Theorems and proofs for Ordinary Differential Equations and Modelling MVE161

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Introduction

This text is written as an aid for those that are taking the course MVE161 Ordinary Differential Equations and Modelling the year of 2015. It contains the recommended theorems and proofs from the year 2015, mainly from from the lecture notes.

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1 Banach's Contraction Principle

Definition 1.1. *Let F be a closed subset in a Banach space X, and let T be an operator such that $T : F \to F$ in X. If for all $x,y \in F$ we have that $||T(x)-T(y)|| \leq \theta||x-y||$ for $\theta < 1$, then we call T a* ***contraction****.*

Theorem 1.1. *If $T : F \to F$ is a contraction on a closed set F in a Banach space X, then $\exists$ a unique fixed point $\overline{x}$ to T in F.*

Proof. Take $x_0 \in F \subset X$ arbitrarily. Consider the successive approximations $x_{n+1} = T(x_n)$ for $\{x_n\}_{n=0}^{\infty}$. Then we have from the definition of a contraction that

$$||x_{n+1}-x_n|| = ||T(x_n)-T(x_{n-1})|| \leq \theta||x_n - x_{n-1}|| = \theta||T(x_{n-1})-T(x_{n-2})|| \leq \\ \leq \theta^2||x_{n-1}-x_{n-2}|| \leq \cdots \leq \theta^n||x_1 - x_0||.$$

We claim that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. We prove this claim by taking $m > n$:

$$\begin{aligned} ||x_m - x_n|| &= ||x_m \underbrace{-x_{m-1} - x_{m-2} + x_{m-2} + x_{m-1}}_{=0} - x_n|| \leq \\ &\leq ||x_m - x_{m-1}|| + \cdots + ||x_{m+1} - x_n|| \leq \\ &\leq \theta^n(1+\theta+\cdots+\theta^{m+n+1})||x_1 - x_0|| \leq \\ &\leq \frac{\theta^n}{1-\theta} \to 0, \text{ as } n \to \infty, \end{aligned}$$

since $\theta < 1$. Hence we have proven the claim since $\exists\, \overline{x} \in X : \lim_{n\to\infty} x_n = \overline{x}$.

Hence, we have that

$$||T(\overline{x}) - \overline{x}|| \to 0 \text{ as } n \to \infty \Rightarrow ||T(\overline{x}) - \overline{x}|| \equiv 0.$$

We have thus shown the existence of a fixed point. To show the uniqueness let $y = T(y), x = T(x)$ for $x,y \in F$ (different fixed points). Thus

$$||x-y|| = ||T(x)-T(y)|| \leq \theta||x-y|| \Rightarrow ||x-y|| = 0 \Rightarrow x = y,$$

which means that x and y are the same fixed points and thus the proof is complete. □

2 Picard-Lindelöf's Theorem

Definition 2.1. *We call $f(x)$ a* ***Lipshitz function*** *in F if*

$$|f(x)-f(y)| \leq L|x-y|, \forall\, x,y \in F$$

for some constant L.

Theorem 2.1. *Let $f(x,t)$ be continuous on $S = \{(t,x) : |t - t_0| \leq a, |x - x_0| \leq b\}$ and let $f(x,t)$ satisfy a Lipshitz condition in x with the Lipshitz constant L. Let $M = \max\{|f(x,t)| : (t,x) \in S\}$. Then $\exists$ a unique solution to the initial value problem (IVP) $x' = f(t,x)$, $x(t_0) = x_0$ on $I = \{t : |t - t_0| \leq \alpha\}$, where $\alpha < \min\{a,b/M,1/L\}$.*

Proof. Let B be the closed subset of $C(I)$ defined by $B = \{\varphi \in C(I) : |\varphi(t) - x_0| \leq b, t \in I\}$. We define a mapping on B by

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s,\varphi(s))\,\mathrm{d}s.$$

First we'll show that T maps B onto B. Since $f(t,\varphi(t))$ is continuous, T maps B onto $C(I)$. If $\varphi \in B$, then

$$\begin{aligned}|T\varphi(t) - x_0| = \left|\int_{t_0}^{t} f(s,\varphi(s))\,\mathrm{d}s\right| &\leq \int_{t_0}^{t} |f(s,\varphi(s))|\,\mathrm{d}s \leq \\ &\leq M|t - t_0| \leq M\alpha < b.\end{aligned}$$

