MATHEMATICS
Univ. of Gothenburg and Chalmers University of Technology
Examination in algebra : MMG500 and MVE 150, 2018-06-01.
No books, written notes or any other aids are allowed.
Telephone 031-772 5325.

1) Let ϕ be a homomorphism from the multiplicative group G to the additive group $\mathbf{Z}_2\times\mathbf{Z}_3\times\mathbf{Z}_4\times\mathbf{Z}_6$. Prove that $g^{12}\in\ker\phi$ for any $g\in G$. 3p

2). Prove that there is an element of order 990 in S_{30} . 4p

3) The center of a ring R is the subset $Z(R)$ of all elements x such that $xy = yx$ for all elements y in R. Show that $Z(R)$ is a subring. 4p

4) A bead is placed at each of the eight vertices of a cube, and each bead is to be painted either red or blue. Under equivalence relative to the group of rotations of the cube, how many distinguishable patterns are there? 4p

5. Formulate and prove Lagrange's theorem. 5p

(You may use general results from set theory, but any result on cosets that is needed should be proved.)

6, Show that a polynomial of degree $n\geq 1$ over a field F has at most n roots in F. 4p

Solutions to examination in algebra 2018-06-07.

MMG500 and MVE 150

1) Let $g \in G$ and $\phi(g) = ([a]_2, [b]_3, [c]_4, [d]_6)$. Then $\phi(g^{12}) = 12\phi(g) = (12[a]_2, 12[b]_3, 12[c]_4, 12[d]_6) = ([0]_2, [0]_3, [0]_4, [0]_6)$ such that $g^{12} \in \ker\phi$ for every $g \in G$.

2) We know by the fundamental theorem of abelian groups that there is a bijection between factorisations of 200 into prime powers (up to ordering) and isomorphism classes of abelian groups of order 200. There are six such factorizations of 200, namely 8×25, $4\times2\times25$, $2\times2\times2\times25$, $8\times5\times5$, $4\times2\times5\times5$ and $2\times2\times2\times5\times5$.

There are thus six isomorphism classes of abelian groups of order 200. They are represented by $\mathbf{Z}_8\times\mathbf{Z}_{25}$, $\mathbf{Z}_4\times\mathbf{Z}_2\times\mathbf{Z}_{25}$, $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_{25}$, $\mathbf{Z}_8\times\mathbf{Z}_5\times\mathbf{Z}_5$, $\mathbf{Z}_4\times\mathbf{Z}_2\times\mathbf{Z}_5\times\mathbf{Z}_5$ and $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_5\times\mathbf{Z}_5$..

3) We use the subring criterion. First, $0 \in \mathrm{Z}(R)$ as $0r=0=r0$ for all $r\in R$.

Next, let $z_1, z_2 \in \mathrm{Z}(R)$ and $r \in R$. Then,

$(z_1+z_2)r = z_1r + z_1r = rz_1 + rz_2 = r(z_1+z_2)$

$(-z_1)r = -z_1r = -rz_1 = r(-z_1)$

$(z_1z_2)r = z_1(z_2r) = z_1(rz_2) = (rz_1)z_2 = (rz_1)z_2 = r(z_1z_2)$.

Hence z_1+z_2, $-z_1$, $z_1z_2 \in \mathrm{Z}(R)$ for all $z_1, z_2 \in \mathrm{Z}(R)$, which shows that $\mathrm{Z}(R)$ is a subring of R

4) There are 24 rotations of the cube which form a group G acting on the set S of 2-colourings of the vertices. For $g\in G$, let $\Psi(g)$ be the number of 2-colourings preserved by g. The number of inequivalent 2-colourings of the vertices is then $\frac{1}{|G|}\sum_{g\in G}\Psi(g)$ by Burnside's lemma. Also, $\Psi(g)=2^{n(g)}$ for the number $n(g)$ of orbits of the action of $<g>$ on the set V of vertices.

The 24 rotations in G are described in Example 57.3 in Durbin's book.

1. The identity.
2. Three 180° rotations around lines joining the centers of opposite faces.
3. Six 90° rotations around lines joining the centers of opposite faces.
4. Six 180° rotations around lines joining the midpoints of opposite edges.
5. Eight 120° rotations around lines joining opposite vertices.

For g of type 1, 2, 3, 4 resp.5 we have the following $<g>$– orbits on V.

1. Eight orbits of length 1.
2. Four orbits of length 2.
3. Two orbits of length 4.
4. Four orbits of length 2.
5. Two orbits of length 1 and two orbits of length 3.

Hence $\Psi(g)=2^8$, 2^4, 2^2, 2^4 resp. 2^4. The number of inequivalent 2-colourings of the vertices is therefore

$\frac{1}{24}(1\times2^8+3\times2^4+6\times2^2+6\times2^4+8\times2^4)=\frac{1}{24}(2^8+17\times2^4+6\times2^2)=\underline{23}$

5) See sections 16 and 17 in Durbin's book.

6) See section 35 in Durbin's book.