant L smaller then one: $L < 1$.

Banach's contraction principle.

Let A be a non-empty closed subset of a Banach space X and $K : A \to A$ be a contraction operator with contraction constant $\theta < 1$.

Then there is a unique fixed point $\overline{x} \in A$, to K such that $K\overline{x} = \overline{x}$ such that

$$\|K^n(x_0) - \overline{x}\|_X \leq \frac{\theta^n}{1-\theta} \|K(x_0) - \overline{x}\|_X$$

for arbitrary $x_0 \in A$. Here $K^n(x_0) = K(K(...K(x_0))...)$ is the operator K applied to itself n times.

Proof (not required at the exam) is based on showing that the sequence of approximations $\{x_n\}_{n=0}^{\infty}$ defined by the equations

$$\begin{aligned} x_1 &= K(x_0) \\ &\dots \\ x_{n+1} &= K(x_n) \end{aligned}$$

with an arbitrary initial approximation $x_0 \in A$, converge to some $\overline{x} \in A$ that is the unique fixed point of K in A.

It follows by induction that

$$\begin{aligned} \|x_{n+1} - x_n\|_X &= \|K(x_n) - K(x_{n-1})\|_X \leq \theta \|x_n - x_{n-1}\|_X \\ &\leq \theta \|K(x_{n-1}) - K(x_{n-2})\|_X \leq \theta^2 \|x_{n-1} - x_{n-2}\|_X \\ &\dots \\ &\leq \theta^n \|x_1 - x_0\|_X \end{aligned}$$

We will show that $\{x_n\}_{n=0}^{\infty}$ is a Chauchy sequence. Let $m > n$.

$$\begin{aligned}
\|x_m - x_n\|_X &= \|x_m - x_{m-1} + x_{m-1} - x_{m-2} + ... + x_{n+1} - x_n\|_X \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + ... + \|x_{n+1} - x_n\| \leq \\
&\leq \left(\theta^n + \theta^{n-1} + ... + \theta^{m-1}\right) \|x_1 - x_0\|_X \\
&= \theta^n \left(1 + \theta + ...\theta^{m-n-1}\right) \|x_1 - x_0\|_X \\
&\leq \theta^n \underbrace{\left(1 + \theta + ...\theta^{m-n-1} + ...\right)}^{=\frac{1}{1-\theta}} \|x_1 - x_0\|_X \\
&\leq \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \to 0, \qquad n \to \infty, \quad \theta < 1
\end{aligned}$$

A is closed, therefore the limit $\lim_{n\to\infty} x_n = \overline{x}$ exists $\overline{x} \in A$.

Claim: $\overline{x}$ is a fixed point to K. We see it by tending to the limit in the expression for x_n:$x_{n+1} = K(x_n)$

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\infty} K(x_n) = K(\lim_{n\infty} x_n) \\
\overline{x} &= K(\overline{x})
\end{aligned}$$

and using continuity of K.

The last question we must answer is the uniqueness of the fixed point to K in A. Suppose that here is another fixed point $\widetilde{x}$ to K in A. Consider the norm of the difference $\overline{x} - \widetilde{x}$:

$$\|\overline{x} - \widetilde{x}\|_X = \|K(\overline{x}) - K(\widetilde{x})\|_X \leq \theta \|\overline{x} - \widetilde{x}\|_X \;, \quad \theta < 1$$

It is possible only if $\overline{x} - \widetilde{x} = 0$.

$$
\begin{aligned}
\|x_m - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
\lim_{m\to\infty} \|x_m - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
\left\| \lim_{m\to\infty} x_m - x_n \right\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
\|\overline{x} - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X
\end{aligned}
$$

■

Elementary exercise on Banach's contraction principle.

Show using Banach's contraction principle that the polynomial $K(x) = x^2 - 0.4$ has a fixed point $K(x) = x$.

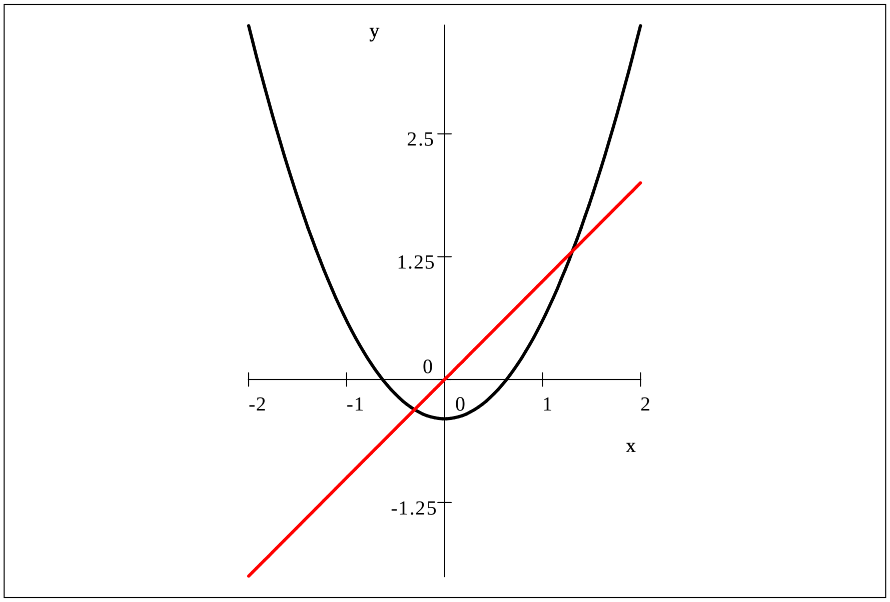
Solution consists of two steps.