Hence $T\varphi \in B$. Next we'll prove that T is a contraction. Let $||x||$ be the sup-norm of x. Then

$$\begin{aligned}|Tx(t) - Ty(t)| &\leq \int_{t_0}^{t} |f(s,x(s)) - f(s,y(s))|\,\mathrm{d}s \leq \\ &\leq L\int_{t_0}^{t} |x(s) - y(s)|\,\mathrm{d}s \leq L||x - y|||t - t_0| \leq \\ &\leq L\alpha||x - y|| = \theta||x - y||,\end{aligned}$$

where $\theta = L\alpha < 1$. Hence $||Tx - Ty|| \leq \theta||x - y||$. Thus the proof is complete by using Banach's contraction principle. □

3 Dependence of solutions on initial data (Th. 2.2.4)

Theorem 3.1 (Grönwalls inequality)**.** *Let $\lambda(t)$ be real and continuous and let $\mu(t) \geq 0$ and continuous on $[a,b]$. If a continuous function $y(t)$ satisfies*

$$y(t) \leq \lambda(t) + \int_{a}^{t} \mu(s)y(s)\,\mathrm{d}s, \quad \text{for } t \in [a,b],$$

then it holds that

$$y(t) \leq \lambda(t) + \int_{a}^{t} \lambda(s)\mu(s)\exp\left(\int_{s}^{t} \mu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

If $\lambda \equiv c$, for a constant c:

$$y(t) \leq \lambda\exp\left(\int_{a}^{t} \mu(s)\,\mathrm{d}s\right).$$

Theorem 3.2. *Let $x_1(t)$ and $x_2(t)$ be two functions on $a \leq t \leq b$ such that*

$$
\begin{aligned}
&|x_1(a) - x_2(b)| \leq \delta \\
&|x_i(t) - f(t)| \leq \epsilon_i, \ i = 1{,}2 \\
&|f(t{,}x_1) - f(t{,}x_2)| \leq L|x_1 - x_2|, \quad \text{i.e } f \text{ Lipschitz.}
\end{aligned}
$$

Then it holds that

$$|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)} + \frac{1}{L}(\epsilon_1 + \epsilon_2)(e^{L(t-a)} - 1)$$

Proof. Consider $\gamma(t) = x_1(t) - x_2(t)$. Thus

$$|\gamma'(t)| = |f(t{,}x_1(t)) - f(t{,}x_2(t))| + \underbrace{\epsilon_1 + \epsilon_2}_{=\epsilon} \leq L|\gamma(t)| + \epsilon.$$

Further,

$$\gamma(t) = \gamma(a) + \int_a^t \gamma'(s)\, \mathrm{d}s,$$

which gives us

$$|\gamma(t)| \leq |\gamma(a)| + \int_a^t (L|\gamma(s)| + \epsilon)\, \mathrm{d}s.$$

Now we use Grönwall's inequality to get

$$|\gamma(t)| \leq \delta + \epsilon(t-a) + \int_a^t L(\delta + \epsilon(s-a))e^{L(t-s)}\, \mathrm{d}s.$$

Now we integrate by parts and we get the desired result. □

4 Global existence of solutions satisfying an a priori estimate

Theorem 4.1. *Let $x_1 = f(t{,}x)$ for $f \in C(D)$ and $\mathbb{R}(M{,}0) \times \mathbb{R} \subset D$. Suppose that any solution to the IVP satisfies the estimate $|x(t)| \leq M$ if $x(t)$ exists. Then it holds that all solutions starting in $B(M{,}0)$ exist $\forall t \in \mathbb{R}$.*

Proof. Local existence and uniqueness follows from the fact that functions $f(t{,}x) = A(t)x + h(t)$ are local Lipschitz because

$$\frac{\partial}{\partial x_k}(A(t)x) = [A_{\ell k}(t)]_{\ell=1}^n, \quad |A_{\ell k}(t)| < L \text{ on } [t_0{,}t_0 + T].$$

