MATHEMATICS University of Gothenburg and Chalmers Univ. of Technology
Examination in algebra: MMG 500 and MVE 150, 2020-03-20.
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1. Let $*$ be the binary operation on $\mathbf{Z}$ for which $m * n = m+n-1$.

a) Is this operation associative ? 2p

b) Is there a neutral element in $\mathbf{Z}$ for this operation $*$? 1p

c) Is $(\mathbf{Z}, *)$ a group ? 2p

2. Determine the number of non-isomorphic abelian groups of order 2020 and write down one group in each isomorphism class. 4p

3. Let G be finite group and N be a normal subgroup of order $o(N)$ relatively prime to its index $[G: N]$. Prove that any element $g \in G$ with $g^{o(N)}=e$ belongs to N. (Hint : Use G/N.) 4p

4. Let R be a commutative ring and r, s be elements of R such that rs is a zero divisor in R. Prove that r or s is a zero divisor in R. 3p

5. Let $\sqrt{-6} = \sqrt{6}i$ and $\mathbf{Z}[\sqrt{-6}] = \{a + b\sqrt{6}i : a,b \in \mathbf{Z}\}$.

a) Show that $\mathbf{Z}[\sqrt{-6}]$ is a subring of $\mathbf{C}$. 2p

b) Determine the units in $\mathbf{Z}[\sqrt{-6}]$. 1p

c) Prove that the numbers $2, 3, \pm\sqrt{-6}$ are irreducible in $\mathbf{Z}[\sqrt{-6}]$ and use this to explain why $\mathbf{Z}[\sqrt{-6}]$ is not a unique factorisation domain. 2p

6. Let K and L be two fields and $K \times L$ be the direct product of these rings.
(Note that $K \times L$ is called the direct sum of K and L in Durbin´s book.)

a) List as many ideals of $K \times L$ as you can and motivate why they are ideals. 2p

b) Can there be more than four ideals in such a ring? 2p

You may use the theorems in Durbin´s book, but all claims that are made must be motivated.

Solutions to exam in algebra: MMG 500 and MVE 150, 2020-03-20.

1a) The operation is associative as $l*(m*n) = l*(m+n-1) = l+(m+n-1) = (l+m-1)+n = (l*m)*n$

b) 1 is a neutral element for the given operation as $m*1 = (m+1)-1 = m$ and $1*m = (1+m)-1 = m$

c) $*$ is an associative operation on $\mathbf{Z}$ with neutral element 1. We have also for any $m\in\mathbf{Z}$ that $2-m$ is inverse to m as $m*(2-m) = (m+(2-m)-1) = 1$ and $(2-m)*m = ((2-m)+m)-1 = 1$. Hence $(\mathbf{Z}, *)$ is a group with 1 as identity.

2. We know by the fundamental theorem for abelian groups that any finite abelian group is isomorphic to a product of cyclic groups of prime power order. As 2020 has prime factorisation $2^2\times5\times101$ we thus conclude that any abelian group of order 2020 is isomorphic to $\mathbf{Z}_4\times\mathbf{Z}_5\times\mathbf{Z}_{101}$ or $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_5\times\mathbf{Z}_{101}$. As by the above theorem these two groups are non-isomorphic, it follows that there are exactly two isomorphism classes of abelian groups of order 2020. (Remark : One can also represent the two isomorphism classes by $\mathbf{Z}_{2020}$ and $\mathbf{Z}_2\times\mathbf{Z}_{1010}$.)

3. As $o(N)$ and $o(G/N)$ are relatively prime we may find integers k and m with $1=o(N)k+o(G/N)m$. Therefore, $Ng=((Ng)^{o(N)})^k((Ng)^{o(G/N)})^m$ in G/N by the power laws. Further, $(Ng)^{o(G/N)}=Ne$ for all $Ng\in G/N$ by a corollary of Lagrange´s theorem. Hence $Ng=((Ng)^{o(G/N)})^m=(N(g^{o(G/N)}))^m=(Ne)^m=N$ and $g\in N$ for any $g\in G$ with $g^{o(N)}\in N$.

4. If rs is a zero divisor in R, then $rs\neq0$ and $(rs)t=0$ for some $t\neq0$ in R. If now $st\neq0$, then r is a zero divisor as $r(st)=0$ and $r\neq0$. If instead $st=0$, then s is a zero divisor as $s\neq0$ and $t\neq0$.