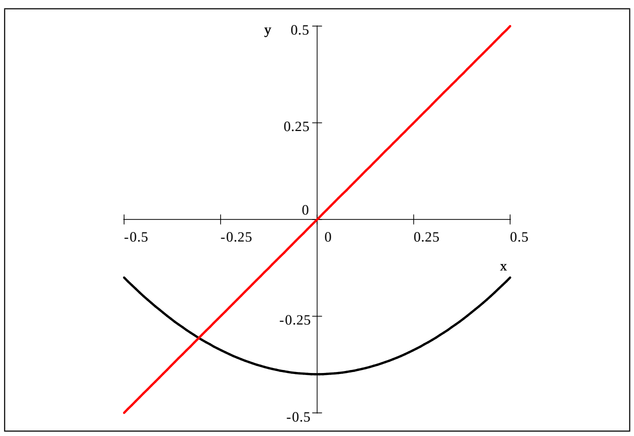
i) Find a set $B \subset \mathbb{R}$ where $K(x)$ has the contraction property: $|K(x) - K(y)| \le \theta\,|x-y|$, $\theta < 1$, for $x, y \in B$

ii) Find a subset $A \subset B$ that the function K maps into itself: $K : A \to A$.

i) $K'(x) = 2x < 1 \Longrightarrow x \in [-0.5 + \delta, 0.5 - \delta]$

ii)The set $[-0.5 + \delta, 0.5 - \delta]$ satisfies the requirement.





Picard-Lindelöf theorem.

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous in $J \times G$. J is an interval, G is an open subset of $\mathbb{R}^n$. Let in addition suppose that f is Lipschitz continuous with respect to the second argument with the Lipschitz constant $L > 0$:

$$\|f(t,x) - f(t,y)\| \leq L\,\|x - y\|\,, \forall x, y \in G$$

(We could suppose a weaker condition that this Lispchitz property is only local, but will not do it because it would make the proof just slightly longer without changing main ideas).

Then for any $(\tau, \xi) \in J \times G$ the initial value problem

$$\begin{aligned} x' &= f(x,t) \\ x(\tau) &= \xi \end{aligned}$$

has a unique solution on some time interval including τ. $\square$

Remark. This local solution can always be extended to a unique maximal solution. We considered maximal extensions earlier in the course.

Proof to the Picard-Lindelöf theorem.

The proof is based on using the integral form of the I.V.P.

$$x(t) = \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

and applying Banachs contraction principle to it. We use the Banach space of continuous functions $x : I \to \mathbb{R}^n$on some compact interval $I \subset J$.

The application of Banach's principle here consists of two steps.

• The first one is to find a time interval I_1 and a subset $A \subset C(I_1)$ such that the operator K defined by

$$K(x)(t) = \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

maps A to itself: $K : A \to A$.

• The second one is to find a time interval I_2 such that the contractness property for the operator would be valid on a subset of $C(I_2)$. Finally we will choose the smallest of I_1 and I_2 for both properties to be valid and will conclude the result.

We consider here the case when an interval $[\tau, \tau + T] \in J$, $T > 0$ and try to find a solution on this time interval (or possibly on a shorter time interval

$[\tau, \tau + \Delta]$ with $\Delta < T$). Considering a time interval backword direction in time is similar.

We choose first a closed ball $\overline{B(\xi, \delta)} = \{x : \|x - \xi\| \leq \delta\}$ such that it belongs to G: $\overline{B(\xi, \delta)} \in G$.

Our intension is to find solution in the set of continuous functions $x : [\tau, \tau + T] \to \mathbb{R}^n$ such that $x(t) \in \overline{B(\xi, \delta)}$ for all $t \in [\tau, \tau + T]$ and therefore $\sup_{t \in [\tau, \tau + T]} \|x(t) - \xi\| \leq \delta$. It is a closed ball

$$A = \|x - \xi\|_{C([\tau, \tau + T])} \leq \delta$$

in the infinitely dimensional space $C([\tau, \tau + T])$.

Our goal in the proof is to find such an interval $[\tau, \tau + T]$ that this set A in $C([\tau, \tau + T])$ and the operator K satisfy conditions in the Banach contraction principle.

The function $f(t, x)$ is continuous on the compact set $V = [\tau, \tau + T] \times \overline{B(\xi, \delta)}$ in $\mathbb{R}^{n+1}$and therefore

$$M = \sup_{(t,x) \in V} \|f(t, x)\| < \infty$$

The constant M controls how large is velocity $f(t, x)$ inside the set $V = [\tau, \tau + T] \times \overline{B(\xi, \delta)}$(yellow in the picture). Correspondingly M controls how fast the (blue) trajectory $x(t)$ can go away from the initial point ξ.