Now we want to prove that the largest existence time b^- is infinite. Consider $\varphi(t) - \varphi(0)$ for some solution $\varphi(t)$:

$$\varphi(t) - \varphi(0) = \int_{t_0}^t f(s{,}\varphi(s))\, \mathrm{d}s = \int_{t_0}^t [f(s{,}\varphi(s)) - f(s{,}x_0)]\, \mathrm{d}s + \int_{t_0}^t f(s{,}x_0)\, \mathrm{d}s.$$

This gives us that

$$\begin{aligned}|\varphi(t)-\varphi(0)| &\leq \int_{t_0}^{t} |A(s)(\varphi(s)-x_0)|\,\mathrm{d}s + \int_{t_0}^{t} |A(s)x_0 + h(s)|\,\mathrm{d}s \leq \\ &\leq \int_{t_0}^{t} ||A(s)||\,|\varphi(s)-x_0|\,\mathrm{d}s + \int_{t_0}^{t} A(s)x_0 + h(s)\,\mathrm{d}s \leq \\ &\leq L\int_{t_0}^{t} |\varphi(s)-x_0|\,\mathrm{d}s + \delta,\end{aligned}$$

where

$$L = \sup_{s\in I} ||A(s)|| \quad \text{and} \quad \delta = T\max_{s\in I}\,[A(s)x_0 + h(s)]$$

for a time interval $I = [t_0, t_0 + T]$. This gives us that

$$|\varphi(t) - x_0| \leq \delta e^{LT}.$$

Suppose that $b^- < \infty$. Take $t_0 + T = b^-$. This implies that the solution can be extended up to the time b^- and further, which is a contradiction. Thus the proof is complete. □

5 Dimension of the solution space

Theorem 5.1. *Consider a homogenous linear system $x' = A(t)x(t)$ where $A(t)$ is a continuous $n \times n$ matrix function on $\mathbb{R}$, $x(t) \in \mathbb{R}^n$. We know that an IVP always has a solution $x(t)$, $t \in \mathbb{R}$. The set V of all solutions to (LH) is an n dimensional vector space.*

Proof. Let $\varphi(t), \psi(t)$ be two solutions to (LH), i.e.:

$$\varphi'(t) = A(t)\varphi(t), \quad \psi'(t) = A(t)\psi(t).$$

We add them to get $(\varphi(t)+\psi(t))' = A(t)(\varphi(t)+\psi(t))$. Thus $\varphi(t)+\psi(t)$ is a solution to (LH) and since $\alpha\varphi(t)$ is a solution to (LH) it means that the solution is a vector space. Now we want to prove that V has dimension n. Consider solutions $\varphi_i(t)$ with initial data $\varphi_i(0) = e_i$ where

$$e_i = \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix},$$

is a vector with a 1 on index i and zero elsewhere. Thus any solution is expressed with $\{e_i\}_{i=1}^n$. Take $\psi(0) = \xi \in \mathbb{R}^n$. So $\psi(t) = \sum_{i=1}^n \xi_i e_i(t)$ is a solution. We must show that $\varphi_i(t)$ are linearly independent $\forall t \in \mathbb{R}$. We prove it by contradiction. Suppose that for some t^* $\exists \{\alpha_i\}_{i=1}^n, \alpha_i \neq 0$ such that

$$\sum_{i=1}^n \alpha_i \varphi_i(t^*) = 0.$$

But $\sum_{i=1}^n \alpha_i \varphi_i(t^*)$ is also a solution to (LH). Now observe that (LH) has trivial solution $\varphi_0 \equiv 0$. Solutions φ_0 and $\sum_{i=1}^n \alpha_i \varphi_i(t)$ coincide in $t = t^*$ and are equal to zero there. Thus

$$\varphi_0 = \sum_{i=1}^n \alpha_i \varphi_i(t) \equiv 0,$$

because of the uniqueness of solutions. Thus

$$\sum_{i=1}^n \alpha_i \varphi_i(0) \neq \sum_{i=1}^n \alpha_i \varphi_i(t) = 0$$

impossible because e_i are linearly independent. This completes the proof. □

6 Liouville's formula

Theorem 6.1. *For $x' = A(t)x$ it holds that*

$$\det \phi(t) = \phi(t_0) \exp\left(\int_{t_0}^t \mathrm{tr}(A(s))\, \mathrm{d}s \right),$$

where ϕ is the matrix valued solution.

