MATHEMATICS
Univ. of Gothenburg and Chalmers University of Technology
Examination in algebra : MMG500 and MVE 150, 2018-03-16.
No aids are allowed. Telephone 031-772 5325.

1a) Let σ=(123) and τ=(145). Compute the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ in S_5. 3p
(The answer should be given in cycle form.)
b) Show that $\sigma\tau\sigma^{-1}\tau^{-1}$ belongs to the subgroup A_5 of even cycles in S_5. 1p

2a) Let ϕ be a homomorphism from **Z** to a finite group G of order n. 3p
Prove that $\langle n\rangle \subseteq \ker \phi$.
b) Show that $\ker \phi = \langle n\rangle$ if and only if ϕ is surjective. 2p

3. Show that the rings $R= \mathbf{Z}[\sqrt{2}]=\{a+b\sqrt{2}: a,b\in\mathbf{Z}\}$ and 4p
$S =\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a,b \in \mathbf{Z}\}$ are isomorphic.

4a) Verify that $1/(3+2\sqrt{2})\in\mathbf{Z}[\sqrt{2}]$. 2p
b) Prove that $R=\mathbf{Z}[\sqrt{2}]$ has infinitely many units. 2p

5. Formulate and prove the fundamental homomorphism 4p
theorem for groups.

6. Prove that any ideal of a polynomial ring $F[x]$ over a field F 4p
is a principal ideal.

The theorems in Durbin's book may be used to solve exercises 1–4,
but all claims that are made must be motivated.

Solutions to examination in algebra: MMG500 and MVE150 2018 -03-16.

1a) If σ=(123) and τ=(145), then σ^{-1}=(132) , τ^{-1}=(154) and

$$\sigma\tau\sigma^{-1}\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} = (124).$$

b) Any 3-cycle (abc) = (ac)(ab). In particular, $\sigma\tau\sigma^{-1}\tau^{-1}$ = (14)(12) is even.

2a) For $nk \in <n>$, then $\phi(nk) = \phi(k)^n$ by a counting rule for homomorphisms. We have also by a corollary of Lagrange's theorem that $\phi(k)^n = e$ as $\phi(k)$ belong to a group of order n. Hence $\phi(nk) = e$ for all $k \in \mathbf{Z}$, thereby proving the assertion.

2b) For $l,m \in \mathbf{Z}$ with $[l]_n = [m]_n$, then $\phi(l)\phi(m)^{-1} = \phi(l-m) = e$ as $<n> \subseteq \ker \phi$. There is thus a well defined map $\theta: \mathbf{Z}_n \to G$, which sends $[m]_n$ to $\phi(m)$. This map is a homomorphism as $\theta([k]_n \oplus [m]_n) = \theta([k+m]_n) = \phi(k+m) = \phi(k)\phi(m) = \theta([k]_n)\theta([m]_n)$. On using that $\ker \theta = \ker \phi/<n>$, $\mathrm{im}\ \theta = \mathrm{im}\ \phi$ and $o(\mathbf{Z}_n) = o(G)$, we have thus $\ker \phi = <n> \Leftrightarrow \ker \theta = \{[0]_n\} \Leftrightarrow \theta$ is one-to one $\Leftrightarrow \theta$ is onto $\Leftrightarrow \phi$ is surjective.

3) Let $\theta: R \to S$ be the bijective map which sends $a+b\sqrt{2}$ to $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$. Then

$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$ is sent to $\begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$

while $(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2}$ is sent to $\begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix} =$

$= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ bc+ad & 2bd+ac \end{pmatrix} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$. Hence θ is additive and mulitplicative, as was to be proved.

4a) As $(3+2\sqrt{2})(3-2\sqrt{2}) = 3^2 - (2\sqrt{2})^2 = 9-8=1$, we get that $1/(3+2\sqrt{2}) = 3-2\sqrt{2}$.

b) We first note that $(3+2\sqrt{2})^n(3-2\sqrt{2})^n = ((3+2\sqrt{2})(3-2\sqrt{2}))^n = 1^n = 1$. Hence as $3+2\sqrt{2}>1$, we have a strictly increasing sequence $(3+2\sqrt{2})^n$, $n \in \mathrm{N}$ of units in $R=\mathbf{Z}[\sqrt{2}]$.

5) See theorem 23.1 in Durbin's book.

6) See theorem 40.3 in Durbin's book.