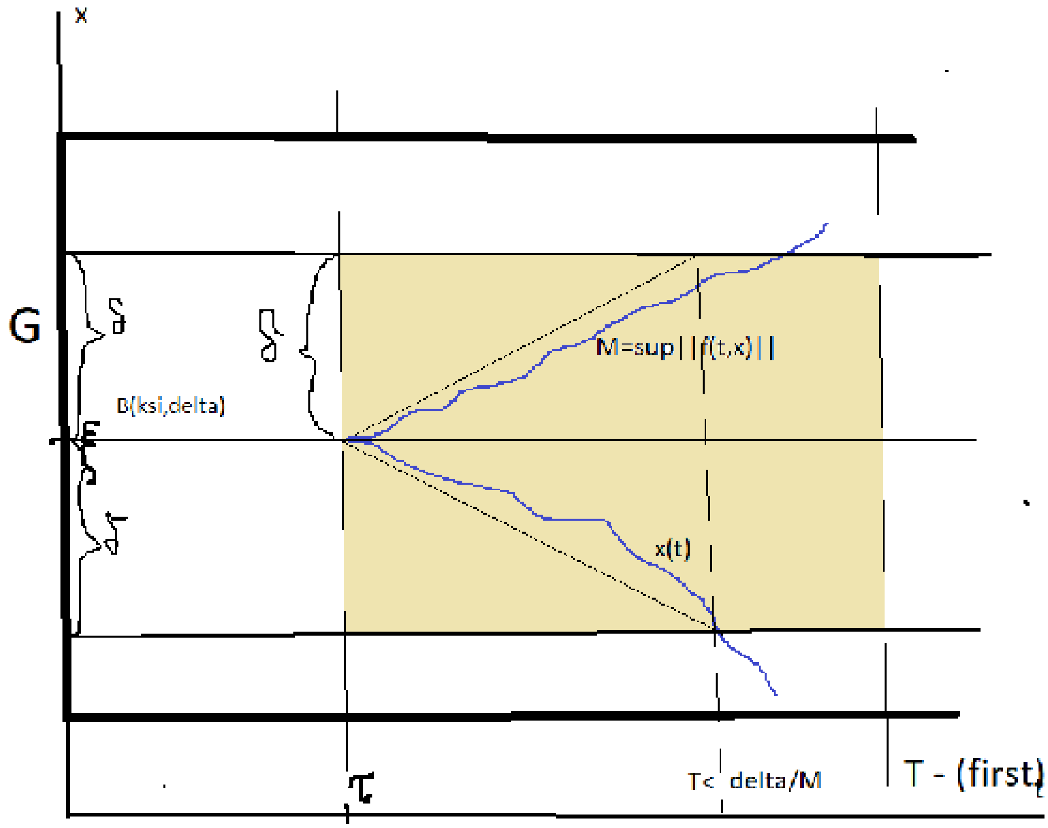
According to the equation $x(t) = K(x)$ must be inside the "angle" bounded by the cone $\|x - \xi\| = M(t - \tau)$.

We give here two pictures illustrating the proof, a one dimensional picture:

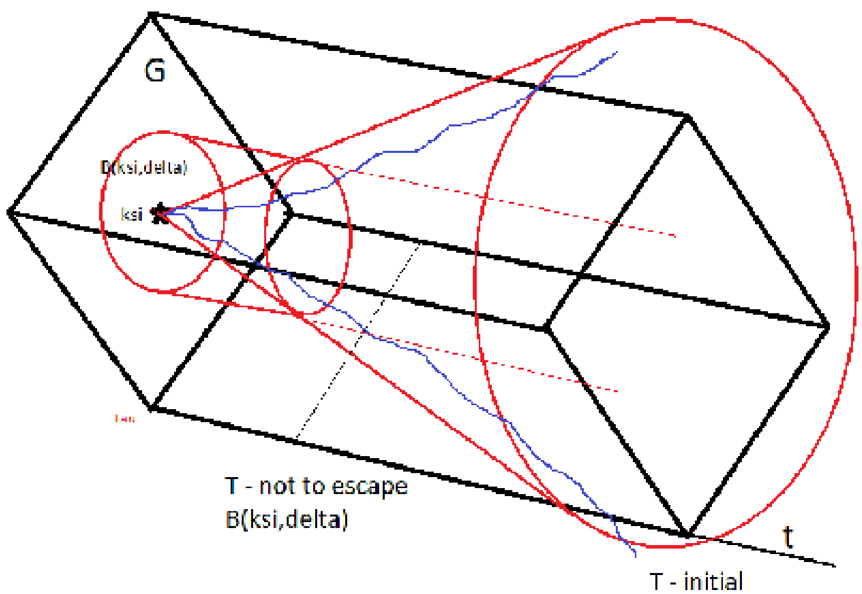
and a two-dimensional picture:

We are going to estimate $\|K(x)(t) - \xi\|$ and choose the length T of the time interval $[\tau, \tau + T]$ in such a way that for any $x(t) \in \overline{B(\xi, \delta)}$ for $t \in [\tau, \tau + T]$, it follows that $K(x(t))$ does not escape the ball $\overline{B(\xi, \delta)}$.

$$\|K(x(t)) - \xi\| \leq \delta$$



x
G
δ
δ
δ
ξ
B(ksi,delta)
M=sup||f(t,x)||
x(t)
τ
T< delta/M
T - (first)


G
B(ksi,delta)
ksi
T - not to escape
B(ksi,delta)
t
T - initial

for $t \in [\tau, \tau + T]$.