Proof. We will prove the theorem in the case where dim$= 2$. The idea is to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det \phi(t)) = \Big(\mathrm{tr}(A(t))\Big) \det \phi(t).$$

First we have that

$$\mathrm{tr}(A(t)) \Leftrightarrow \phi(t+\epsilon) = \phi(t) + \epsilon A(t)\phi(t) + O(\epsilon^2) = \phi(t)[I + \epsilon A(t) + O(\epsilon^2)].$$

Further

$$\det(\phi(t+\epsilon)) = \det(\phi(t))\Big(\det(I + \epsilon A + O(\epsilon^2)\Big),$$

where

$$\det(I+\epsilon A+O(\epsilon^2)) = \det \begin{pmatrix} 1+\epsilon a_{11} + O(\epsilon^2) & \epsilon a_{12} + O(\epsilon^2) \\ \epsilon a_{21} + O(\epsilon^2) & 1 + \epsilon a_{22} + O(\epsilon^2) \end{pmatrix} = 1+\epsilon(a_{11}+a_{22})+O(\epsilon^2),$$

and

$$\det\phi(t+\epsilon) = (1+\epsilon\,\mathrm{tr}(A)+O(\epsilon^2))\det\phi(t).$$

The last equation gives us

$$\frac{1}{\epsilon}\det\phi(t+\epsilon)-\det\phi(t) = \mathrm{tr}\ (A)\det\phi(t)+\underbrace{O(\epsilon)}_{\to 0,\epsilon\to 0}.$$

Solving:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det\phi(t)) = \mathrm{tr}\ (A)(\det\phi(t)).$$

□

7 Stability of solutions with constant matrix

Theorem 7.1. *(i) If all eigenvalues λ to A satisfy $Re(\lambda) < 0$, then*

$$\exists\alpha > 0, k > 0 : ||e^{At}|| \leq ke^{-\alpha t}.$$

(It implies that $|x(t)| = |e^{At}x_0| \leq ke^{-\alpha t}|x_0| \to 0, t\to\infty$.)

(ii) If $Re(\lambda) \leq 0$ and all eigenvalues λ with $Re(\lambda)$=0 have algebraic and geometric multiplicity equal ($m_i = n_i$), then

$$\exists M > 0 : ||e^{At}|| \leq M, \forall t \geq 0.$$

(It implies that $|x(t)| \leq M|x_0|, \forall t$.)

Proof. (i) We begin by writing $e^{At} = P^{-1}e^{Jt}P$ for some constant matrix P, where J is a Jordan matrix. Thus we have that

$$||e^{At}|| \leq ||P||\,||P^{-1}||\,||e^{Jt}||.$$

Further it holds that

$$||e^{Jt}|| = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & \frac{t^k}{k!}e^{\lambda_1 t} \\ & \ddots & \ddots & \vdots \\ & & \ddots & te^{\lambda_1 t} \\ 0 & & & e^{\lambda_1 t}\end{pmatrix} \leq e^{-\alpha t}.$$

We can choose $\alpha = \min(-\mathrm{Re}(\lambda)) - \varepsilon$. Hence $|e^{-\varepsilon t}t^p| < k$ for any $p > 0$ and $\varepsilon > 0$. Thus

$$||e^{At}|| \leq ||P||\,||P^{-1}||\,e^{-\alpha t}\underbrace{||e^{-\varepsilon t}t^p||}_{<\infty} \leq ke^{-\alpha t}.$$

(ii) If $\mathrm{Re}(\lambda) \leq 0$ for all λ and $m_i = n_i$ for those eigenvalues that are equal to zero, then we get that $||e^{At}|| \leq M\,\forall t$. □

