MATHEMATICS
University of Gothenburg and Chalmers University of Technology.
Examination in algebra : MMG500 and MVE 150, 2019-03-22.
No aids are allowed. Telephone 031-772 5325.

1) Let R be an integral domain with four elements $0,1,a$ and b where 1 is the neutral element for multiplication. Prove the following statements in R.

a) $1+1=0$. 2p

b) $a+1=b$. 1p

c) $a^2=b$. 2p

2) Let R be the set of all rational numbers of the form $m/2^n$ for integers m and n.

a) Show that R is a subring of $\mathbf{Q}$. 2p

b) Which of the integers 2, 4 and 6 are irreducible in R ? 2p

Which of these three integers are units in R ?

3a) Determine the number of elements of order 5 in S_5. 2p

3b) Determine the number of subgroups of order 5 in S_5. 2p

4) Let T be the set of subgroups of order 5 of S_5. If $\sigma\in S_5$, let $\pi_\sigma: T\rightarrow T$ be the bijective map which sends a subgroup H of order 5 to the subgroup $\sigma H\sigma^{-1}:=\{\sigma h\sigma^{-1} : h\in H\}$ in T.

a) Prove that the map $\pi: S_5\rightarrow \mathrm{Sym}(T)$, which sends $\sigma\in S_5$ to $\pi_\sigma\in \mathrm{Sym}(T)$ gives an action of S_5 on T. 2p

b) The above action of S_5 on T is transitive by a theorem of Sylow. Use this to show that π is injective. 2p

5a) Show that each group has at most one neutral element. 2p

5b) Show that each element of a group has at most one inverse. 2p

6) Prove that congruence modulo n is an equivalence relation on $\mathbf{Z}$ for each positive integer n. 4p

You may use the theorems in Durbin's book to solve the first 4 exercises.
But all claims should be motivated !

Solutions to algebra exam: MMG500 and MVE 150, 2019-03-23.

1a) On applying a corollary of Lagrange's theorem to the underlying additive group we get that $(1+1)^2=1+1+1+1=0$. Hence $1+1=0$ as we are in an integral domain.

b) The group equation $x+1=b$ has a unique solution $x=b-1$. But $0+1=1\neq b$, $1+1=0\neq b$ and $b+1\neq b$. Hence $x\notin\{0,1,b\}$, which implies that $x=a$ and $a+1=b$.

c) A finite integral domain is a field and hence $\{1, a, b\}$ a multiplicative group of prime order. We have thus by a corollary of Lagrange's theorem that $\{1, a, b\}=\{1, a, a^2\}$. So $a^2=b$.

2a) If $k.l.m.n\in$Z, then $k/2^l\pm m/2^n = (2^{n-l}k \pm m)/2^n= (k\pm 2^{l-n}m)/2^l\in R$ as $2^{n-l}k \pm m\in \mathbf{Z}$ or $k \pm 2^{l-n}m\in \mathbf{Z}$. Hence R is closed under addition and subtraction. . R is also closed under multiplication as $(k/2^l)(m/2^n) = km/2^{l+n}\in R$. Hence R is a subring of $\mathbf{Q}$ by the subring criterion.

b) 2 and 4 are units in R with inverses 1/2 resp. 1/4 in R. They are thus not irreducible in R.

6 is not a unit in R as $1/6\notin R$. To decide if 6 is irreducible or nor we use that all elements of R are of the form $m/2^n$ with $n\geq 0$. If now $3=(k/2^l)(m/2^n)$ with $l,n\geq 0$, then $2^{l+n}3=kl$ in N. Hence k or l is then a 2-power and $k/2^l$ or $m/2^n$ a unit. This means that 6 is irreducible in R.

3a) An element of order 5 in S_5 can only have one cycle of order 5. It is therefore of the form $(1\ a\ b\ c\ d)$ with $\{a,b,c,d\}=\{2,3,4,5\}$. There are 4!=24 orderings of {2,3,4,5} and hence 24 elements of order 5 in S_5 .

b) The intersection of two different subgroups of order 5 is of order 1 by Lagrange's theorem.

We have thus a partition of the 24 elements of order 5 into disjoint subsets with 4 elements,

where each such subset consists of the 4 non-neutral elements in a subgroup of order 5. There are thus 6=24/4 such subsets and 6 subgroups of order 5.in S_5.

4a) Let $\sigma,\tau\in S_5$ and H be a subgroup of order 5 in S_5. Then,

$\pi_{\sigma\tau}(H)=(\sigma\tau) H(\sigma\tau)^{-1}=\{(\sigma\tau)h(\sigma\tau)^{-1} : h\in H\}=\{(\sigma\tau)h(\tau^{-1}\sigma^{-1}) : h\in H\}$,

$\pi_\sigma(\pi_\tau (H)) = \sigma(\tau H\tau^{-1})\sigma^{-1}=\{\sigma(\tau h\tau^{-1})\ \sigma^{-1} : h\in H\}=\{(\sigma\tau)h(\tau^{-1}\sigma^{-1}) : h\in H\}$.

Hence $\pi_{\sigma\tau}=\pi_\sigma\pi_\tau$ in Sym(T) for all $\sigma,\tau\in S_5$, which means that π is an action of S_5 on T.

b) K=ker π is contained in the stabilizer $N(H)=\{\sigma\in S_5 : \sigma H\sigma^{-1}=H\}$ of any $H\in T$. If the normal subgroup K contained a Sylow 5-subgroup of S_5, then it would contain all such subgroups as they are conjugate in S_5. It would thus then be of order >24 as it contains all elements of order 5 in S_5. But this is impossible as K is a subgroup of $N(H)$ for any $H\in T$ and $o(N(H))= o(S_5)/\text{Card } T=20$ as S_5 acts transitively on T. The order of K can therefore not be divisible by 5 as it contains no Syolw-5-subgroup. It is thus by Lagrange's theorem of order 1,2 or 4 as a subgroup of $N(H)$. If $o(K)=2$ or 4, then there is an element $\sigma\in K$ of order 2. But all conjugates of σ are then also in K as it is normal in S_5. We have thus then that (12) and all its 9 conjugates are in K or that (12)(34) and all its 14 conjugates are in K. But this is not possible that $o(K)\leq 4$. Hence $o(K)=1$ and π injective.