It would imply that

$$\sup_{t \in [\tau, \tau+T]} \|K(x)(t) - \xi\| = \|K(x) - \xi\|_{C([\tau,\tau+T])} \leq \delta$$

for $\|x - \xi\|_{C([\tau,\tau+T])} \leq \delta$. We start with proving the first inequality:

$$\|K(x)(t) - \xi\| = \left\| \int_\tau^t f(s, x(s))\, ds \right\| \leq \int_\tau^t \|f(s, x(s))\|\, ds \leq TM$$

We observe that choosing $T < \delta/M$ we get that $\|K(x)(t) - \xi\| \leq \delta$ for $t \in [\tau, \tau + T]$. Taking supremum of the left hand side over $t \in [\tau, \tau + T]$ arrive to

$$\|K(x) - \xi\|_{C([\tau,\tau+T])} \leq \delta$$

It means that for this time interval trajectories do not gun out of the yellow domain on the picture. In turn it means that the operator K maps the closed ball A in $C([\tau, \tau + T])$ defined by the inequality $\|x - \xi\|_{C([\tau,\tau+T])} \leq \delta$, with

$$T < \delta/M$$

into itself:

$$K : A \to A$$

Now we check conditions (again choosing the length of the time interval) such that the operator K would be contraction on the set A with once again suitably adjusted time interval T. Consider first the difference $\|K(x)(t) - K(y)(t)\|$, for arbitrary $t \in [\tau, \tau + T]$. We apply the triangle inequality, the Lipschitz property of the function f , and estimate the integral by the length of the interval times maximum of the function under it.

$$
\begin{aligned}
\|K(x)(t)-K(y)(t)\| &= \left\|\int_\tau^t f(s,x(s))-f(s,y(s))\,ds\right\| \overset{\text{triangle inequality}}{\leq} \\
&\leq \int_\tau^t \|f(s,x(s))-f(s,y(s))\|\,ds \\
\overset{\text{Lipschitz property}}{\leq} L\int_\tau^t \|x(s)-y(s)\|\,ds &\leq \\
\overset{\sup}{\leq} LT\sup_{s\in[\tau,\tau+T]}\|x(s)-y(s)\| &= LT\,\|x-y\|_{C([\tau,\tau+T])}
\end{aligned}
$$

Calculating supremum over $t\in[\tau,\tau+T]$ of the left hand side we arrive to the inequality

$$
\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq LT\,\|x-y\|_{C([\tau,\tau+T])}
$$

It implies that choosing the length f the time interval $T<1/L$ we get the contraction property.

$$
\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq \theta\,\|x-y\|_{C([\tau,\tau+T])}\,, \quad 0<\theta<1
$$

Now choosing the time interval $T<\min(1/L,\,\delta/M)$ we conclude that the operator K maps the closed ball A in $C([\tau,\tau+T])$ defined by

$$
A=\left\{x\in C([\tau,\tau+T]),\ \|x-\xi\|_{C([\tau,\tau+T])}\leq\delta\right\}
$$

into itself: $K:A\to A$ and that K is a contraction on A: $\|K(x)-K(y)\|_{C([\tau,\tau+T])}\leq \theta\,\|x-y\|_{C([\tau,\tau+T])}$, $\theta<1$, for any $x,y\in A$.

By the Banach contraction principle K has for $T<\min(1/L,\,\delta/M)$ a unique fixed point $\overline{x}$ in A that is the solution to the integral equation (3) and to the original initial value problem.■

Example. Banach's contraction principle applied to a non-linear integral operator.(exam 2019 june)

Consider the following (nonlinear!) operator

$$K(x)(t) = \int_0^2 B(t,s)\,[x(s)]^2\,ds + g(t),$$

Fixed point problem to solve:

$$x = K(x)$$

acting on the Banach space $C([0,2])$ of continuous functions with norm $\|x\|_{C([0,2])} = \|x\|_C = \sup\limits_{t\in[0,2]} |x(t)|$. Here $B(t,s)$ and $g(t)$ are continuous functions and $|B(t,s)| < 0.5$ for all $t, s \in [0,2]$. Estimate the norm $\|K(x) - K(y)\|_{C([0,2])}$ for the operator $K(x)(t)$. Find requirements on the function $g(t)$ such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point.

Solution.

Banach's contraction principle. Let B be a nonempty closed subset of a Banach space X and let the non-linear operator $K : B \to B$ be a contraction.

$$\|K(x) - K(y)\|_X \leq \theta \|x - y\|_X, \quad \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x_0) - \overline{x}\|_X \leq \frac{\theta^n}{1-\theta}$$

for any $x_0 \in B$. Here $K^n(x_0) = (K(K(...K(x_0)...))$ is the n -fold superposition of the operator K with itself.

We like to have the estimate $\|K(x) - K(y)\| \leq \theta \|x - y\|$ for x, y in some closed subset B of $C([0,2])$.

$$
\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \left| \int_0^2 |B(t,s)| \left| [x(s)]^2 - [y(s)]^2 \right| ds \right| \\
&= \left| \int_0^2 |B(t,s)| \cdot |x(s) - y(s)| \cdot |x(s) + y(s)| \, ds \right| \overset{\text{taking} \sup\limits_{t,s\in[0,2]}}{\leq} \\
&\leq \int_0^2 ds \sup_{t,s\in[0,2]} |B(t,s)| \sup_{s\in[0,2]} |x(s) - y(s|) \left(\sup_{s\in[0,2]} |x(s)| + \sup_{s\in[0,2]} |y(s)| \right) = \\
&= 2 \cdot 0.5 \, \|x - y\|_C \left(\|x\|_C + \|y\|_C \right) = \|x - y\|_C \left(\|x\|_C + \|y\|_C \right)
\end{aligned}
$$

We take supremum ower $t \in [0,2]$ of the left hand side and get

$$
\|K(x) - K(y)\|_C \leq \|x - y\|_C \left(\|x\|_C + \|y\|_C \right)
$$

We can choose a ball $B \subset C([0,2])$ such that for any x, $y \in B$ it follows $\|x\|_C + \|y\|_C \leq \theta < 1$, for example B can be taken as a set of continuous functions with $\|x\|_C \leq \theta/2$. On this set K will be a contraction because

$$
\|K(x) - K(y)\|_C \leq \theta \, \|x - y\|_C \,, \quad \theta < 1.
$$

To apply Banach's principle we need also that K maps B into itself, namely that $\|K(x)\|_C \leq \theta/2$ for $\|x\|_C \leq \theta/2$.