8 General solution for a linear ODE with constant matrix

Theorem 8.1. *Let $\lambda_1, \ldots, \lambda_s$ be all distinct eigenvalues to A, and $M(\lambda_1, A), \ldots, M(\lambda_s, A)$ be corresponding generalised eigenspaces. Then for $x_0 = \sum_{j=1}^{s} x^{0,j}$; $x^{0,j} \in M(\lambda_j, A)$ it holds that*

$$x(t) = e^{At}x_0 = \sum_{j=1}^{s} \left(\sum_{k=0}^{k_j-1} (A - \lambda_j I)^k \frac{t^k}{k!} x^{0,j} e^{\lambda_j t} \right).$$

Proof. Consider

$$e^{At}x_0 = e^{(A-\lambda I)t}e^{\lambda t}x_0 = \sum_{k=0}^{\alpha} \frac{(A-\lambda I)^k t^k}{k!} x_0 e^{\lambda t} = \\ = \sum_{k=0}^{n_\lambda - 1} \frac{(A-\lambda I)^k t^k}{k!} x_0 e^{\lambda t},$$

if $x_0 \in M(\lambda, A)$. This yields

$$x(t) = \sum_{j=1}^{s} \left[\sum_{k=0}^{n_\lambda - 1} (A - \lambda_j I)^k \frac{t^k}{k!} x^{0,j} e^{\lambda t} \right].$$

□

9 Floquet theorem

Theorem 9.1. *Let $\Phi(t)$ be a fundamental matrix of $x' = A(t)x$, for $A(t+T) = A(t), A(t)$ continuous. Then*

(i) $\Phi(t+T)$ *is also a fundamental matrix*

(ii) $\exists P(t) \in \mathbb{C}^{n \times n}$*, periodic and non singular. Further,* $\exists R \in \mathbb{C}^{n \times n}$ *such that* $\Phi(t) = P(t)e^{tR}$.

Proof. (*i*) Let $\Psi(t) = \Phi(t+T)$. Then

$$\Psi'(t) = \Phi'(t+T) = \{\Phi(t) \text{ non singular}, \Phi'(t) = A(t)\Phi(t)\} = A(t+T)\Phi(t+T) = \\ = \{A \text{ periodic}\} = A(t)\Phi(t+T) = A(t)\Psi(t).$$

(*ii*) $\exists C$ constant matrix such that $\Phi(t+T) = \Phi(t)C$, C independent of time. The matrix C "maps" states at time t to states at time $t+T$. C is non singular, which gives us that $\exists R \in \mathbb{C}^{n\times n}$ such that $C = e^{TR}$ to mimic a shift in time on T. We get: $\Phi(t+T) = \Phi(T)e^{TR}$. The goal is to find $P(t)$ such that $\Phi(t) = P(t)e^{tR}$. Take $P(t) = \Phi(t)e^{-tR}$. Check if $P(t)$ is periodic:

$$P(t+T) = \Phi(t+T)e^{-(t+T)R} = \Phi(t)e^{TR}e^{-tR}e^{-TR} = \Phi(t)e^{-tR} = P(t).$$

□

10 Stability of periodic linear systems

Theorem 10.1. *$x' = A(t)x, A(t) = A(t+T)$.*

(i) If all characteristic multipliers (Floquet) $|\lambda_i| < 1$, then all solutions $x(t) \to 0$ as $t \to \infty$. This is equivalent to that all characteristic (Floquet) exponents ρ_i have $Re(\rho_i) < 0$.

(ii) If $|\lambda_i| \leq 1(Re(\rho_i) \leq 0)$ and all λ_i with $|\lambda_i| = 1$ have algebraic multiplicity equal to geometric multiplicity, $n_i = m_i$, then $|x(t)| \leq C$ (bounded).

