MATHEMATICS Univ. of Gothenburg and Chalmers University of Technology
Examination in algebra : MMG500 and MVE 150, 2017-03-17.
No books, written notes or any other aids are allowed.
Telephone 031-772 5325.

1a) Give an example of a non-cyclic abelian group of order 12.
2p

b) Give an example of a non-abelian group of order 12.
2p

2. Let $\varphi: S_n \rightarrow H$ be a homomorphism from the symmetric group S_n to a group H or order 3

a) What can be said about the order of $\varphi(\tau)\in H$ for a transposition $\tau \in S_n$?
2p

b) Use this information to deduce that S_n has no normal subgroup of index 3.
2p

3. Let K be a field with 4 elements and $\alpha\in K$ an element with $\alpha\neq 0$ and $\alpha\neq 1$.
a) Show that $\alpha^{2017}=\alpha$ in K.
2p
b) Show that $\alpha^{2018}+\alpha^{2017}+1=0$ in K.
2p
c) Is the polynomial $x^{2018}+x^{2017}+1$ irreducible over K ?
1p

4. Let m,n be two positive integers where m divides n There is then a natural
4p

map $\phi : U(\mathbf{Z}_n) \rightarrow U(\mathbf{Z}_m)$ between the unit groups of $\mathbf{Z}_n$ and $\mathbf{Z}_m$, which sends $[a]_n$ to $[a]_m$ for any integer a with $(a, n) =1$. Show that this map is surjective.

(Hint : Study first the case where m is a prime power.)

5. Let $* : G\times G \rightarrow G$ be an associative binary operation on a set G.
4p

a) Show that $(G, *)$ has at most one neutral element.

b) Show that each element of G has at most one inverse with respect to $*$.

6. Show that the kernel of a ring homomorphism θ: $R \rightarrow S$ is an ideal of R.
4p

(You should verify all conditions for a subset of R being an ideal.)

The theorems in Durbin's book may be used to solve exercises 1−4, but all claims that are made must be motivated.

MATHEMATIK Göteborgs universitet och Chalmers tekniska högskola
Examen I algebra : MMG500 och MVE 150, 2017-03-17.
Inga hjälpmedel är tillåtna
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1a) Ge exampel på en abelsk grupp av ordning 12,. som ej är cyklisk. .
2p

b) Ge exampel på en icke abelsk grupp of order 12.
2p

2. Låt φ: $S_n \rightarrow H$ vara en homomorfi från den symmetriska gruppen S_n till en grupp H av ordning 3.

a) Vad kan man saga om ordningen av bilden $\varphi(\tau)$ av en transposition $\tau \in S_n$? 2p

b) Härled från denna information att S_n saknar normala delgrupper av index 3. 2p

3. Låt K vara en kropp med 4 element och $\alpha \in K$ att element där $\alpha \neq 0$ och $\alpha \neq 1$.
a) Visa at $\alpha^{2017} = \alpha$ i K.
2p
b) Vsia att $\alpha^{2018} + \alpha^{2017} + 1 = 0$ i K.
2p
c) Är polynomet $x^{2018} + x^{2017} + 1$ irreducibelt över K ?
1p

4. Låt m,vara två psoitiva heltal där m delar n och $\phi : U(\mathbf{Z}_n) \rightarrow U(\mathbf{Z}_m)$ den
4p naturlig mellan enhetsgrupperna $\mathbf{Z}_n$ and $\mathbf{Z}_m$, som skickar $[a]_n$ to $[a]_m$ för varje heltal a med $(a, n) = 1$. Visa att denna avbildning är surjektiv.

(Ledning : Studera först fallet när m är en primpotens.)

5. Låt $*$: $G \times G \to G$ vara associativ binär operation på en mängd G.
4p

a) Visa att $(G, *)$ har högst ett neutralt element.

b) Show that each element of G has at most one inverse with respect to $*$.

6. Show that the kernel of a ring homomorphism θ: $R \to S$ is an ideal of R.
4p

(You should verify a<u>ll c</u>onditions for a subset of R being an ideal.)

The theorems in Durbin's book may be used to solve exercises 1−4, but all claims that are made must be motivated.

Solutions to exam in algebra MMG500

2017-03-17

1a) $\mathbf{Z}_6\times\mathbf{Z}_2$ is abelian of order 12, but not cyclic as $6a=0$ for every $a\in \mathbf{Z}_6\times\mathbf{Z}_2$.
b) $S_3\times\mathbf{Z}_2$ is non-abelian of order 12 as S_3 is non-abelian of order 6.

2a) $\varphi(\tau)^2= \varphi(\tau^2)=\varphi(\mathrm{id})=e$ as φ is a homomorphism. As $o(H)=3$, we have also that $\varphi(\tau)^3=e$ by a theorem. So $\varphi(\tau)=e$ such that τ of order one.

b) As any element in S_n is a product of transpositions we deduce from a) that Im $\varphi=\{e\}$ for any homomorphism $\varphi:S_n\to H$ to a group of order 3. But the quotient map $\varphi:S_n\to S_n/N$ of a normal subgroup is always surjective. There can thus be no normal subgroup of index 3 in S_n as otherwise we would have a contradiction.

3. We first note that $(2a)^2+3(2b)^2\equiv 0 \pmod 4$ as $(2a)^2\equiv(2b)^2\equiv 0 \pmod 4$ when $2a$ and $2b$ are even and as $(2a)^2\equiv(2b)^2\equiv 1 \pmod 4$ when $2a$ and $2b$ are odd. Hence $N(z):=|z|^2=a^2+3b^2\in\mathbf{N}$ for any $z=a+b\sqrt{3}i\in R$ different from 0. So if z is a unit in R with inverse w, then $N(z)=1$ as $N(z)N(w)=N(zw)=N(1)=1$. Hence $(2a)^2+3(2b)^2= N(2z)= N(2)N(z)=4$, which implies that $(2a)^2=4$ and $(2b)^2=0$ or that $(2a)^2=(2b)^2=1$. There are thus at most six possible units given by ± 1 and $\pm 1/2\pm\sqrt{3}i/2$. These are all units in R as they correspond to the six roots of the equation $z^6=1$.

4a) α belongs to the multiplicative group $U(K)=K\setminus\{0\}$ of order 3. Hence $\alpha^3=1$ and $\alpha^{2017}=\alpha^{2016}\alpha=(\alpha^3)^{672}\alpha=\alpha$.

b) $\alpha^{2018}+\alpha^{2017}+1=\alpha^{2017}\alpha+\alpha+1=\alpha^2+\alpha+1=(\alpha^3-1)/(\alpha-1)=0$ as $\alpha\notin\{0.1\}$.

c) It follows from the factor theorem that $x^{2018}+x^{2017}+1$ is divisible by $x-\alpha$ as it has a zero at $\alpha\in K$. The polynomial is thus not irreducible over K

5. See Durbin's book.

6. See Durbin's book.