It gives a requirement on function $g(t)$.

$$
\begin{aligned}
K(x)(t) &= \int_0^2 B(t,s) \, [x(s)]^2 \, ds + g(t), \\
\|K(x)\|_C &\leq 2 \times 0.5 \times \|x\|_C^2 + \|g\|_C \leq (\theta/2)^2 + \|g\|_C \leq \theta/2
\end{aligned}
$$

Conclusion is that $\|g\|_C = \sup_{t\in[0,2]} |g(t)| \leq \theta/2 - (\theta/2)^2 = \theta/2 \, (1 - \theta/2)$ implies that $K : B \to B$, where $B = \{x(t) : |x(t)| \leq \theta/2\}$, $x(t)$ - continuous.

Therefore K has a unique fixed point in the ball B in $C([0,2])$. ∎

Example. (exam. 2018 january)

1. Consider the following initial value problem: $y' = \sin(y)t^2$; $y(1) = 2$.

   a) Reduce the initial value problem to an integral equation and give a general description of iterations approximating the solution as in the proof to the existence and uniqueness theorem by Picard and Lindelöf. **(2p)**

   b) Find a time interval such that these approximations converge to the solution of the initial value problem. **(2p)**

   Solution.

   We introduce an integral equation equivalent to the ODE $y' = f(t, y)$ by the integration of the right and left hand sides in the equation:

$$y(t) = y(1) + \int_1^t f(s, y(s))ds.$$

   Taking $y_0(t) = y(1)$ we define Picard iterations by the recurrense relation

$$\begin{aligned} y_{n+1}(t) &= y(1) + \int_1^t f(s, y_n(s))ds. \\ y_{n+1} &= \mathbb{K}(y_n) \end{aligned}$$

   For the particular equation it looks as

$$y_{n+1}(t) = y(1) + \int_1^t \sin(y_n(s))s^2 ds = \mathbb{K}(y_n, t).$$

   Fixed point problem:

$$y = \mathbb{K}(y)$$

   The Banach contraction principle gives existence and uniqueness of solutions by showing that the operator $\mathbb{K}$ is a contraction on some closed set B of a Banach space X, such that $\mathbb{K}$ maps B into itself.

A hidden question here is that we must find this Banach space X and this set B where these conditions are satisfied.

One proves the existence and uniqueness theorem by showing that at some time interval the integral operator $\mathbb{K}(y,t) = y(1)+\int_1^t \sin(y(s))s^2 \mathrm{d}s$ in the right hand side is a contraction in $C([1,T])$:

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} \stackrel{def}{=} \sup_{t\in[1,T]} |\mathbb{K}(w,t) - \mathbb{K}(u,t)| < \alpha \sup_{t\in[1,T]} |w(t) - u(t)| = \alpha \|w - u\|_{C([1,T]}$$

$\alpha < 1$, in some ball $\|w - y(1)\|_{C([1,T]} = \sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ in the space $C([1,T])$ of continuous functions on $[1,T]$, and maps this ball into itself:

$$\sup_{t\in[1,T]} |\mathbb{K}(w,t) - y(1)| \leq R$$

and applying the Banach contraction theorem to $\mathbb{K}(y,t)$.

We estimate first $\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} = \sup_{t\in[1,T]} |\mathbb{K}(w,t) - \mathbb{K}(u,t)|$ for continuous functions u and w such that $\sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ and

$\|w - y(1)\|_{C([1,T]} = \sup_{t\in[1,T]} |u(t) - y(1)| \leq R$. Point out that $\sup_{t\in[1,T]} |w(t)| \leq y(1) + R$. We will find T such that the contraction property is valid:

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} = \sup_{t\in[1,T]} \left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| \leq \alpha \sup_{t\in[1,T]} |w(t) - u(t)|,$$

We carry out elementary estimates using the triangle inequality and intermediate value theorem for sin. $\left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| \leq \int_1^t |(\sin(w(s)) - \sin(u(s))| s^2 ds =$

$\int_1^t |(w(s)-u(s))\cos(\theta(s))|\, s^2 ds \leq (T-1)\, T^2 \cdot 1 \cdot \sup_{t\in[1,T]} |w(s)-u(s)|$

$$\|\mathbb{K}(w)-\mathbb{K}(u)\|_{C([1,T]} \leq (T-1)\, T^2\, \|w(s)-u(s)\|_{C([1,T]}$$

The argument $\theta(s)$ above is a number between $w(s)$ and $u(s)$ that exists according the intermediate value theorem. It was also used above that $|\cos(\theta)| \leq 1$. Therefore to have the contraction property we need to have $(T-1)\, T^2 < 1$.

For a function w with $\|w(s)\|_{C([1,T]} = \sup_{t\in[1,T]} |w(t)-y(1)| \leq R$ we like to have that $|\mathbb{K}(w,t)-y(1)| \leq R$ meaning that $\mathbb{K}$ maps this ball in $C([1,T])$ into itself. For this particualr case it is not necessary because he equation is defined in the whole $\not{R}$ and the contraction property is bvalid in the whole $C([1,T])$. But this checking might be necessary if the contraction property is valid only locally, not in thew whole $C([1,T])$.

