1. Let $\ast$ be the binary operation on $\mathbb{Z}$ for which $m \ast n = m+n-1$.
   a) Is this operation associative?  
   b) Is there a neutral element in $\mathbb{Z}$ for this operation $\ast$?  
   c) Is $(\mathbb{Z}, \ast)$ a group?

2. Determine the number of non-isomorphic abelian groups of order 2020 and write down one group in each isomorphism class.

3. Let $G$ be finite group and $N$ be a normal subgroup of order $o(N)$ relatively prime to its index $[G: N]$. Prove that any element $g \in G$ with $g^{o(N)} = e$ belongs to $N$. (Hint : Use $G/N$.)

4. Let $R$ be a commutative ring and $r, s$ be elements of $R$ such that $rs$ is a zero divisor in $R$. Prove that $r$ or $s$ is a zero divisor in $R$.

5. Let $\sqrt{-6} = \sqrt{6}i$ and $\mathbb{Z}[\sqrt{-6}] = \{a+b\sqrt{6}i : a, b \in \mathbb{Z}\}$.
   a) Show that $\mathbb{Z}[\sqrt{-6}]$ is a subring of $\mathbb{C}$.  
   b) Determine the units in $\mathbb{Z}[\sqrt{-6}]$.  
   c) Prove that the numbers 2, 3, $\pm \sqrt{-6}$ are irreducible in $\mathbb{Z}[\sqrt{-6}]$ and use this to explain why $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorisation domain.

6. Let $K$ and $L$ be two fields and $K \times L$ be the direct product of these rings.  
   a) List as many ideals of $K \times L$ as you can and motivate why they are ideals.  
   b) Can there be more than four ideals in such a ring?

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You may use the theorems in Durbin’s book, but all claims that are made must be motivated.
1a) The operation is associative as \( l*(m*n) = l*(m+n-1) = (l+m)+n = (l+m)*n \)
b) 1 is a neutral element for the given operation as \( m*1 = (m+1)-1 = m \) and \( 1*m = (1+m)-1 = m \)
c) \( * \) is an associative operation on \( \mathbb{Z} \) with neutral element 1. We have also for any \( m \in \mathbb{Z} \) that \( 2-m \) is inverse to \( m \) as \( m*(2-m) = (m + (2-m)) - 1 = 1 \) and \( (2-m)*m = (m(2-m)) + m = -1 \).

2. We know by the fundamental theorem for abelian groups that any finite abelian group is isomorphic to a product of cyclic groups of prime power order.

3. As \( o(N) \) and \( o(GN) \) are relatively prime we may find integers \( k \) and \( m \) with \( 1 = o(N)k + o(GN)m \). Therefore, \( Ng=((Ng)^(mN))^k \cdot ((Ng)^{mGN})^m \) in \( GN \) by the power laws. Further, \( (Ng)^{o(GN)} = Ne \) for all \( Ng \in GN \) by a corollary of Lagrange’s theorem. Hence \( Ng=((Ng)^{o(GN)})^k \cdot (Ne)^m \) is isomorphic to \( Z_{2 \times Z_{1010}} \). As by the above theorem these two groups are non-isomorphic, it follows that there are exactly two isomorphism classes of abelian groups of order 20.

4. If \( rs \) is a zero divisor in \( R \), then \( rs \equiv 0 \) and \( (rs) \equiv 0 \) for some \( t \equiv 0 \) in \( R \). If now \( st \equiv 0 \), then \( r \) is a zero divisor as \( (rs) \equiv 0 \) and \( rt \equiv 0 \).

5a) This follows from the subring criterion as \( (a+b\sqrt{6}i) \pm (c+d\sqrt{6}i) = ((a \pm c) + (b \pm d)\sqrt{6}i) \in \mathbb{Z}[\sqrt{-6}] \) and \( (a+b\sqrt{6}i)(c+d\sqrt{6}i) = (ac-6bd) + (ad+bc)\sqrt{6}i \in \mathbb{Z}[\sqrt{-6}] \) for all \( a,b,c,d \in \mathbb{Z} \).

