MATHEMATICS Univ.of Gothenburg and Chalmers University of Technology
Examination in algebra: MMG500 and MVE 150, 2019-06-10.
No books, written notes or any other aids are allowed.
Telephone 031-772 5325.

1. Which or the following statements are correct?
   a) Every integral domain of characteristic $p>0$ is a field. 1p
   b) Every field is an integral domain. 1p
   c) Every finite commutative ring is an integral domain. 1p

2. Let H be a subgroup of a group G and m,n be two integers with $(m,n)=1$. 4p
Let $g\in G$ be an element such that $g^m\in H$ and $g^n\in H$. Prove that $g\in H$.

3. Let $I_1\subseteq I_2 \subseteq \ldots \subseteq I_k\subseteq\ldots$ be an infinite chain of (possibly equal) ideals in 4p
$\mathbf{C}[x]$. Prove that this chain becomes stationary $I_n= I_{n+1}= \ldots$ after some $n\in$N.

4 The sides of a cube are coloured in blue, green and red and two coloured 5p
cubes are identified if they are related by a rotational symmetry. Show that
there are exactly 57 such cubes.
(Only solutions based on group theory will receive points.)

5. Let $* : G\times G \to G$ be an associative binary operation on a set G. 4p
a) Show that $(G, *)$ has at most one neutral element.
b) Show that each element of G has at most one inverse with respect to $*$.

6. Formulate and prove Lagrange's theorem. 5p
(The proof should be complete and not based on any unproved lemmas except that you
may use the fact that the equivalence classes of an equivalence relation form a partition.)

The theorems in Durbin's book may be used to solve the first four exercises. But all claims that are made should be ***motivated****.*

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Brief solutions to examination in algebra: MMG500 and MVE 150, 2019-06-10.

1a) False. If p is a prime, then $\mathbf{Z}_p[x]$ is an integral domain of characteristic p, but not a field.

1b) True. If $a\neq 0,b$ are elements in a field K, then $ab=0 \Rightarrow a^{-1}(ab)=0 \Rightarrow (a^{-1}a)b=0 \Rightarrow b=0$.

1c) False. $\mathbf{Z}_6$ is a finite commutative ring with $[2][3]=[0]$ in $\mathbf{Z}_6$.

2) There exists by Euclid´s algorithm integers a,b with $ma+nb= (m, n)$. We have, therefore, if $g^m, g^n \in H$ that $g^{(m, n)} = g^{ma+nb} = (g^m)^a(g^n)^b \in H$. So if m and n are coprime, then $g\in H$.

3) If all $I_m=\{0\}$, then there is nothing to prove. We may hence assume that $I_k \neq\{0\}$for some k. Then I_m is generated by a monic polynomial for all $m\geq k$ with f_{m+1} dividing f_m as $I_m \subseteq I_{m+1}$. We have therefore a sequence of non-negative integers $\deg f_k \geq \deg f_{k+1} \geq \ldots$,which will become stationary $\deg f_n= \deg f_{n+1}=\ldots$ after some $n\geq k$. As f_{m+1} divides f_m for all m we have thus that $f_n= f_{n+1}=\ldots$ and $I_n= I_{n+1}=\ldots$, as was to be proved.

4) There are 24 rotations of the cube. They are (see Example 57.3 in Durbin's book)

1. The identity.
2. Three 180° rotations around lines joining the centers of opposite faces.
3. Six 90° rotations around lines joining the centers of opposite faces.
4. Six 180° rotations around lines joining the midpoints of opposite edges.
5. Eight 120° rotations around lines joining opposite vertices.

These rotations form a group G acting on the set T of 3-colourings of the sides. For $g\in G$, let $\Psi(g)$ be the number of 3-colourings preserved by g. It is equal to $3^{n(g)}$ for the number $n(g)$ of orbits of the action of $\langle g\rangle$ on the set S of the six sides of the cube.

We have for $\underline{g}$ of type 1, 2, 3, 4 resp. 5 the following $\langle g\rangle$–orbits on S.

1. Six orbits of length 1.
2. Two orbits of length 1 and two orbits of length 2.
3. Two orbits of length 1 and one orbit of length 4.
4. Three orbits of length 2.
5. Two orbits of length 3.

In particular, $\Psi(g)=3^6$, 3^4 , 3^4 , 3^3 resp. 3^2 such that $o(G)^{-1}\sum_{g\in G}\Psi(g)= \frac{1}{24} \sum_{g\in G}3^{n(g)} =$
$\frac{1}{24}(1\times 3^6 +3\times 3^4 +6\times 3^3 +6\times 3^3 +8\times 3^2) = \frac{3^2}{24}(1\times 3^4 +3\times 3^2 +6\times 3^1 +6\times 3^1 +8\times 3^0) = \frac{3}{8}\, 152 =57$.

There are thus by Burnside's lemma 57 inequivalent 3-colourings of the six sides.

5 See Durbin's book

6 See Durbin's book