The following estimate leads to another bound for T: $\sup_{t\in[1,T]} |\mathbb{K}(w,t)-y(1)| \leq \sup_{t\in[1,T]} \left|\int_1^t \sin(w(s))s^2 ds\right| \leq (T-1)\, T^2 \leq R.$

Therefore the time interval must satisfy estimates $(T-1)\, T^2 < 1$ and $(T-1)\, T^2 < R$ to have convergence of Picard iterations in the ball $\sup_{t\in[1,T]} |w(t)-y(0)| \leq R$. Taking $R=1$ we get an optimal estimate $(T-1)\, T^2 < 1$ that is satisfied for example for $T=1.4$:

$\alpha = 0.4(1.4)(1.4) = 0.784$

Introduction to bifurcations.

Considering differential equations where the right hand side includes a parameter:

$$x' = f(t,x,\mu)$$

we can observe qualitative changes in the phase portrait of the system at certain values of the parameter $\mu = \mu_0$.

Examples of bifurcations.

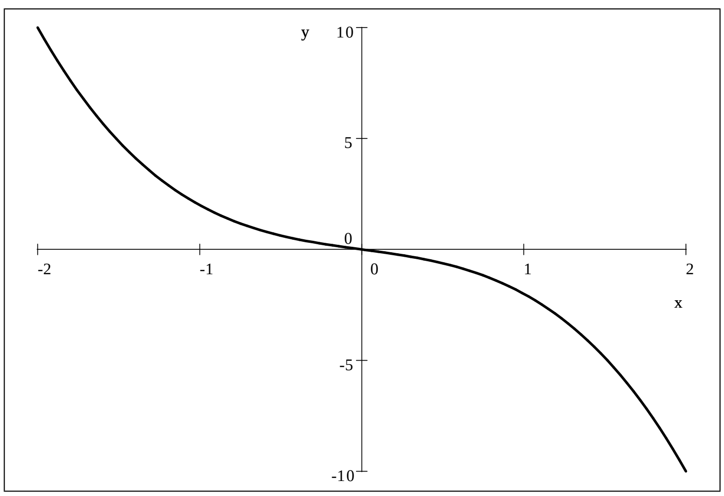
Pitchforc bifurcation

The equation

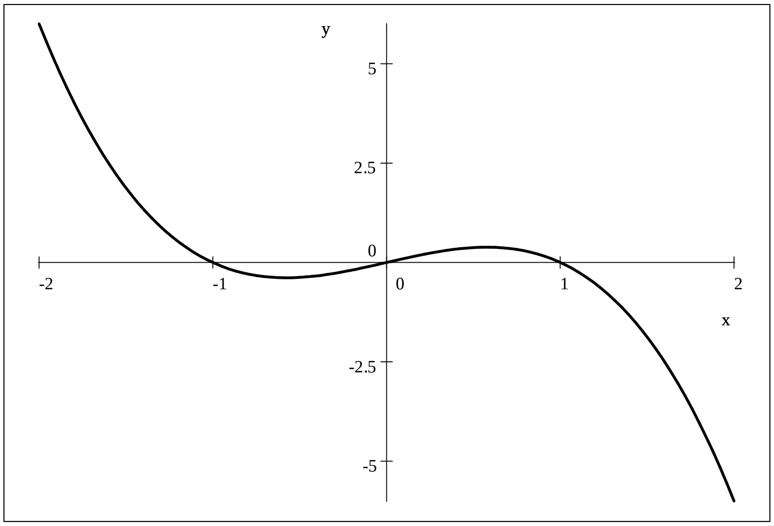
$$x' = \mu x - x^3$$

has one stable equilibrium point $x = 0$ for $\mu \leq 0$, that becomes unstable and splits into two stable equilibrium points at $\mu = 0$.

$f(x) = -x - x^3, \mu < 0$



$f(x) = x - x^3, \mu > 0$



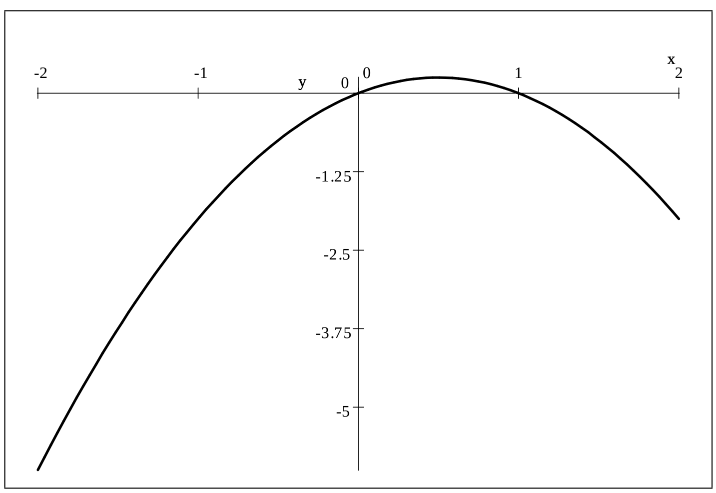
Transcritical bifurcation.

The equation

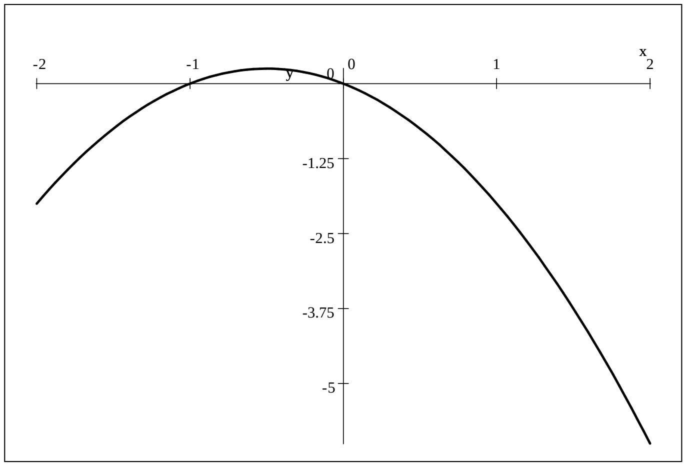
$$x' = \mu x - x^2$$

has two fixed points for $\mu \neq 0$ which collide and exchange stability at $\mu = 0$.