Proof. We begin by making a change of variables $x(t) = P(t)y(t)$, where $P(t) = P(t+T)$ such that the monodromy matrix $\Phi(t+T) = \Phi(t)e^{TR}, \Phi(t) = P(t)e^{tR}$. We begin by showing that $y' = Ry$. We compute $x'(t) = P'(t)y + P(t)y'$. We also have that $x'(t) = Ax = AP(t)y$. This gives us that

$$AP(t)y = P'(t)y + P(t)y'. \tag{1}$$

Now we need $P'(t)$. To get it we calculate

$$\Phi'(t) = A\Phi(t) = \{\Phi(t) = P(t)e^{tR}\} = P'(t)e^{tR} + P(t)Re^{tR}.$$

Multiplication by e^{-tR} yields

$$A\Phi(t)e^{-tR} = P'(t) + P(t)R.$$

Thus $P'(t) = A\Phi e^{-tR} - P(t)R$. Equation (1) gives us that

$$AP(t)y = A\Phi e^{-tR}y - P(t)Ry + P(t)y'.$$

Therefore

$$0 = -P(t)Ry + P(t)y' \Rightarrow y' = Ry \Rightarrow y(t) = e^{Rt}y(0).$$

Finally

$$x(t) = P(t)y(t) = P(t)e^{Rt}y(0).$$

Note that $P(t)$ is bounded, periodic and continuous. We have that ρ_i are eigenvalues to R. Thus

$$(i) \;\; \text{Re}(\rho_i) < 0 \Rightarrow |e^{Rt}y(0)| \to 0, t \to \infty,$$

and therefore $|x(t)| \to 0, t \to \infty$, $\forall y(0)$.

(ii) If $\text{Re}(\rho_i) \leq 0$ and those with $\text{Re}(\rho_i) = 0$ having $n_i = m_i$, we get that $||e^{tR}|| \leq C$. □

11 Stablity of autonomous non-linear ODEs by linearization

Theorem 11.1. *Let x^* be an equilibrium of the system $x' = f(x), f(x^*) = 0$. Let $D_x f(x^*)$ be the Jacobi matrix of f in x^*. Let $f \in C^2(V)$, V neighbourhood of x^*. If all Re(λ) eigenvalues to $D_x f(x^*)$ are negative, then x^* is an asymptotically stable equilibrium point.*

Proof. Taylor expansion around x^* gives us

$$f(x^* + y) = f(x^*) + D_x(x^*)y + g(y), \frac{g(y)}{|y|} \to 0, |y| \to 0.$$

$$\begin{aligned} y' &= Ay + g(y), \\ y &= e^{A(t-t_0)}y(t_0) + \int_{t_0}^t e^{A(t-s)} g(y(s))\,\mathrm{d}s. \end{aligned}$$

We use that $|e^{A(t-t_0)}y(t_0)| \leq Me^{-\sigma t}|y(t_0)|$ since all $\text{Re}(\lambda) < 0$. Thus

$$|y(t)| \leq Me^{-\alpha t}|y(t_0)| + \int_{t_0}^t Me^{-\sigma(t-s)}|g(y(s))|\,\mathrm{d}s.$$

The fact that $g(y) = O(y)$ means that $\forall \varepsilon > 0 \, \exists \, \delta(\varepsilon) > 0 \, : \, |g(y)| < |y|\varepsilon$ for $|y| < \delta(\varepsilon)$. Thus

$$|y(t)| \leq Me^{-\alpha t}|y(t_0)| + M\varepsilon \int_{t_0}^t e^{-\sigma(t-s)}|y(s)|\,\mathrm{d}s.$$

Consider $z(t) = y(t)e^{\sigma t}$.

$$|y(t)|e^{\sigma t} \leq M(|y(t_0)|e^{\sigma t_0}) + M\varepsilon \int_{t_0}^t |y(s)|e^{\sigma s}\,\mathrm{d}s.$$

This yields

$$z(t) \leq Mz(t_0) + M\varepsilon \int_{t_0}^t z(s)\,\mathrm{d}s.$$

Grönwalls inequality gives us that

$$z(t) \leq Mz(t_0)e^{M\varepsilon(t-t_0)} \Rightarrow |y(t)| \leq M(|y(t_0)|e^{\sigma t_0})e^{-\sigma t+M\varepsilon t-M\varepsilon t_0} = \\ = M(e^{\sigma t_0 - M\varepsilon t_0})|y(t_0)|e^{-(\sigma - M\varepsilon)t}.$$