5a). This follows from the subring criterion as $(a+b\sqrt{6}i)\pm(c+d\sqrt{6}i) = (a\pm c)+(b\pm d)\sqrt{6}i \in \mathbf{Z}[\sqrt{-6}]$ and $(a+b\sqrt{6}i)(c+d\sqrt{6}i) = (ac-6bd)+(ad+bc)\sqrt{6}i \in \mathbf{Z}[\sqrt{-6}]$ for all a,b,c,d in $\mathbf{Z}$.

b) Let N be the norm map with $N(a+b\sqrt{6}i)=a^2+6b^2$. This map is multiplicative as $N(a+b\sqrt{6}i)=|a+b\sqrt{6}i|^2$. So if $a+b\sqrt{6}i$ is inverse to $c+d\sqrt{6}i$ in $\mathbf{Z}[\sqrt{-6}]$ then $(a^2+6b^2)(c^2+6d^2) = N((a+b\sqrt{6}i)(c+d\sqrt{6}i)) = N(1) = 1$. But then $a^2+6b^2=1$ as $a^2+6b^2\in\mathbf{N}$ and $c^2+6d^2\in\mathbf{N}$ **.** We have thus for any unit $a+b\sqrt{6}i$ that $a^2=1-6b^2\in(0,1]\cap\mathbf{Z}=\{1\}$, which implies that $a=\pm1$ and $b=0$. There are therefore just two units in $\mathbf{Z}[\sqrt{-6}]$ given by ±1.

c) If $a+b\sqrt{6}i$ and $c+d\sqrt{6}i$ are in $\mathbf{Z}[\sqrt{-6}]$, then $N((a+b\sqrt{6}i)(c+d\sqrt{6}i))=(a^2+6b^2)(c^2+6d^2)$- Also, if $b\neq0$ then $a^2+6b^2\geq6b^2\geq6$ and if $b=0$ then $a^2+6b^2=a^2\geq4$ if not $a=\pm1$. We have therefore, if of $a+b\sqrt{6}i$ and $c+d\sqrt{6}i$ are non-units that $a^2+6b^2\geq4$, $c+d\sqrt{6}i\geq4$ and $N((a+b\sqrt{6}i)(c+d\sqrt{6}i))\geq16$. The non-units 2,3, $\sqrt{6}i$, $-\sqrt{6}i$ are thus all irreducible as $N(2)=4$, $N(3)=9$ and $N(\sqrt{6}i)=N(-\sqrt{6}i)=6$. We have thus two different prime factorisations of 6 given by 2×3 and $(\sqrt{6}i)(-\sqrt{6}i)$ in $\mathbf{Z}[\sqrt{-6}]$, where 2 and 3 are not associates to $\pm\sqrt{6}i$. So $\mathbf{Z}[\sqrt{-6}]$ is not a UFD.

6a). If $I\subseteq R$ and $J\subseteq S$ are ideals in the rings R and S, then it follow from the subgroup criterion that $I\times J$ is an additive subgroup of $R\times S$. Further, if $(r,s)\in R\times S$ and $(i,j)\in I\times J$, then $(r,s)(i,j)=(ri,sj)$ and $(i,j)(r,s)=(ir,js)$ are both in $I\times J$, such that $I\times J$ is an ideal of $R\times S$. If we apply this to the two trivial ideals in R and S, then we get four ideals $R\times S$, $R\times\{0_S\}$, $\{0_R\}\times S$, $\{0_R\}\times\{0_S\}$ in $R\times S$ for any two rings and in particular for fields $R=K$ and $S=L$.

b) Let I be an ideal in $K\times L$ for two fields K and L. There are then four cases.

Case 1: There exists $(a,b)\in I$ with $a\neq0_R$ and $b\neq0_S$. Then $(a^{-1},b^{-1})(a,b)=(1_R,1_S)\in I$ such that $I=K\times L$ as any $(r,s)=(r,s)(1_R,1_S)\in I$.

Case 2: $I\subseteq K\times\{0_L\}$, but $I\neq\{0_K\}\times\{0_L\}$. We have then that $(a,0)\in I$ for some $a\neq0$ and hence that $(a^{-1},0)(a,0))=(1_R,0_L)\in I$. But then $I=K\times\{0_L\}$ as any element $(r,0)\in K\times\{0_L\}$lies in I as $(r,0)=(r,0)(1,0)\in I$.

Case 3: $I\subseteq\{0_K\}\times L$, but $I\neq\{0_K\}\times\{0_L\}$. Then $(0_R,1_L)\in I$ and $I=\{0_K\}\times L$ by the same arguments as in case 2.

Case 4: $I=\{0_K\}\times\{0_L\}$.

There are thus no other ideals in $K\times L$ than $K\times L$, $K\times\{0_L\}$, $\{0_K\}\times L$ and $\{0_K\}\times\{0_L\}$ if K and L are fields.