$f(x) = x - x^2, \mu = 1$



$f(x) = -x - x^2$, $\mu = -1$



Saddle point bifurcation.

The equation

$$x' = \mu + x^2$$

has one stable and one unstable equilibrium point for $\mu < 0$ which collide at $\mu = 0$ and vanish when $\mu > 0$.

Hopf bifurcation

One impressive example is the so called Hopf bifurcation where an asymptotically stable equilibrium becomes unstable equilibrium surrounded by a unique limit cycle, a periodic solution attracting sorrounding trajectories.

The theorem blow gives a possibility to show the existence of a **unique periodic solution** surrounding an equilibrium that is a repeller.

Theorem on Hopf bifurcation. Let the system of differential equations in plane:

$$\begin{aligned} x_1' &= f_1(x_1, x_2, \mu) \\ x_2' &= f_2(x_1, x_2, \mu) \end{aligned}$$

have an equilibrium point in the origin for all real values of the parameter μ.

Suppose that for the linearized system of equation around the origin eigenvalues are purely imaginary for $\mu = \mu_0$. Suppose also that for real part part of eigenvalues $\operatorname{Re}(\lambda_1(\mu)) = \operatorname{Re}(\lambda_2(\mu))$ the condition

$$\frac{d}{d\mu}\ \{\operatorname{Re}(\lambda_1(\mu))\}|_{\mu=\mu_0} > 0$$

and that the origin is asymptotically stable for $\mu = \mu_0$.

Then

i) $\mu = \mu_0$ is a bifurcation point for the system

ii) there is an interval (μ_1, μ_0) such that the origin is a stable spiral(focus)

iii) there is an interval (μ_0, μ_2) such that the origin is an unstable spiral(focus), surrounded by a limit cycle (periodic orbit) with size increasing with increasing of μ.

Example. Show that the following system undergoes Hopf bifurcation at $\mu = 0$.

$$\begin{aligned} x_1' &= \mu x_1 - 2x_2 - 2x_1(x_1^2 + x_2^2)^2 \\ x_2' &= 2x_1 + \mu x_2 - x_2(x_1^2 + x_2^2)^2 \end{aligned}$$

Linearized equations are the following:

$$\begin{aligned} x_1' &= \mu x_1 - 2x_2 \\ x_2' &= 2x_1 - \mu x_2 \end{aligned}$$

with matrix $\begin{bmatrix} \mu & -2 \\ 2 & \mu \end{bmatrix}$ with eigenvalues $\lambda_{1,2}(\mu) = \mu \pm 2i$. Therefore $\lambda_{1,2}(0) = \pm 2i$ are purely imaginary.

$\operatorname{Re}\lambda(\mu) = \mu$. and $\frac{d}{d\mu}\operatorname{Re}\lambda(\mu) = 1 > 0$.

The system has a strong Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$ for $\mu = 0$.

$$V_f(x_1, x_2) = -2\left(2x_1^2 + x_2^2\right)(x_1^2 + x_2^2)^2 < 0, (x_1, x_2) \neq (0, 0)$$

that makes the origin asymptotically stable for $\mu = 0$. Then according to the Hopf theorem the system undergoues a bifurcation at $\mu = 0$ and at some small $\mu > 0$ it has instable spiral in the origin, surrounded by a periodic orbit. If it is difficult to find a strong Lyapunov function, one can apply LaSalle's invariance principle.

Exercise.

Show that the equation $x'' + (x^2 - \mu)x' + 2x + x^3 = 0$ has a Hopf bifurcation at $\mu = 0$.

Bifurcations will not be at the exam!!!

Main ideas and tools in the course in ODE

1. Integral form of I.V.P. to ODEs
2. Grönwall's inequality for showing uniqueness and continuity with respect to data.
3. Generalised eigenspaces of matrices. Basis of generalized eigenvectors.
4. Jordan form of matrices. Matrix exponent and logarithm.
5. Transfer matrix. Monodromy matrix.
6. Stability and instability of equilibrium points.
7. Linearization and Grobman Hartman theorem. (iff $\mathrm{Re}(\lambda) \neq 0$)
8. Lyapunov functions.(for stability and for hunting positively invariant sets)
9. LaSalle's invariance principle for hunting ω - limit sets.
10. Idea of solving integral equations by iterations.

Examples

1) Solve the initial value problem

$$\dot{x} = t\, x^3, \qquad x\,(1) = \xi$$

and find maximal intervals for solutions. Give a sketch of the domain for $x(t) = \varphi(t, 1, \xi)$ in the (t, x) plane.