b) Let \( N \) be the norm map with \( N(a+b\sqrt{6}i) = a^2 + 6b^2 \). This map is multiplicative as \( N((a+b\sqrt{6}i) \pm (c+d\sqrt{6}i)) = (a^2+6b^2) \pm (c^2+6d^2) \). So if \( a+b\sqrt{6}i \) is inverse to \( c+d\sqrt{6}i \) in \( \mathbb{Z}[\sqrt{-6}] \) then \( (a^2+6b^2)(c^2+6d^2) = N((a+b\sqrt{6}i)(c+d\sqrt{6}i)) = N(1) = 1 \). But then \( a^2 + 6b^2 = 1 \) and \( c^2 + 6d^2 \in \mathbb{N} \). We have thus for any unit \( a+b\sqrt{6}i \) that \( a^2 - 1 = 6b^2 \) if \( 0 \in \mathbb{Z} \), which implies that \( a \equiv =1 \) and \( b \equiv =0 \). There are therefore just two units in \( \mathbb{Z}[\sqrt{-6}] \) given by \( \pm 1 \).

c) If \( a+b\sqrt{6}i \) and \( c+d\sqrt{6}i \) are in \( \mathbb{Z}[\sqrt{-6}] \), then \( N((a+b\sqrt{6}i)(c+d\sqrt{6}i)) = (a^2+6b^2)(c^2+6d^2) \). Also, if \( b \equiv =0 \) then \( a^2 + 6b^2 \equiv =0 \) and \( b \equiv =0 \) then \( a^2 + 6b^2 \equiv =0 \) if not \( a \equiv =1 \). We have therefore, if of \( a+b\sqrt{6}i \) and \( c+d\sqrt{6}i \) are units that \( a^2 + 6b^2 \equiv =0 \) and \( c^2 + 6d^2 \equiv =0 \) and \( N((a+b\sqrt{6}i)(c+d\sqrt{6}i)) \equiv >1 \). The non-units \( 2, 3, \sqrt{-6}i, -\sqrt{-6}i \) are thus all irreducible as \( N(2)=4, N(3)=9 \) and \( N(\sqrt{-6}i) = N(-\sqrt{-6}i) = 6 \). We have thus two different prime factorisations of 6 given by \( 2 \times 3 \) and \( (\sqrt{-6}i)(-\sqrt{-6}i) \) in \( \mathbb{Z}[\sqrt{-6}] \), where 2 and 3 are not associates to \( \pm \sqrt{-6}i \) so \( \mathbb{Z}[\sqrt{-6}] \) is not a UFD.

6a). If \( I \subseteq R \) and \( J \subseteq S \) are ideals in the rings \( R \) and \( S \), then it follow from the subgroup criterion that \( I \cap J \) is an additive subgroup of \( R \times S \). Further, if \( (r,s),(r',s) \in I \cap J \) then \( (r,s)(r',s) = (rr',ss) \) and \( (i,j)(r,s) = (ir,js) \) are both in \( I \cap J \), such that \( I \cap J \) is an ideal of \( R \times S \). If we apply this to the two trivial ideals in \( R \) and \( S \), then we get four ideals \( R \times \{0 \} \), \( \{0 \} \times S \), \( \{0_k \} \times \{0_k \} \) and \( \{0 \} \times \{0 \} \) in \( R \times S \) for any two rings and in particular for fields \( R = K \) and \( S = L \).

b) Let \( I \) be an ideal in \( K \times L \) for two fields \( K \) and \( L \). There are then four cases.

Case 1: There exists \( (a,b) \in I \) with \( a \equiv =0 \) and \( b \equiv =0 \). Then \( (a',b')(a,b) = (1a,1b) \in I \) such that \( I = K \times L \), as any \( (r,s) = (r,s)(1,1) \in I \).

Case 2: \( I \subseteq K \times \{0 \} \), but \( I \not= \{0 \} \times \{0 \} \). We have then that \( (a,0) \in I \) for some \( a \equiv =0 \) and hence that \( (a',0)(a,0) = (1a,0) \in I \). But then \( I = K \times \{0_k \} \) as any element \( (r,0) \in K \times \{0 \} \) lies in \( I \) as \( (r,0)(1,0) \in I \).

Case 3: \( I \subseteq \{0_k \} \times L \), but \( I \not= \{0 \} \times L \). Then \( (0_k,1a) \in I \) and \( I = \{0_k \} \times L \) by the same arguments as in case 2.

Case 4: \( I = \{0 \} \times \{0 \} \).

There are thus no other ideals in \( K \times L \) than \( K \times L \), \( K \times \{0 \} \), \( \{0_k \} \times L \) and \( \{0_k \} \times \{0 \} \) if \( K \) and \( L \) are fields.