MATHEMATICS Univ.of Gothenburg and Chalmers University of Technology
Examination in algebra: MMG500 and MVE 150, 2017-08-16.
No books, written notes or any other aids are allowed.
Telephone 031-772 5325.

1 Prove or disprove the following statements

a) Every group of order at most 4 is cyclic. 2p

b) Every group of order at most 6 is abelian 2p

2a) Define the characteristic of a ring. . 1p

2b). Determine the characteristic of the ring $2\mathbf{Z}$ of even integers. 1p

2c) Determine the characteristic of the ring $2\mathbf{Z} \times \mathbf{Z}_3$ 1p

2d). Determine the characteristic of the infinite direct product 1p

$\prod \mathbf{Z}_p = \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times \ldots$ over all primes p.

3a) Explain why the non-zero elements in $\mathbf{Z}_{2017}$ form a multiplicative group. 1p

3b) Explain why the equation $x^n = 1$ cannot have more than n roots in $\mathbf{Z}_{2017}$. 2p

3c) Show by means of theorems in group theory that the equation $x^{32} = 1$ has exactly 32 roots in $\mathbf{Z}_{2017}$. 2p

(Note that $2016 = 2^5 3^3\, 7$ while 2017 is not divisible by any prime $p<45$.)

4 The sides of a cube are marked with one to six dots to form a die and two marked cubes give the same dice if and only if they are related by a rotational symmetry. Show that there are exactly 30 dice. 4p

(Only solutions based on group theory will receive points.)

5. Formulate and prove Lagrange's theorem. 4p

6. Show that any finite integral domain is a field. 4p

The theorems in Durbin's book may be used to solve exercises 1–4, but all claims that are made must be motivated.

Solutions to examination in algebra 2017-08-16.
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1a. The assertion is false as $\mathbf{Z}_2\times\mathbf{Z}_2$ is a group of order 4, which is not cyclic.

1b. The assertion is false as S_3 is a group of order 6, which is not abelian.

2a. The characteristic of a ring R is 0 if for some $r\in R$ all finite sums $r+\ldots+r\neq 0$.

Otherwise the characteristic of R is the smallest positive integer n such that $nr=0$ for all $r\in R$.

2b. The characteristic of $R=2\mathbf{Z}$ is 0 as all finite sums $2+\ldots+2\neq 0$.

2c. The characteristic of $R=2\mathbf{Z}\times\mathbf{Z}_3$ is 0 as all finite sums of copies of $(2,[0]_3)\in R$ are different from $(0,[0]_3)$

2d. Any multiple $n([1]_2,[1]_3,[1]_5,\ldots)$, $n\in\mathbf{N}$ of $([1]_2,[1]_3,[1]_5,\ldots)\in\prod\mathbf{Z}_p$ is $\neq 0$ as $n[1]_p\neq[0]_p$ for any prime p not dividing n. The ring is thus of characteristic 0.

3a. The non-zero elements in $\mathbf{Z}_{2017}$ form a multiplicative group as 2017 is a prime and $\mathbf{Z}_p$ is a field for any prime p. (Theorem in Durbin's book.)

3b. We have by a theorem in Durbin's book that a polynomial $f(x)\in K[x]$ over a field has at most n zeroes in K. Hence as $\mathbf{Z}_{2017}$ is a field $x^n-1=0$ cannot have more than n zeroes in $\mathbf{Z}_{2017}$

3c. Let $U(\mathbf{Z}_{2017})$ be the multiplicative group of units in $\mathbf{Z}_{2017}$. As this group is of order $2016=2^5 3^3\, 7$, it contains a Sylow 2-subgroup G of order 32 in $U(\mathbf{Z}_{2017})$. By a corollary of Lagrange's theorem we have therefore that $x^{32}=1$ for any $x\in G$. There are thus at least 32 elements $x\in\mathbf{Z}_{2017}$ with $x^{32}=1$ and hence exactly 32 such elements by 3b.

4 The symmetry group of the cube consists of 24 rotations (see section 57 in Durbin's book). This group acts on the set of 6! =720 numberings by 1,2,3,4,5,6 dots of the sides of the cube. But none of these numberings will be fixed by any other rotation than the identity. We have therefore by Burnside's counting theorem (see op.cit.) that the number of non-equivalent dice is 720/24=30.

5. See Durbin's book.

6. See Durbin's book