MATHEMATICS University of Gothenburg and Chalmers University of Technology
Examination in algebra: MMG500 and MVE 150, 2014-08-20.
No books, written notes or any other aids are allowed.
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1. A relation $\sim$ on a set S is said to be Euclidean if for all $a,b,c \in S$ with $a \sim c$ and $b \sim c$ we have that $a \sim b$.

a) Show that any equivalence relation $\sim$ is Euclidean. 2p

b) Show that any reflexive and Euclidean relation $\sim$ is an equivalence relation. 2p

2. Let G be a group and H be a subgroup. Let $\sim$ be the relation on G such that $a \sim b$ if and only if $ab^{-1} \in H$. Give a direct proof that $\sim$ is reflexive and Euclidean. (It is not enough to refer to the theorem that $\sim$ is an equivalence relation.) 3p

3. Let G be a multiplicative group and S its underlying set.

a) Prove that $G \times G$ acts on S by $\sigma(s) = asb^{-1}$ for $\sigma = (a,b) \in G \times G$ and $s \in S$. 3p

b) Prove that this action of $G \times G$ on S is transitive and determine the stabilizer of the neutral element $e \in S$ under this action. 2p

4. Let R be a ring such that $r^3 = r$ for all $r \in R$.

a) Prove that $r+r+r+r+r+r=0$ for all $r \in R$. 1,5p

b) Prove that $2(xy^2+yxy+y^2x)=0$ for all $x,y \in R$. 1,5p

c) Prove that $2xy = 2yx$ for all $x, y \in R$. 2p

(Hint : To solve c), start with b) and deduce new identities by multiplication from the left and from the right.)

5. Prove that any ideal in $\mathbf{Z}$ is a principal ideal . 4p

6. Show that any finite integral domain is a field. 4p

All claims that are made must be motivated.

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1a) Suppose that $a\sim c$ and $b\sim c$ for an equivalence relation $\sim$. Then $b\sim c \Rightarrow c\sim b$ as $\sim$ is symmetric. So $a\sim c$ and $c\sim b$. Hence by the transitivity of $\sim$ we get that $a\sim b$, which proves that $\sim$ is Euclidean.

1b) Suppose that $b\sim a$ and that $a=c$. Then $b\sim c$. Further, as $\sim$ is reflexive we get $a\sim c$. Finally, as $\sim$ is Euclidean we deduce from $a\sim c$ and $b\sim c$ that $a\sim b$. Hence $b\sim a \Rightarrow a\sim b$, which proves that $\sim$ is symmetric.

To see that $\sim$ is transitive, suppose that $a\sim c$ and $c\sim b$. Then, as $\sim$ is symmetric, we get from $c\sim b$ that $b\sim c$. But then as $\sim$ is Euclidean, we conclude from $a\sim c$ and $b\sim c$, that $a\sim b$, which proves that $\sim$ is transitive.

2. $aa^{-1}=e\in H \Rightarrow a\sim a$ for all $a\in G$ such that $\sim$ is reflexive. To prove that $\sim$ is Euclidean, suppose $a\sim c$ and $b\sim c$ for all $a,b,c\in G$. Then $ac^{-1}\in H$, $bc^{-1}\in H$ and $(ac^{-1})(bc^{-1})^{-1}\in H$. But $(bc^{-1})(cb^{-1})=(cb^{-1})(bc^{-1})=e$. Hence $(bc^{-1})^{-1}=(cb^{-1})$ such that $(ac^{-1})(cb^{-1})\in H$. Hence $ab^{-1}\in H$ and $a\sim b$, thereby proving that $\sim$ is Euclidean.

3a) Let $\sigma=(a,b)\in G\times G$, $\tau=(c,d)\in G\times G$ and $s\in S$. Then $\sigma\tau=(ac, bd)$ and

$$\sigma(\tau(s)) = \sigma(csd^{-1}) = a(csd^{-1})b^{-1} = (ac)s(bd)^{-1} = (\sigma\tau)(s)$$

b) Let $s,t\in S=G$ and $\sigma=(t,s)\in G\times G$. Then $\sigma(s) = tss^{-1} = t$, thereby proving that the action is transitive. Further $\sigma=(g,h)\in G\times G$ is in the stabilizer of e if and only if $\sigma(e) = geh^{-1} = e$, that is if and only if $g=h$. Hence the stabilizer of e is the diagonal Subgroup $G^{\Delta}\subset G\times G$ of all pairs (g, g) , $g\in G$.

4a) $r+r+r+r+r+r= r^3+r^3+ r^3+r^3+ r^3+r^3= (r+r)^3-(r^3+r^3)= (r+r)^3-(r+r)= 0.$

b) $2(xy^2+yxy+y^2x)= (x+y)^3 +(x-y)^3-2x^3= ((x+y)^3-(x+y)) + ((x-y)^3-(x-y))= 0.$

c) $2 (xy^2+yxy+y^2x)y = 0 = 2 y(xy^2+yxy+y^2x)x \Rightarrow$
$2(xy^3+yxy^2+y^2xy)=2(yxy^2+y^2xy+yx^3) \Rightarrow 2xy^3 = 2yx^3 \Rightarrow 2xy = 2yx$

5. See Durbin's book.

6. See Durbin's book.