M is independent of g, ε can be chosen arbitrarily small by choosing $|y(s)|$ small enough and $|y(t_0)|$ is the definition of initial data from the equilibrium point x^*. Thus

$$e^{-(\sigma-M\varepsilon)t} \to 0, t \to \infty,$$

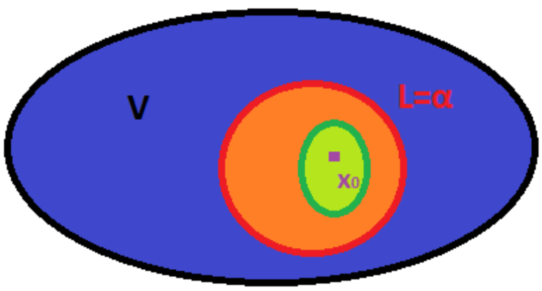
if $M\varepsilon$ is chosen such that $M\varepsilon < 0$. □

12 Stability of fixed points to autonomous ODE by Lyapunov functions

Theorem 12.1. *Suppose the equation $x' = f(x)$ has a fixed point $x_0 \in M$, open. Further $f \in C^1$ and $f : M \to \mathbb{R}^n$. If there is a Lyapunov function around x_0, then x_0 is a stable equilibrium point.*

Proof. Take $\delta > 0$ so small that $B_\delta(x_0) \subset V$. Let $S_\delta(x_0)$ be the boundary of $B_\delta(x_0)$. Further, $\alpha = \min(L(x), x \in S_\delta(x_0)) > 0$, because $L(x) > 0$ outside of x_0. Let $V_1 = \{x \in B_\delta(x_0) : L(x) < \alpha\}$. Thus V_1 is an open set as inverse picture of an open set $(-\infty,\alpha)$ by a continous function. Also $x_0 \in V_1, L(x_0) = 0$. Thus V_1 is a neighbourhood of x_0.

No one trajectory starting in V_1 will leave $B_\delta(x_0) \Rightarrow x_0$ is a stable equilibrium point, because $\forall B_\delta(x_0)\, \exists V_1$-neighbourhood of x_0 such that $\varphi(t,x) \in B_\delta(x_0)$ if $x \in V_1$. □



13 Instability of fixed points to autonomous ODE by Lyapunovs method

Theorem 13.1. *If $\exists$ a neighbourhood U of 0 and $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $0 \in \Omega$, such that V and V' are positive definite on $U \cap \Omega \backslash \{0\}$. Then the equilibrium 0 is completely unstable, i.e. 0 is a repeller.*

Proof. Suppose 0 is not a repeller. Then there exists a neighbourhood Ω_0 such that $\varphi(t,x_0)$ stays in $U \cap \Omega_0 \backslash \{0\}$ for some $x_0 \in U \cap \Omega_0 \backslash \{0\}$. Then $V(\varphi(t,x_0)) \geq V(x_0) > 0$ for all $t \geq 0$. Let $\alpha = \inf\{V'(x) : x \in U \cap \Omega_0, V(x) \geq V(x_0)\} > 0$. Then

$$V(\varphi(t,x_0)) = V(x_0) + \int_0^t V'(\varphi(s,x_0)) \,\mathrm{d}s \geq V(x_0) + \alpha t.$$

Since $\varphi(t,x_0)$ remains in $U \cap \Omega_0 \backslash \{0\} \forall t \geq 0$, $V(\varphi(t,x_0))$ is bounded for $t \geq 0$. This gives us that for t sufficiently large we obtain a contradiction from the above inequality. □

14 Bendixson's negative criterion

Theorem 14.1. *Consider the system*

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y). \end{cases}$$

If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is of the same sign in a simple-connected domain D in $\mathbb{R}^2$, then there are no periodic orbits on D.

Proof. Suppose there is a periodic orbit $C = \{x(t),y(t)\}_{0 \leq t \leq T}$. Green's theorem gives

$$\begin{aligned} \int\int_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) &= \oint_C -g(x,y)\,\mathrm{d}x + f(x,y)\,\mathrm{d}y = \\ &= \int_0^T \left[-g(x(t),y(t))x'(t) + f(x(t),y(t))y'(t)\right] \,\mathrm{d}t = 0. \end{aligned}$$

But $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is of the same sign, thus

$$\int\int_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)$$

is either positive or negative. This is a contradiction. □