2) Can one conclude which maximal interval have solutions to the similar equation

$$\dot{x} = t^3 x$$

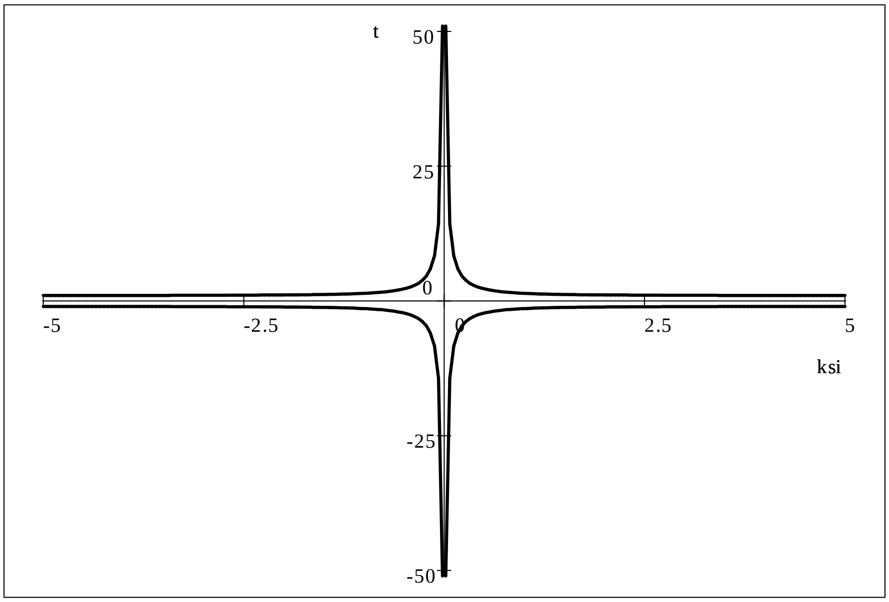
without solving it?

1. **Solution.**

1) It is the equation with separable variables.

$$\begin{aligned}
\frac{dx}{dt} &= tx^3; \qquad x(1) = \xi \\
\int \frac{dx}{x^3} &= \int tdt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2\xi^2} = \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{\xi^2 t^2}{2\xi^2} - \frac{1+\xi^2}{2\xi^2} = \frac{\xi^2 t^2 - \left(1+\xi^2\right)}{2\xi^2} \\
x^2 &= \frac{\xi^2}{\left(1+\xi^2\right) - \xi^2 t^2} = \frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2} \\
x &= \sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi > 0 \\
x &= -\sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi < 0 \\
x &\equiv 0, \quad \xi = 0, \quad \mathit{equilibrium} \\
\left(1+\xi^2\right)/\left(\xi^2\right) &> t^2; \quad t \in \left(-\sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}, \sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}\right) \mathit{OPEN}!!!
\end{aligned}$$

1. The maximal intervals for these solutions are open in accordance with the general theory. One solution $x \equiv 0$ is defined on the whole $\mathbb{R}$. We draw boundaries of the domain for $\varphi(t, 1, \xi)$.



The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $\left|t^3 x\right| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

Estimating Lyapunov functions V and their derivatives $V_f = \nabla V \cdot f$ along trajectories.

Investigation of positivity of functions V and $V_f = \nabla V \cdot f$.

Choosing a Lyapunov function: it must be positive definite: $V(0) = 0$, $V(x) > 0$, $x \neq 0$.

We like to have $V_f = \nabla V \cdot f$ negative definite $V_f < 0$ or at least $\nabla V \cdot f \leq 0$.

Example.

1. Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

   Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

   Hint: applying Lyapunovs theorem, use the elementary inequality

   $$|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

   to estimate indefinite terms with xy.

A more general inequality can be useful for polynomials of higher degree in f:

$$|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}; \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$$

1. **Solution.** Choose a test function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$

   $V_f = x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2$
   $= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + xy \leq -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + 0.5x^2 + 0.5y^2$

   We apply the inequality $2xy \leq \left(x^2 + y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

   $V_f \leq -x^2\left(0.5 - y^2\right) - y^2\left(2.5 - y^2\right)$

   It implies that $V_f < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the origin is asymptotically stable.

   The attracting region is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely $\left(x^2 + y^2\right) < 1/2$.

   The second idea for choosing Lyapunov functions is choice of V of **polynomilas with arbitrary even powers and arbitrary coefficients.**

   Another more clever choice of a test function as

   $$V(x,y) = ax^m + by^n$$

   in particular $V(x,y) = 3x^2 + 2y^2$ works in this particular case:

   $V_f = 6x(-x-2y+xy^2)+4y(3x-3y+y^3) = 4y^4-12y^2-6x^2+6x^2y^2 = -4y^2\left(3 - y^2\right) - 6x^2\left(1 - y^2\right) < 0$

   for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

   One can also observe the asymptotic stability of the origin here by linearization with variational matrix

   $A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2 + 4\lambda + 9 = 0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the domain of attraction.

Poincare - Bendixson theorem and
testing absence of equilibrium points in the positive invariant set.

We try to find an ring shaped domain that is positively invariant and need to check three conditions:

i) The outer boundary of the ring (using a level set of a test function, or a polygon shaped domain testing velosities on each segment of it's boundary)

ii) The inner boundary of the ring (using a level set of a test function, or linearization for identifying a repeller inside a large postively invariant set by applying the Grobman - Hartman theorem)

iii) Check that no equilibrium points exist inside of the ring (missed often by students)

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x, y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x, y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - (2x^2 + y^2)\right)(2x^2 + y^2)$.

It implies that the elliptic shaped ring: $R = \left\{(x, y) : 0.5 \leq (2x^2 + y) \leq 2\right\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system. It is not so easy to see from the system of equations itself. But one can see it easier by cheching first zeroes of $V_f(x, y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x, y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot F(x, y) = 0$ only if $(x, y) = (0, 0)$ or if $(2x^2 + y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x, y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x, y) = (0, 0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

1. $$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

   By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.$\blacksquare$